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A counterexample of the Euler condition: the Appell–Hamel dynamical system on a horizontally moving plate*

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As a counterexample of the Euler condition for nonholonomic constraint problems [H. C. Shen, *Acta Phys. Sin.* **54**, 2468 (2005)], we investigate the Apell–Hamel dynamical system on a horizontally moving plate. The inconsistency of the results with Newton mechanics suggests that the Euler condition is not a universal model for nonlinear nonholonomic systems. This is attributed to the fact that the virtual displacements so obtained are not normal to the constraint forces.

Keywords: Euler condition, nonholonomic constraint, Appell–Hamel dynamical system, Newton solution

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1. Introduction

The linear equations of virtual displacements are of fundamental importance for analytical mechanics, through which differential equations of motion for a mechanical system may be derived. For the holonomic constraints, the virtual displacements can be obtained by a straightforward variation of the constraint equations. The situation for the nonholonomic constraints is however different: Chetaev and others[1−4] expressed the virtual displacement equation as a sum of virtual displacements weighted by the gradient of the constraint with respect to velocities. Such a condition indeed has never been justified by the fundamental principle of mechanics. As an alternative approach, Kozlov[5−8] obtained the virtual displacements by varying the constraint equation with respect to the generalized coordinates and velocities. Li *et al*[9,10] showed that in this model the virtual displacements are not normal to the constraint forces and the virtual work in this case is unreasonably omitted. Vâlcovici[11,12] suggested that at first one should express the constraint equation in an infinitesimal form, and then replace each differential by the variation and let the time variation vanish. Mei[12,13] indicated, however, that the virtual displacements in this case are not unique. Recently, Shen[14] put forward a further treatment by expanding the constraint into a MacLaurin series of velocities before applying the replacement procedure. Shen named this virtual displacement condition the Euler condition, for it is derived by the Euler theorem.

To test the Euler condition, in this work we present a counterexample: the Appell–Hamel dynamical system[12,15−19] in the frame of a horizontally moving plate. The organization of this paper is as follows: In Section 2, we solve the problem according to the Euler condition. In Section 3, the same problem is solved in the framework of Newton mechanics which may be taken as a criterion to judge various models. Conclusions are summarized in Section 4.

2. Solution by the Euler condition

For a mechanical system described by the Lagrangian $L(q, \dot{q}, t)$ (q represent the generalized coordi-

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nates $q_1, \ldots, q_n$) subject to the following constraints:

$$f_j(q, \dot{q}, t) = 0, \quad (j = 1, \ldots, k), \tag{1}$$

expanding the constraint equation (1) in a MacLaurin series in velocities, and replacing each differential by a variation and then making the time variation vanish ($\delta t = 0$), yields the so-called Euler condition:[14]

$$\sum_{\alpha} b_{j\alpha}(q, \dot{q}, t)\delta q_\alpha = 0, \tag{2}$$

where

$$\begin{aligned} & b_{j\alpha}(q, \dot{q}, t) \\ = & \left(\frac{\partial f_j}{\partial \dot{q}_\alpha}\right)_{\dot{q}=0} + \frac{1}{2!}\sum_{\alpha_1=1}^{n}\left(\frac{\partial^2 f_j}{\partial \dot{q}_\alpha \partial \dot{q}_{\alpha_1}}\right)_{\dot{q}=0}\dot{q}_\beta \\ & + \frac{1}{3!}\sum_{\alpha_1,\alpha_2=1}^{n}\left(\frac{\partial^3 f_j}{\partial \dot{q}_\alpha \partial \dot{q}_{\alpha_1}\partial \dot{q}_{\alpha_2}}\right)_{\dot{q}=0}\dot{q}_{\alpha_1}\dot{q}_{\alpha_2} + \ldots \end{aligned} \tag{3}$$

The Routh equation based on the virtual displacements of Eq.(2) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = Q_\alpha + \sum_{j=1}^{k}\lambda_j b_{j\alpha}, \ (\alpha = 1, \ldots, n), \tag{4}$$

where $\lambda_1, \ldots, \lambda_k$ are multipliers.

As an example to test Eq.(2), we set up the Appell–Hamel problem as shown in Fig.1: A block of mass m is attached to the end of a thread which passes over two small pulleys, and is then wound around a drum of radius b. The drum is rigidly attached to a wheel of radius a which may roll on a plate without slip. The plate moves in the xy-plane with a velocity $\dot{s}(t)$ along the direction $\cos\theta_0 \boldsymbol{e}_x + \sin\theta_0 \boldsymbol{e}_y$. The mass of the wheel is M, and the moment of inertia of the wheel about its center and about the vertical symmetry axis are respectively I and I'. The legs of the frame supporting the pulleys may slide on the plate without friction. The masses of the pulleys and the frame are negligible. A nonvanishing force $\boldsymbol{F}$ is applied in the direction tangent to the wheel trace. Such a force may be accomplished, for instance, by a small rocket oriented along the tangent direction. Let x, y, z be the coordinates of the mass m, and θ the angle between the plane of wheel and the x-axis. The angle ϕ describes the rotation of the wheel in its own plane. The distance between the centers of the two pulleys is ρ. In the case $\rho \to 0$, the constraints read,

$$[\dot{x} - \dot{s}(t)\cos\theta_0]\sin\theta - [\dot{y} - \dot{s}(t)\sin\theta_0]\cos\theta = 0, \tag{5}$$

$$[\dot{x} - \dot{s}(t)\cos\theta_0]\cos\theta + [\dot{y} - \dot{s}(t)\sin\theta_0]\sin\theta = a\dot{\phi}, \tag{6}$$

$$\dot{z} = -b\dot{\phi}. \tag{7}$$

From these equations, we obtain a constraint equation which is nonlinear in velocities:

$$[\dot{x} - \dot{s}(t)\cos\theta_0]^2 + [\dot{y} - \dot{s}(t)\sin\theta_0]^2 = \frac{a^2}{b^2}\dot{z}^2. \tag{8}$$

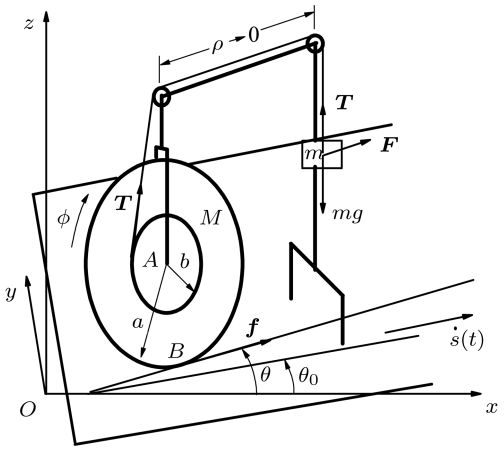


Fig.1. The Appel–Hamel dynamical system on a horizontally moving plate.

For simplicity, we may assume all the mass of the wheel and drum to be at its center A, which implies $I = I' = 0$, and let both the plate and the dynamical system move in the same direction, i.e., $\theta = \theta_0$. In this case, the Lagrangian of the system reads,

$$L = \frac{1}{2}(m + M)(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\dot{z}^2 - mgz, \tag{9}$$

and the nonconservative generalized forces are respectively,

$$Q_x = F\cos\theta, \quad Q_y = F\sin\theta, \quad Q_z = 0. \tag{10}$$

Using the constraint equation (5)–(7), equation (8) can be rewritten as

$$\dot{x}^2 + \dot{y}^2 = \frac{a^2}{b^2}\left[\dot{z} - \frac{b}{a}\dot{s}(t)\right]^2. \tag{11}$$

Expanding this equation into a MacLaurin series in velocities leads to the Euler condition (see Eqs.(2) and (3)):

$$\dot{x}\delta x + \dot{y}\delta y = \frac{a^2}{b^2}\left[\dot{z} - 2\frac{b}{a}\dot{s}(t)\right]\delta z. \tag{12}$$

Then the Routh equation (4) reads,

$$(m + M)\ddot{x} = F\cos\theta_0 + \lambda\dot{x}, \tag{13}$$

$$(m + M)\ddot{y} = F\sin\theta_0 + \lambda\dot{y}, \tag{14}$$

$$m\ddot{z} = -mg - \lambda\frac{a^2}{b^2}\left[\dot{z} - 2\frac{b}{a}\dot{s}(t)\right]. \tag{15}$$

Now we solve for the acceleration tangent to the wheel trace

$$A_\tau = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\dot{x}^2+\dot{y}^2} = \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}. \tag{16}$$

Let

$$\dot{\xi} = \dot{z} - \frac{b}{a}\dot{s}(t), \tag{17}$$

$$g' = g + \frac{b}{a}\ddot{s}(t), \tag{18}$$

then equation (15) becomes

$$m\ddot{\xi} = -mg' - \lambda\frac{a^2}{b^2}\left[\dot{\xi} - \frac{b}{a}\dot{s}(t)\right], \tag{19}$$

Differentiating the constraint equation (11) with respect to time yields,

$$\ddot{\xi} = \frac{b^2}{a^2\dot{\xi}}(\dot{x}\ddot{x}+\dot{y}\ddot{y}). \tag{20}$$

Inserting Eq.(20) into Eq.(19), we obtain

$$m\frac{b^2}{a^2\dot{\xi}}(\dot{x}\ddot{x}+\dot{y}\ddot{y}) = -mg' - \lambda\frac{a^2}{b^2}\left[\dot{\xi} - \frac{b}{a}\dot{s}(t)\right], \tag{21}$$

thus,

$$\lambda = -m\frac{b^2}{a^2}\frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\dot{x}^2+\dot{y}^2+\dot{s}(t)\sqrt{\dot{x}^2+\dot{y}^2}} + mg'\frac{b}{a}\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}+\dot{s}(t)}, \tag{22}$$

here we have used $\dot{\xi} = -\frac{b}{a}\sqrt{\dot{x}^2+\dot{y}^2}$. Inserting Eq.(22) into Eqs.(13) and (14) yields,

$$\ddot{x} + \frac{m}{m+M}\frac{b^2}{a^2}\frac{\dot{x}(\dot{x}\ddot{x}+\dot{y}\ddot{y})}{\dot{x}^2+\dot{y}^2+\dot{s}(t)\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{m}{m+M}g'\frac{b}{a}\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}+\dot{s}(t)} + \frac{F}{m+M}\cos\theta_0, \tag{23}$$

$$\ddot{y} + \frac{m}{m+M}\frac{b^2}{a^2}\frac{\dot{y}(\dot{x}\ddot{x}+\dot{y}\ddot{y})}{\dot{x}^2+\dot{y}^2+\dot{s}(t)\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{m}{m+M}g'\frac{b}{a}\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}+\dot{s}(t)} + \frac{F}{m+M}\sin\theta_0. \tag{24}$$

By using the the relation

$$\dot{x}\cos\theta_0 + \dot{y}\sin\theta_0 = \sqrt{\dot{x}^2+\dot{y}^2},$$

equations (23) and (24) lead to

$$\dot{x}\ddot{x}+\dot{y}\ddot{y} = \frac{mg'b\dfrac{\sqrt{\dot{x}^2+\dot{y}^2}}{\sqrt{\dot{x}^2+\dot{y}^2}+\dot{s}(t)} + Fa}{(m+M)a^2 + mb^2\dfrac{\sqrt{\dot{x}^2+\dot{y}^2}}{\sqrt{\dot{x}^2+\dot{y}^2}+\dot{s}(t)}}a\sqrt{\dot{x}^2+\dot{y}^2}, \tag{25}$$

therefore

$$A_\tau = \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{mg'b\dfrac{\sqrt{\dot{x}^2+\dot{y}^2}}{\sqrt{\dot{x}^2+\dot{y}^2}+\dot{s}(t)} + Fa}{(m+M)a^2 + mb^2\dfrac{\sqrt{\dot{x}^2+\dot{y}^2}}{\sqrt{\dot{x}^2+\dot{y}^2}+\dot{s}(t)}}a. \tag{26}$$

This is the tangent acceleration based on the Euler virtual displacements.

3. The Newton solution

Now we solve the same problem in the framework of Newton mechanics for the general case.

Suppose that the tension in the rope is $\boldsymbol{T}$, the friction in the contact point B is $\boldsymbol{f}$, and the torque along the vertical symmetry axis vanishes, then the Newton equations of the dynamical system read,

$$F + f = (M+m)A_\tau, \tag{27}$$

$$T - mg = m\ddot{z}, \tag{28}$$

$$Tb - fa = I\ddot{\phi}, \tag{29}$$

$$I'\ddot{\theta} = 0. \tag{30}$$

By considering Eqs.(5) and (6), the tangent acceleration can be written as

$$A_\tau = a\ddot{\phi} + B_\tau, \tag{31}$$

where

$$B_\tau = \cos\theta_0 \frac{\mathrm{d}}{\mathrm{d}t}[\dot{s}(t)\cos\theta] + \sin\theta_0 \frac{\mathrm{d}}{\mathrm{d}t}[\dot{s}(t)\sin\theta]. \tag{32}$$

Solving Eqs.(27)–(31), we obtain

$$\ddot{\phi} = \frac{[F-(m+M)B_\tau]a + mgb}{I + (m+M)a^2 + mb^2}. \tag{33}$$

Therefore,

$$A_\tau = \frac{[F-(m+M)B_\tau]a + mgb}{I + (m+M)a^2 + mb^2}a + B_\tau. \tag{34}$$

In a simplified case with $I = I' = 0$, equation (34) becomes

$$A_\tau = \frac{mb(g+B_\tau) + Fa}{(m+M)a^2 + mb^2}a. \tag{35}$$

The solution (26) based on the Euler condition is obviously inconsistent with Newton solution (35) except for $\dot{s}(t) = 0$. Even when the constraint degenerates to the holonomic case ($\theta = \theta_0 = $ const.), the inconsistency cannot be removed. This may be attributed to the fact that the virtual displacements in the Euler condition are not normal to the constraint forces. Thus the Euler condition cannot be used to describe the actual motion of such a dynamical system.

4. Conclusions and final remarks

In summary, we discussed the Appell–Hamel dynamical system in a horizontally moving plate using the Euler condition. The solution is inconsistent with the Newton mechanics even for the holonomic case, which means that the Euler condition is not a universal model for nonlinear nonholonomic systems. The correct virtual displacement condition for nonholonomic constraint problems may be obtained by introducing a special reference frame and a set of velocity-determined virtual displacements in that frame.[20]

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