Vol 18 No 8, August 2009
1674-1056/2009/18(08)/3115-07

Chinese Physics B

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Some characteristics of three exact solutions of Einstein equations minimally coupled to a Quintessence field

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(Received 8 October 2008; revised manuscript received 20 April 2009)

We show some characteristics of three exact solutions to Einstein's gravity equation minimally coupled to a Quintessence field. Besides eternal inflation, several other interesting inflationary processes, such as transitory inflation, are obtained in these solutions. Singularity is avoided in some special cases.

Keywords: inflation, Quintessence, solution, singularity
PACC: 0240, 0420J

1. Introduction

Since the universe was found to be undergoing an accelerated and expanding process,[1−4] which is called inflation, many methods have been put forward to explain this phenomenon. Typical methods, the so-called dark energy models which can be described as Quintessence,[5] Phantom,[6] K-essence,[7] and so on (see review in Ref.[8]), are being used to achieve an inflationary universe. The Lagrangian of a scalar field is minimally coupled to gravity, which is named Quintessence, and given as

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{1}$$

where V is the suitable potential. On the assumption that the universe is homogeneous and isotropic, the corresponding pressure and density of the scalar field can be written as

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \text{ and } \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2}$$

where an overdot denotes the differential with respect to time. We choose metric as the spatially flat Friedman–Lemaitre–Robertson–Walker form, $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}\boldsymbol{x}^2$, where the 3D line-element $\mathrm{d}\boldsymbol{x}^2$ is a flat 3D space, and take a perfect fluid energy-momentum tensor $T^M_{\mu\nu} = (\rho_M + P_M)U_\mu U_\nu - P_M g_{\mu\nu}$, for which the curvature scalar is

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = 6\left(\dot{H} + 2H^2\right), \tag{3}$$

where the Hubble parameter $H = \dot{a}/a$, and $a(t)$ is the scale factor. The cosmological Einstein equations are

$$3M_p^2H^2 - \frac{1}{2}\dot{\phi}^2 - V = 0, \tag{4}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial\phi} = 0, \tag{5}$$

where $M_p = (8\pi G)^{-1/2}$. The above two equations are complicated and difficult to exactly show the details about cosmological evolution by solving them generally. The de Sitter solution: $a(t) \propto \exp(Ht)$ $(H > 0)$ and power-law solution $a(t) \propto t^A$ $(A > 1)$ have been investigated extensively in Refs.[9]–[12]. In addition, a complex solution with a sixth-degree potential was obtained by Islam,[13] and other interesting solutions were shown in Refs.[14]–[17].

Generally speaking, an inflationary solution, which is important to avoid flatness, horizon and isotropic problems, means an accelerated and expanding universe, in which, we need

$$\ddot{a}(t) > 0, \tag{6}$$

where one should note that the expanding universe condition $H > 0$ should be satisfied in the meantime. This condition is easily satisfied by the de Sitter solution and power-law solution. We suppose that condition (6) is satisfied in a region

$$0 \le t_b \le t \le t_e. \tag{7}$$

In inequality (7), if $t_b > 0$, we call this process the transitory inflation; if $t_b = 0$, we call it the imme-

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diate inflation because inflation occurs immediately at the beginning of the universe; and if $t_b = 0$ and $t_e \to \infty$, we call it the eternal inflation. In this paper, we discuss three exact solutions which contain several interesting inflationary processes.

2. Three exact solutions

2.1. Solution one

$$a(t) = a_0 \exp\left[-\alpha t \ln\left(\frac{t}{t_0}\right) + \beta t\right], \tag{8}$$

$$\phi = \pm 2\sqrt{2\alpha} M_p \sqrt{t} + \phi_0, \tag{9}$$

$$V(\phi) = M_p^2 \alpha^2 \left\{3\left[1 - \frac{\beta}{\alpha} + 2\ln\left(\frac{\phi - \phi_0}{\phi_p}\right)\right]^2 - \frac{8M_p^2}{(\phi - \phi_0)^2}\right\}, \tag{10}$$

where $a_0 > 0, \alpha \geq 0, t_0 > 0$, β and ϕ_0 are constants, and $\phi_p = 2M_p\sqrt{2\alpha t_0}$. A simple method to prove that *solution one* and the following two solutions are indeed solutions of Eqs.(4) and (5) is shown in appendix A. If $\alpha = 0$, this solution is reduced to a de Sitter (or anti-de Sitter) solution. Moreover, several characteristics of this solution can be found.

(i) Expanding process. The Hubble parameter is

$$H = \beta - \alpha\left[1 + \ln\left(\frac{t}{t_0}\right)\right]. \tag{11}$$

Let $\xi = \mathrm{e}^{-1+\beta/\alpha}$, it is easy to find that expanding universe condition $H > 0$ induces

$$t < \xi t_0. \tag{12}$$

At $t = \xi t_0$, the universe reaches its biggest extension $a_{\max} = a_0 \mathrm{e}^{\beta t_0 \xi} \xi^{-\alpha t_0 \xi}$.

(ii) Inflationary process. Inflationary condition $\ddot{a} > 0$ leads to

$$t \ln^2\left(\frac{t}{\xi t_0}\right) - \frac{1}{\alpha} > 0. \tag{13}$$

Now, we introduce two new parameters $x = t(\xi t_0)^{-1}$ and $\eta = (\alpha \xi t_0)^{-1}$, then one can find that condition (13) will be reduced to a simple state in which there are only two parameters

$$x \ln^2(x) > \eta. \tag{14}$$

Two parameters make sure that a general picture of inequality (14) can be shown in a planar figure. We let $f(x) = x\ln^2(x)$ and show it in Fig.1. It is easy to find that $f(0) = f(1) = 0$ and $f(x)$ reaches its local maximum $f_{\max} = 4\mathrm{e}^{-2}$ at $x = \mathrm{e}^{-2}$. Noting that expanding condition $H > 0$ satisfies $t < \xi t_0$ $(x < 1)$ and $f(x)$ reaches its minimum at $t = \xi t_0$ $(x = 1)$, thus the existence of inflation requires $\eta < 4\mathrm{e}^{-2}$. Let x_1, x_2 and x_3 $(x_1 < x_2 < x_3)$ be the three solutions of $f(x) = \eta$ with $\eta < 4\mathrm{e}^{-2}$, transitory inflation occurs only in the range $x_1 < x < x_2$ $(x_1\xi t_0 < t < x_2 \xi t_0)$. Whether $\eta < 4\mathrm{e}^{-2}$ or $\eta \geq 4\mathrm{e}^{-2}$, the universe will change from expanding to collapsing when it crosses $t = \xi t_0$. Therefore, this solution provides us with an interesting evolution process which is different from the de Sitter solution and power-law solution, in which inflation occurs immediately and forever.

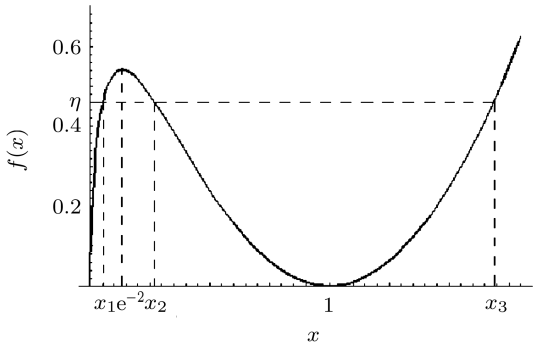


Fig.1. In this figure, x_1, x_2 and x_3 are the three solutions of $f(x) = \eta$ with $\eta < 4\mathrm{e}^{-2}$. Expanding universe satisfies $x < 1$ $(t < \xi t_0)$ and transitory inflation occurs in the range $x_1 < x < x_2$ $(x_1\xi t_0 < t < x_2\xi t_0)$.

(iii) Singularity. Submitting expression (9) into expression (3), the curvature scalar is

$$R = 6\alpha\left[2\alpha\ln^2\left(\frac{t}{\xi t_0}\right) - \frac{1}{t}\right]. \tag{15}$$

To locate singularities in this solution, we should calculate the Kretschmann scalar $K = R_{abcd}R^{abcd}$, which is

$$K = 24\left\{\left[\alpha - \beta + \alpha\ln\left(\frac{t}{t_0}\right)\right]^4 - \frac{\alpha}{t}\left[\alpha - \beta + \alpha\ln\left(\frac{t}{t_0}\right)\right]^2 + \frac{\alpha^2}{2t^2}\right\}. \tag{16}$$

It indicates that there are two singularities in this solution. One is at the initial time $t = 0$ and the other will occur in the infinite future.

(iv) State parameter. For the state parameter

$$\omega = \frac{P}{\rho} = -1 + \frac{2\eta}{3x\ln^2 x}, \tag{17}$$

it reaches its minimum

$$\omega_{\min} = -1 + \frac{\mathrm{e}^2\eta}{6} \tag{18}$$

at $x = \mathrm{e}^{-2}$. Nearby $x = \mathrm{e}^{-2}$, it is just the inflation process (see Fig.1).

2.2. Solution two

$$a(t) = a_0 \exp\left(\mu t - \lambda t^n\right), \tag{19}$$

$$\phi = \pm\frac{2}{n}\sqrt{2n(n-1)\lambda}M_p t^{n/2} + \phi_0, \tag{20}$$

$$V(\phi) = 3M_p^2\left\{\mu - 2^{3/n-3}n\lambda\left[\frac{n(\phi-\phi_0)}{\phi_p}\right]^{2-2/n}\right\}^2 - 2^{6/n-3}(n-1)n\lambda M_p^2\left[\frac{n(\phi-\phi_0)}{\phi_p}\right]^{2-4/n}, \tag{21}$$

where $a_0 > 0, \mu, \lambda$ and ϕ_0 are constants, and $\phi_p = 4M_p\sqrt{(n-1)n\lambda}$. This solution is given by Barrow and Liddle in Ref.[18]. In expression (20), we need $(n-1)n\lambda \geq 0$, which leads to the following conditions:

$$\lambda > 0,\ n < 0; \tag{22}$$

$$\lambda > 0,\ n > 1; \tag{23}$$

$$\lambda < 0,\ 0 < n < 1; \tag{24}$$

$$(n-1)n\lambda = 0. \tag{25}$$

Several characteristics of this solution are shown as follows.

i) De Sitter solution and anti-de Sitter solution. This solution can also be reduced to a de Sitter solution in the following two cases:

$$\mu > 0,\ n\lambda = 0; \tag{26}$$

$$\mu > \lambda,\ n = 1. \tag{27}$$

Anti-de Sitter solution can be attained in two states

$$\mu < 0,\ n\lambda = 0; \tag{28}$$

$$\mu < \lambda,\ n = 1. \tag{29}$$

ii) Expanding process. The Hubble parameter is

$$H = -n\lambda t^{n-1} + \mu. \tag{30}$$

Considering conditions (22)–(25), the expanding universe condition $H > 0$ is valid in the following cases:

$$t \geq 0 \text{ for } \lambda > 0, n < 0, \mu \geq 0; \tag{31}$$

$$t < \left(\frac{\mu}{n\lambda}\right)^{1/(n-1)} \text{ for } \lambda > 0, n < 0, \mu < 0; \tag{32}$$

$$t < \left(\frac{\mu}{n\lambda}\right)^{1/(n-1)} \text{ for } \lambda > 0, n > 1, \mu > 0; \tag{33}$$

$$t \geq 0 \text{ for } \lambda < 0, 0 < n < 1, \mu \geq 0; \tag{34}$$

$$t < \left(\frac{\mu}{n\lambda}\right)^{1/(n-1)} \text{ for } \lambda < 0, 0 < n < 1, \mu < 0; \tag{35}$$

$$t \geq 0 \text{ for } \mu > 0,\ n\lambda = 0; \tag{36}$$

$$t \geq 0 \text{ for } \mu > \lambda,\ n = 1. \tag{37}$$

In the above cases, conditions (31), (34), (36) and (37) are the external expanding states, and the universe will change from expanding to collapsing at $t = \left(\frac{\mu}{n\lambda}\right)^{1/(n-1)}$ in the other states.

iii) Inflationary process. Inflationary condition $\ddot{a} > 0$ is reduced to

$$n^2\lambda^2 t^{2n} + \mu^2 t^2 - n\lambda t^n(2\mu t + n - 1) > 0. \tag{38}$$

Unlike solution one, many constants make it difficult to give a general picture of the valid region of condition (38). We will give an example in the later part of this section, in which inflation condition is expressed in apparent state.

iv) Singularity. The curvature scalar is

$$R = 6\left[(1-n)n\lambda t^{n-2} + 2\left(\mu - n\lambda t^{n-1}\right)\right], \tag{39}$$

and the Kretschmann scalar is

$$K = 12\left\{\left[\left(n\lambda t^{n-1}\right)^2 + \mu^2 - n\lambda t^{n-2}(2\mu t + n - 1)\right]^2 + \left(\mu - n\lambda t^{n-1}\right)^4\right\}. \tag{40}$$

If $n < 1$, the universe begins with an initial singularity it will evolve to a de Sitter (or anti-de Sitter) phase as $H \simeq \mu (R \simeq 12\mu)$ when $t \to \infty$. Specially, $n = 1$ is the de Sitter (or anti-de Sitter) spacetime. Else if $1 < n < 2$ the universe begins with an initial singularity and it will evolve to another end-singularity. If $n \geq 2$, the universe begins with de Sitter spacetime and it will evolve to a singularity in the infinite future.

Now, we consider an example for $n = -1$. Then this solution can be written as

$$a(t) = a_0 \exp\left(\mu t - \frac{\lambda}{t}\right), \tag{41}$$

$$\phi = \pm 4M_p\sqrt{\lambda/t} + \phi_0, \tag{42}$$

$$V(\phi) = M_p^2\left\{3\left[\mu + \left(\frac{\phi-\phi_0}{\phi_p}\right)^4\right]^2 - \left(\frac{\phi-\phi_0}{\phi_p}\right)^6\right\}, \tag{43}$$

where $\lambda \geq 0$. Several characteristics of this solution can be found as follows.

(I) Expanding process. The Hubble parameter is

$$H = \mu + \frac{\lambda}{t^2}. \tag{44}$$

For $n = -1$, valid expanding universe conditions (31), (32) and (36) can be simplified into

$$t \geq 0 \text{ for } \lambda > 0, \mu \geq 0; \tag{45}$$

$$t < \sqrt{-\lambda/\mu} \text{ for } \lambda > 0, \mu < 0; \tag{46}$$

$$t \geq 0 \text{ for } \lambda = 0, \mu > 0. \tag{47}$$

(II) Inflationary process. Condition $\ddot{a} > 0$ in expression (38) can be simplified into

$$(\mu t^2 + \lambda)^2 - 2\lambda t > 0. \tag{48}$$

On the left-hand side of inequality (48) is a high-order function of t, which is difficult to solve generally. However, by letting $x = \mu t^2/\lambda$ and $\delta = 4/\mu\lambda$ ($\mu\lambda \neq 0$), condition (48) can be reduced to the following simple forms in which there are only two parameters

$$(1+x)^4/x > \delta \text{ for } \mu > 0; \tag{49}$$

$$(1+x)^4/x < \delta \text{ for } \mu < 0. \tag{50}$$

Specially, when $\mu\lambda = 0$, we have

$$t < \lambda/2 \text{ for } \mu = 0, \lambda > 0; \tag{51}$$

$$t \geq 0 \text{ for } \mu > 0, \lambda = 0. \tag{52}$$

Noting that expanding conditions (45)–(47) should be satisfied in the meantime. Combining the above conditions (45)–(47) with conditions (49)–(52) yields the valid inflationary regions

$$t < \lambda/2 \text{ for } \mu = 0, \lambda > 0; \tag{53}$$

$$t \geq 0 \text{ for } \mu > 0, \lambda = 0; \tag{54}$$

$$(1+x)^4/x > \delta \text{ for } \mu > 0, \lambda > 0; \tag{55}$$

$$(1+x)^4/x < \delta \text{ and } x > -1 \text{ for } \mu < 0, \lambda > 0. \tag{56}$$

In the above cases, inequality (53) is immediate inflationary state and inequality (54) is a de Sitter inflationary process. Let $f(x) = (1+x)^4/x$, a general picture of conditions (55) and (56) can be obtained, which is shown in Fig.2. One can find that $f(x)$ has a local minimum $f_{\min} = 256/27$ at $x = 1/3$ and a local maximum $f_{\max} = 0$ at $x = -1$. When $\delta > 0$ ($\mu > 0$), if $\delta < 256/27$ ($\mu\lambda > 27/64$) condition (55) is always satisfied and inflation will continue forever; if $\delta \geq 256/27$ ($\mu\lambda \leq 27/64$), there are two inflationary processes: the immediate inflation when $x < x_1$ ($t < \sqrt{\lambda x_1/\mu}$) and the late-time inflation when $x > x_2$ ($t > \sqrt{\lambda x_2/\mu}$), where x_1 and x_2 ($x_1 \leq x_2$) are the two solutions for $f(x) = \delta$. When $\delta < 0$ ($\mu < 0$), letting x_3 and x_4 ($x_3 < x_4$) be the two solutions of $f(x) = \delta$, considering condition (56), one finds that there is only one inflationary process: the immediate inflation when $x > x_4$ ($t < \sqrt{\lambda x_4/\mu}$).

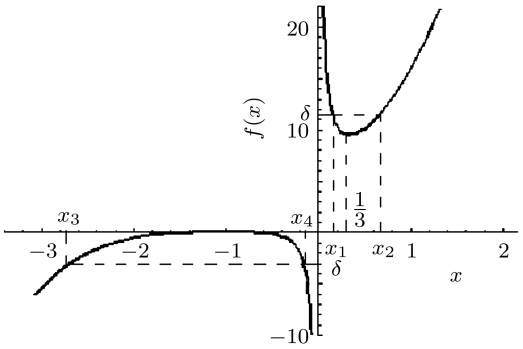


Fig.2. In this figure, x_1 and x_2 are the two solutions of $f(x) = \delta$ with $\delta \geq 256/27$ ($\mu\lambda \leq 27/64$), while x_3 and x_4 are the two solutions of $f(x) = \delta$ with $\delta < 0$ ($\mu < 0$). Immediate inflation occurs when $x < x_1$ ($t < \sqrt{\lambda x_1/\mu}$) for $\delta \geq 256/27$ and when $x > x_4$ ($t < \sqrt{\lambda x_4/\mu}$) for $\delta < 0$; late-time inflation occurs when $x > x_2$ ($t > \sqrt{\lambda x_2/\mu}$) for $\delta \geq 256/27$.

(III) State parameter. The corresponding state parameter is

$$\omega = -1 + \frac{2\sqrt{\delta x}}{3(1+x)^2}. \tag{57}$$

If $\delta > 0$ ($\mu > 0$), one finds that ω reaches its maximum

$$\omega_{\max} = -1 + \frac{\sqrt{3\delta}}{8} \tag{58}$$

at $x = 1/3$ ($t = \sqrt{\lambda/(3\mu)}$). If $\delta < 0$ ($\mu < 0$), one obtains the state parameter $\omega \to \infty$ when $x \to -1$ ($t \to \sqrt{-\lambda/\mu}$). Specially when $\mu = 0$, one can easily find that ω increases linearly with the evolvement of t. We now turn to investigating the evolvement of the potential in expression (43). By solving $\mathrm{d}V/\mathrm{d}\phi = 0$, we obtain

$$\phi = \phi_0, \text{ and } \phi = \phi_0 \pm 2M_p\sqrt{1 \pm \sqrt{1 - 16\lambda\mu}}\ . \tag{59}$$

Choosing suitable constants, we will obtain several different evolvements of the potential. Figure 3 shows an example of $V(\phi)$ in expression (43) with $0 < 16\lambda\mu < 1$. From it we can see that the curve has three turning

points at $\phi = \phi_0, \phi = \phi_1$ and $\phi = \phi_2$ separately. The universe begins with $\phi = \phi_0$ and clams up to the local maximum of $V(\phi)$ at which ϕ reaches ϕ_1. After that, the universe will roll down from the maximum of $V(\phi)$ to the minimum of $V(\phi)$ at $\phi = \phi_2$ or it will roll back to ϕ_0. This process is very similar to the one shown in Ref.[13].

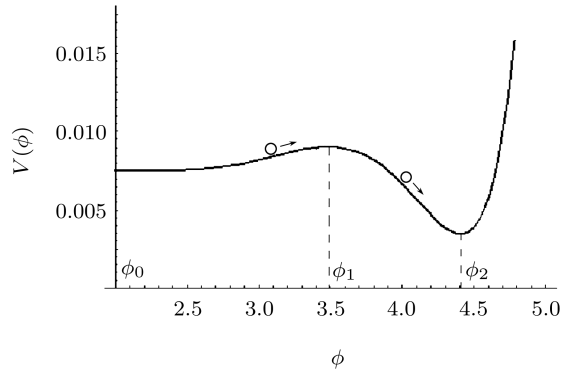


Fig.3. Curve of $V(\phi)$ given by expression (43) with $M_p = 1, \mu = 0.005, \lambda = 1$ and $\phi_0 = 2$. The curve has three turning points at $\phi = \phi_0, \phi = \phi_1$ and $\phi = \phi_2$ separately. This process is similar to the one shown in Ref.[13].

2.3. Solution three

$$a(t) = \sigma \left(\mathrm{e}^{\mu t} - \tau \mathrm{e}^{-\mu t}\right)^n, \tag{60}$$

where $\sigma > 0, \tau, \mu$ and n are suitable constants, and we need

$$\mu = \text{any value}, \tau \leq 0, n \leq 0; \tag{61}$$

$$\mu > 0, 0 < \tau \leq 1, n > 0. \tag{62}$$

Here, $(\mu = \text{any value}, \tau \leq 0)$ or $(\mu > 0, 0 < \tau \leq 1)$ will make sure $\mathrm{e}^{\mu t} - \tau \mathrm{e}^{-\mu t} \geq 0$ in expression (60) in the region $0 \leq t < \infty$. For $0 < \tau \leq 1$ and $n > 0$, we have

$$\phi = \pm\sqrt{2n} M_p \log\left(\frac{\mathrm{e}^{\mu t} - \sqrt{\tau}}{\mathrm{e}^{\mu t} + \sqrt{\tau}}\right) + \phi_0, \tag{63}$$

$$V(\phi) = \frac{n}{2}\mu^2 M_p^2 \times \left[1 + 3n + (3n-1)\cosh\left(\frac{\phi - \phi_0}{\phi_p}\right)\right]. \tag{64}$$

In this case, the state parameter

$$\omega = -1 + \frac{8\tau \mathrm{e}^{2\mu t}}{3n(\tau + \mathrm{e}^{2\mu t})^2}. \tag{65}$$

It reaches its maximum $\omega_{\max} = -1 + 2/3n$ at $t = \ln\tau/2\mu$. For $\tau < 0$ and $n < 0$, we attain

$$\phi = \pm 2\sqrt{-2n} M_p \arctan\left(\mathrm{e}^{\mu t}/\sqrt{-\tau}\right) + \phi_0, \tag{66}$$

$$V(\phi) = \frac{n}{2}\mu^2 M_p^2 \left[1 + 3n + (3n-1)\cos\left(\frac{\phi - \phi_0}{\phi_p}\right)\right]. \tag{67}$$

Then the corresponding state parameter is

$$\omega = -1 + \frac{8\tau \mathrm{e}^{2\mu t}}{\mathrm{e}^{4\mu t} - 3(n-1)\mathrm{e}^{2\mu t} + 3n\tau^2}. \tag{68}$$

Here in expressions (64) and (67), ϕ_0 and ϕ_p are constants and we need $\phi_p = M_p 2^{-|n|/2n}\sqrt{|n|}$. Apparently, a de Sitter (or anti-de Sitter) solution is obtained if $\tau = 0$. Moreover, this solution reduces to the 'sinh' solution if $\tau = 1$ and $n = 1$ and to the 'cosh' solution if $\tau = -1$ and $n = 1$ [14] (there is a small difference, we take the flat metric, but it is not in Ref.[14]). Actually, the potentials (64) and (67) have been obtained by Spalinski.[16] Now we examine several characteristics of this solution.

1) Expanding process. The Hubble parameter is

$$H = \frac{n\mu\left(\mathrm{e}^{\mu t} + \tau \mathrm{e}^{-\mu t}\right)}{\mathrm{e}^{\mu t} - \tau \mathrm{e}^{-\mu t}}. \tag{69}$$

By combining condition (61) with condition (62) and noting $\mathrm{e}^{\mu t} - \tau \mathrm{e}^{-\mu t} > 0$, the valid regions of expanding condition $H >$ are

$$t \geq 0 \quad \text{for} \quad \mu > 0, 0 \leq \tau \leq 1, n > 0; \tag{70}$$

$$t < \frac{\ln(-\tau)}{2\mu} \quad \text{for} \quad \mu > 0, \tau < -1, n < 0; \tag{71}$$

$$t < \frac{\ln(-\tau)}{2\mu} \quad \text{for} \quad \mu < 0, -1 < \tau < 0, n < 0. \tag{72}$$

In the case (70), the universe will expand forever. In cases (71) and (72), the expanding process stops and the universe will begin to collapse at $t = \ln(-\tau)/2\mu$.

2) Inflationary process. $\ddot{a} > 0$ induces

$$\left(\mathrm{e}^{\mu t} + \tau \mathrm{e}^{-\mu t}\right)^2 - \frac{4\tau}{n} > 0. \tag{73}$$

Noting that $n\tau \geq 0$ in conditions (61) and (62), the above inequality is changed into the following two cases:

$$\mathrm{e}^{2\mu t} - 2\sqrt{\tau/n}\, \mathrm{e}^{\mu t} + \tau > 0; \tag{74}$$

$$\mathrm{e}^{2\mu t} + 2\sqrt{\tau/n}\, \mathrm{e}^{\mu t} + \tau < 0. \tag{75}$$

Although the above two inequalities are linear after the parameter transformation: $x = \mathrm{e}^{\mu t}$, considering

expanding conditions (70)–(72), many constants make it difficult to find the valid inflationary regions. We will show an example in the later part of this section.

3) Singularity. The curvature scalar is

$$R = \frac{12n\mu^2\left[n\mathrm{e}^{4\mu t} + 2\tau(n-1)\mathrm{e}^{2\mu t} + n\tau^2\right]}{\left(\mathrm{e}^{2\mu t} - \tau\right)^2}, \quad (76)$$

Specially, if $n = 1/2$, the curvature scalar is a constant $R = 3\mu^2$. The Kretschmann scalar is

$$\begin{aligned} K \;=\; & 24n^2\mu^4(\mathrm{e}^{2\mu t} - \tau)^{-4}\Big[n^2\mathrm{e}^{8\mu t} + 4n\tau(n-1)\mathrm{e}^{6\mu t} \\ & + 2(3n^2 - 4n + 4)\tau^2\mathrm{e}^{4\mu t} \\ & + 4n(n-1)\tau^3\mathrm{e}^{2\mu t} + n^2\tau^4\Big]. \end{aligned} \quad (77)$$

Considering the conditions (61) and (62), if $\mu =$ any value and $\tau < 0$, or $\mu > 0$ and $0 < \tau < 1$, there is no singularity in this solution; if $\mu > 0$ and $\tau = 1$, the universe begins with an initial singularity and will evolve into de Sitter spacetime as $H \simeq n\mu$ and $R \simeq 12n^2\mu^2$ when $t \to \infty$. Specially, $\tau = 0$ is de Sitter (or anti-de Sitter) spacetime.

Now we give an example for $n = 1$. Then the solution is reduced to

$$a(t) = \sigma\left(\mathrm{e}^{\mu t} - \tau\mathrm{e}^{-\mu t}\right), \quad (78)$$

$$\phi = \pm\sqrt{2}M_p \log\left(\frac{\mathrm{e}^{\mu t} - \sqrt{\tau}}{\mathrm{e}^{\mu t} + \sqrt{\tau}}\right) + \phi_0, \quad (79)$$

$$V(\phi) = \mu^2 M_p^2\left[2 + \cosh\left(\frac{\sqrt{2}(\phi - \phi_0)}{M_p}\right)\right]. \quad (80)$$

For $n = 1$, condition (62) is simplified into

$$\mu > 0 \text{ and } 0 < \tau \leq 1. \quad (81)$$

The valid expanding region is only condition (70)

$$t \geq 0,\ 0 < \tau \leq 1,\ \mu > 0. \quad (82)$$

The above conditions (81) and (82) indicate that the universe will expand forever. Inflationary condition (74) can be written as

$$(\mathrm{e}^{\mu t} - \sqrt{\tau})^2 > 0. \quad (83)$$

Clearly, inflation will continue forever in this case.

3. Conclusions

The non-linear flow equation is a widely-used approach for exploring the inflationary behaviour of the universe, and our discussion provides us with several interesting evolutionary models of the universe. According to our discussion, the fate of the universe depends strongly on those constants: different constants correspond to different cosmological evolvements. The most prominent characteristics of those solutions are that inflation does not take place forever in some special cases, which means that those solutions need to be studied deeply because the current investigations do not support typical internal inflation.[19]

In Ref.[20], Parsons and Barrow gave a way to generate a family of new exact solutions. In their method, if there is a solution $a(t) = \exp[f(t)]$, where $f(t)$ is an arbitrary function, we can obtain another exact solution in the form $a(t) = \exp[At + f(t)]$. Specifically, by choosing $f(t) = -\alpha t\ln(t/t_0)$, (note that $a(t) = \exp[-\alpha t\ln(t/t_0)]$ is an exact solution), we obtain solution one. For solution two, there is $f(t) = -\lambda t^n$, which corresponds to the intermediate inflationary: $a(t) = \exp(At^n)$, $0 < n < 1$ in Refs.[10], [21] and [22]. In this paper, we present the full valid regions of those constants and give some useful information about this solution. However, solution three cannot be obtained in this way. It can be written as $a(t) = \exp[\ln\delta + n\mu t + n\ln(1 - \tau\mathrm{e}^{-2\mu t})]$, and it is easy to find that $a(t) = \exp[n\ln(1-\tau\mathrm{e}^{-2\mu t})]$ is not an exact solution.

Recently, Spalinski found a way,[16] which is different from the standard truncating method,[23] to attain an exact solution. In his solutions, the Hubble parameter H and potential V are both functions of ϕ, but it is difficult to acquire the scalar factor $a(t)$ with t as a variable. In this paper, solution three gives us an analytical function of $a(t)$ in expression (60) and the corresponding $H(t)$ in expression (69). Moreover, in this solution, the periodic potential in expression (67) is unlike the other potentials in expressions (10), (21) and in (64) which will go to infinite value when $\phi = \phi_0$ or $\phi \to \infty$, and singularity is avoided in some special cases, which all lead it to become a recommendable solution.

Acknowledgment

We thank professor Barrow for his earnest help and useful suggestions and also thank Wang Jing for her helpful correction of our manuscript.

Appendix A

In this appendix, we give a simple way to prove that solution one is indeed a solution of Eqs.(4) and (5). The other two solutions in this paper can be tested in this way.

Differentiating Eq.(4) with respect to t, and combining Eq.(5) yield

$$\phi = \pm M_p \int \sqrt{-2\dot{H}}\mathrm{d}t, \tag{A1}$$

$$V = M_p^2(3H + \dot{H}), \tag{A2}$$

where one should note $H = \dot{a}/a$ and $\dot{H} = (\ddot{a}a - \dot{a}^2)/a^2$. In the general case, we suppose that the primitive function on the right-hand side of expression (A1) is existent for a given function $a(t)$, and we obtain an analytical function $\phi = f(t)$ with t as the variable. Then we need to attain the inverse solution of $\phi = f(t)$ as follows:

$$t = f^{-1}(\phi), \tag{A3}$$

where the superscript -1 denotes the corresponding inverse function. Substituting expression (A3) into expression (A2), one obtains V with ϕ as the variable. The above process can be used to validate the three solutions in this paper. Now we consider an example. Inserting expression (8) into expression (A1), one obtains expression (9). Then one obtains the inverse solution from expression (9)

$$t = \frac{1}{8\alpha} M_p^{-2}(\phi - \phi_0)^2. \tag{A4}$$

Inserting expression (8) into expression (A2), one has

$$V = M_p^2 \left\{ 3\left[\alpha - \beta + \alpha \ln\left(\frac{t}{t_0}\right)\right]^2 - \frac{\alpha}{t} \right\}. \tag{A5}$$

Substituting expression (A4) into expression (A5), one will obtain V with ϕ as the variable as shown in expression (10).

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