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Variational solutions for Hermite–Gaussian solitons in nonlocal nonlinear media

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The physical features exhibited by Hermite–Gaussian (HG) beams propagating in nonlocal nonlinear media with Gaussian-shaped response are discussed with an approximate variational method. Using direct numerical simulations, we find that the beam properties in the normalized system are different with the change of the degree of nonlocality. It is shown that initial HG profiles break up into several individual beams with propagation when the degree of nonlocality α is small. HG beams can propagate stably when α is large enough.

Keywords: nonlocal nonlinear media, variational approach, Hermite–Gaussian solitons
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1. Introduction

Spatial optical solitons(SOS) and vortex solitons are stationary solitary wave structures which form when diffraction spreading is balanced exactly by nonlinearity. Nonlocal spatial optical solitons (NSOS) have attracted much attention during recent years for their rich phenomena and potential applications in signal processing and optical communication. The propagation of the NSOS is governed by the nonlocal nonlinear Schrödinger equation (NNLSE).[1,2] Nonlocal nonlinear response of the media can prevent a catastrophic collapse which often occurs in local self-fousing media when the power of the wave packet exceeds some critical value.[3] Early in 1997, Snyder and Mitchell[1] simplified the NNLSE in the strongly nonlocal case and a linear model, called Snyder–Mitchell model, was presented to describe the propagations of these solitons (*named accessible solitons*), and found that two coherent interacting strongly NSOSs are always attractive, independent of their relative phase. A new approximate model to the NNLSE was presented by means of Taylor's expansion to the second order of the response function in the strongly nonlocal case, and an exact analytical solution of the NSOS with a different phenomenon of large phase shift was revealed comparable to its local counterpart.[4] The NSOS in the sub-strongly nonlocal case was studied with an approximate variational approach.[5] Optical beams in lossy strongly nonlocal Kerr media do not possess the adiabatic property anymore.[6] Lately, it was revealed[7,8] that the NSOS in the nematic liquid crystals (NLC), called *nematicons*,[9] were exactly the *accessible solitons* for the NLC exhibiting strongly nonlocal nonlinearity. The first observation of modulational instability in NLCs which have a nonlocal behaviour was reported in Ref.[10]. The possibility of applications for all-optical switching and logic gating was demonstrated experimentally through the collisional behavior of the *nematicons*.[11,12] The interaction between NSOSs in the NLC can be controlled experimentally through bias voltage on the NLC, and the solitons behave like their local counterpart when the degree of the nonlocality is below a critical value.[13] Recently, the fundamental and some other higher order (1+1)-D and (1+2)-D sub-strongly NSOSs with a Gaussian-type response function and an exponential-decay response function were presented with a perturbation method.[14,15] In local nonlinear media, high-order optical solitons and vortices are unstable. Nonlocality can support multisoliton bound states and the existence of higher-order mode solitons travelled stably was demonstrated experimentally.[16,17] Laguerre and Hermite soliton clusters in nonlocal nonlinear media were studied.[18] Exact analytical solutions in the Hermite–Gaussian (HG) form in the strongly nonlocal nonlinear media was derived;[19] and it was demonstrated that HG functions can be expressed as a linear superposition of individual Gaussian functions with π phase difference under appropriate conditions.[20]

In this paper, we present the HG mode solution of the 1+1D NNLSE with an approximate variational

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method and numerical simulations. We reveal that the beams properties in the normalied system become different with the change of the degree of nonlocality. It is shown that initial HG profiles break up into several individual beams with propagation when the degree of nonlocality α is small. HG beams can propagate stably when α is large enough.

2. Basic equations

The propagation of an optical beam in nonlocal Kerr media with nonlocal Gaussian response is governed by the NNLSE,[1,2] which reads in the (1+1)-D case

$$\mathrm{i}\frac{\partial \Psi}{\partial Z}+\mu\frac{\partial^2 \Psi}{\partial X^2}+\rho\Psi\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}w_{\mathrm{m}}} \times\exp\left[-\frac{(X-X')^2}{2w_{\mathrm{m}}^2}\right]|\Psi(X',Z)|^2\mathrm{d}X'=0, \quad (1)$$

where $\Psi(X,Z)$ is the complex amplitude envelope of the optical beam, X and Z are the transverse and longitudinal coordinates, respectively. $\mu=1/2k$, $\rho=k\eta$, k is the wave number in the media for the linear case, η is a material constant ($\eta>0$ for a focusing material, $\eta<0$ for a defocusing one), w_{m} is the characteristic length of the material. The nonlocality of the model can be defined as $\alpha=w_{\mathrm{m}}/w_0$, where w_0 is the beam width. Here, in this paper the normalized real symmetrical nonlocal response function kernel modelled by the Gaussian shape kernel is

$$R(X)=\frac{1}{\sqrt{2\pi}w_{\mathrm{m}}}\exp\left[-\frac{X^2}{2w_{\mathrm{m}}^2}\right]. \quad (2)$$

In the case of strong nonlocality, $R(X)$ is much wider than the width of the beam. While in the case of weak nonlocality, $R(X)$ is much narrower than the width of the beam. The nonlinear term is expressed in the nonlocal form $N=\int_{-\infty}^{+\infty}R(X-X')|\Psi(X',Z)|^2\mathrm{d}X'$. We nomalize Eq.(1) by letting

$$z=\frac{Z}{4kw_{\mathrm{m}}^2}, \quad (3)$$

$$x=\frac{X}{\sqrt{2}w_{\mathrm{m}}}, \quad (4)$$

$$\psi=\frac{2}{\pi^{1/4}}k\eta^{1/2}w_{\mathrm{m}}\Psi. \quad (5)$$

Thus the NNLSE can be deduced as

$$\mathrm{i}\frac{\partial \Psi}{\partial z}+\frac{\partial^2 \Psi}{\partial x^2}+\psi\int_{-\infty}^{+\infty}\mathrm{e}^{-(x-x')^2}|\Psi(x',z)|^2\mathrm{d}x'=0. \quad (6)$$

Because the response function is real (no nonlinear loss or gain) and symmetric, among the conserved quantities of system (6) are the energy flow P and the Hamiltonian H

$$P=\int_{-\infty}^{+\infty}|\psi|^2\mathrm{d}x, \quad (7)$$

$$H=\int_{-\infty}^{+\infty}\left[\left|\frac{\partial \Psi}{\partial x}\right|^2-\frac{1}{2}|\psi|^2\int_{-\infty}^{+\infty}\mathrm{e}^{-(x-x')^2} \times|\psi(x',z)|^2\mathrm{d}x'\right]\mathrm{d}x. \quad (8)$$

3. Variational approach

We search for the stationary solutions of Eq.(6) in the form

$$\psi(x,z)=u(x)\exp(\mathrm{i}\lambda z), \quad (9)$$

where $u(x)$ is a real function and λ is a real propagation constant, so the real function $u(x)$ obeys the following equation

$$-\lambda u+\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+u\int_{-\infty}^{+\infty}\mathrm{e}^{-(x-x')^2}|u(x')|^2\mathrm{d}x'=0. \quad (10)$$

In accordance with the variational procedure,[21] we construct the Lagrangian density

$$L=-\lambda u^2-\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+\frac{1}{2}u^2\int_{-\infty}^{+\infty}\mathrm{e}^{-(x-x')^2}|u(x')|^2\mathrm{d}x', \quad (11)$$

where Eq.(11) is the variational equation corresponding to the variational principle

$$\delta\int_{-\infty}^{+\infty}L\mathrm{d}x\mathrm{d}z=0. \quad (12)$$

Since nonlocality allows us to compensate the soliton repulsion stabilizing multihump solitons,[22] the trial-function of evolution we use can resort to the HG modes

$$u(x)=A\exp\left[-\frac{x^2}{2a^2}\right]H_n\left(\frac{x}{b}\right), \quad (13)$$

where the integer index n determines the number of nodes across the Gaussian envelope, and $H(x)=(-1)^n\mathrm{e}^{x^2}\mathrm{d}^n\mathrm{e}^{-x^2}/\mathrm{d}x^n$ is the Hermite polynomial. From Eqs.(4) and (13), we have

$$\frac{x^2}{2a^2}=\frac{X^2}{4a^2w_{\mathrm{m}}^2}=\frac{X^2}{2w_0^2}, \quad (14)$$

so the degree of nonlocality of the model reads

$$\alpha=\frac{w_{\mathrm{m}}}{w_0}=\frac{1}{\sqrt{2}a}. \quad (15)$$

From Eq.(15), we know that the degree of nonlocality α increases with decreasing a. By inserting Eq.(11) into the variational principle (12) and performing the integration with respect to x, we arrive at the reduced variational problem

$$\delta \int_{-\infty}^{+\infty} \mathcal{L} \mathrm{d}z = 0, \tag{16}$$

where

$$\mathcal{L} = \int_{-\infty}^{+\infty} L \mathrm{d}x. \tag{17}$$

Here, we figure out

$$\begin{aligned} L = & -\lambda A^2 \exp\left[-\frac{x^2}{a^2}\right] H_n^2\left(\frac{x}{b}\right) - \left\{\frac{2An}{b} \exp\left[-\frac{x^2}{2a^2}\right] \times H_{n-1}\left(\frac{x}{b}\right) - \frac{A^2}{a^2} \exp\left[-\frac{x^2}{2a^2}\right] x H_n\left(\frac{x}{b}\right)\right\}^2 \\ & + A^4 \exp\left[-\frac{x^2}{1+a^2} - \frac{x^2}{a^2}\right] \frac{\sqrt{1+a^2}}{a} \sqrt{\pi} H_n^2\left(\frac{x}{b}\right) \times \sum_{\lambda=0}^{n} 2^\lambda \lambda! \left[\frac{n!}{\lambda!(n-\lambda)!}\right]^2 \left[1 - \frac{a^2}{b^2(1+a^2)}\right]^{n-\lambda} \\ & \times H_{2n-2\lambda}\left[\frac{a^2 x}{((1+a^2)(b^2+a^2b^2-a^2))^{1/2}}\right]. \end{aligned} \tag{18}$$

Substituting Eq.(18) into the Langrangian (17), we obtain a system of equations for A, a, and b from the corresponding variational problem $\partial\mathcal{L}/\partial a_j = 0$, where $a_j = \{A, a, b\}$.

When $n = 0$, the equations about the parameters A, a of the zeroth-order HG solitons are as follows:

$$-\frac{A\sqrt{\pi}}{a} - 2aA\lambda\sqrt{\pi} + \frac{2A^3\pi a^2}{\sqrt{1+2a^2}} = 0, \tag{19}$$

$$\begin{aligned} & -\frac{A^2\sqrt{\pi}}{2a^2} - A^2\lambda\sqrt{\pi} + \frac{A^4\pi a}{2(1+a^2)\sqrt{1+2a^2}} \\ & + \frac{\pi A^4 a(2a^4 - 2a^2 + 1)}{2(1+2a^2)^{3/2}} = 0. \end{aligned} \tag{20}$$

From Eqs.(19) and (20), we can derive

$$\begin{aligned} A = & \frac{1}{4\pi^{1/4}} \left[(1+\sqrt{9+16\lambda}) \left(\frac{3+\sqrt{9+16\lambda}}{\lambda}\right)\right. \\ & \times(-3+8\lambda+\sqrt{9+16\lambda})^2 \\ & \left.\times \frac{\lambda(3+2\lambda+\sqrt{9+16\lambda})}{9+14\lambda+(3+2\lambda)\sqrt{9+16\lambda)}}\right]^{1/4}, \end{aligned} \tag{21}$$

$$a = \left(\frac{3+\sqrt{9+16\lambda}}{4\lambda}\right)^{1/2}. \tag{22}$$

When $\lambda \to \infty$, $a \to 0$, from Eq.(15) we know this case corresponding to the strongly nonlocal one, then Eqs.(21) and (22) can be reduced into

$$A = \frac{1}{\pi^{1/4}} \lambda^{5/8}, \tag{23}$$

$$a = (1/\lambda)^{1/4}. \tag{24}$$

And, the zeroth-order HG soliton solution is

$$\psi_s(x,z) = \frac{\exp(\mathrm{i}\lambda z)}{\pi^{1/4}} a^{5/2} \exp\left[-\frac{x^2}{2a^2}\right], \tag{25}$$

which is in good agreement with the soliton expression for the 1+1D case in Ref.[4] in the same normalized systems.

The equations with parameters A, a, and b for several high-order HG mode solitons cannot be easily solved analytically, so we solve these equations numerically. We use variational solutions as input for the direct simulation of the beam propagation as shown in Fig.1, which illustrates the evolution of (1+1)-dimensional (a)–(c)first-, (d)–(f)second-, and (g)–(i)third-order-mode HG beams up to a distance of $z = 10$ in a nonlocal medium. When λ is very small, from the results of numerical simulation (see column (a), (d), (g) of Fig.1) we can see that initial HG profiles break up into several individual Gaussian beams with propagation, these are not solitons. In fact, the parameter λ is the quantitative measure for the degree of nonlocality,[3] according to the transformation of the physical variables and the progress of analytical solution. Since the larger λ corresponds to the smaller a, the degree of nonlocality α increases with increasing λ. When a is very large such that α is very small and can be considered as zero, it corresponds to a local case. We know that the solution of NLSE is of hyperbolic secant profile in local media, while higher-order optical solitons are unstable in local and weakly nonlocal media. For larger λ, HG solitons can propagate stably over several distances without change (see

column (b), (e), (h) of Fig.1). These are solitons, in which diffraction is balanced by nonlinearity. From the results of the analytical solutions we find that a approaches to b when λ is large. When λ is very large so that a is very small, this case corresponds to the highly nonlocal case (see column (c), (f), (i) of Fig.1), and we know that HG beams can propagate stably over huge distances.[20] The initial HG beam width is $a_{0n} = \sqrt{2n+1}a_0$, which can be obtained by using the definition of the second order moment for beam width. According to Fig.1, with the increasing degree of nonlocality α, HG beam width a_{0n} decreases and the intensity increases.

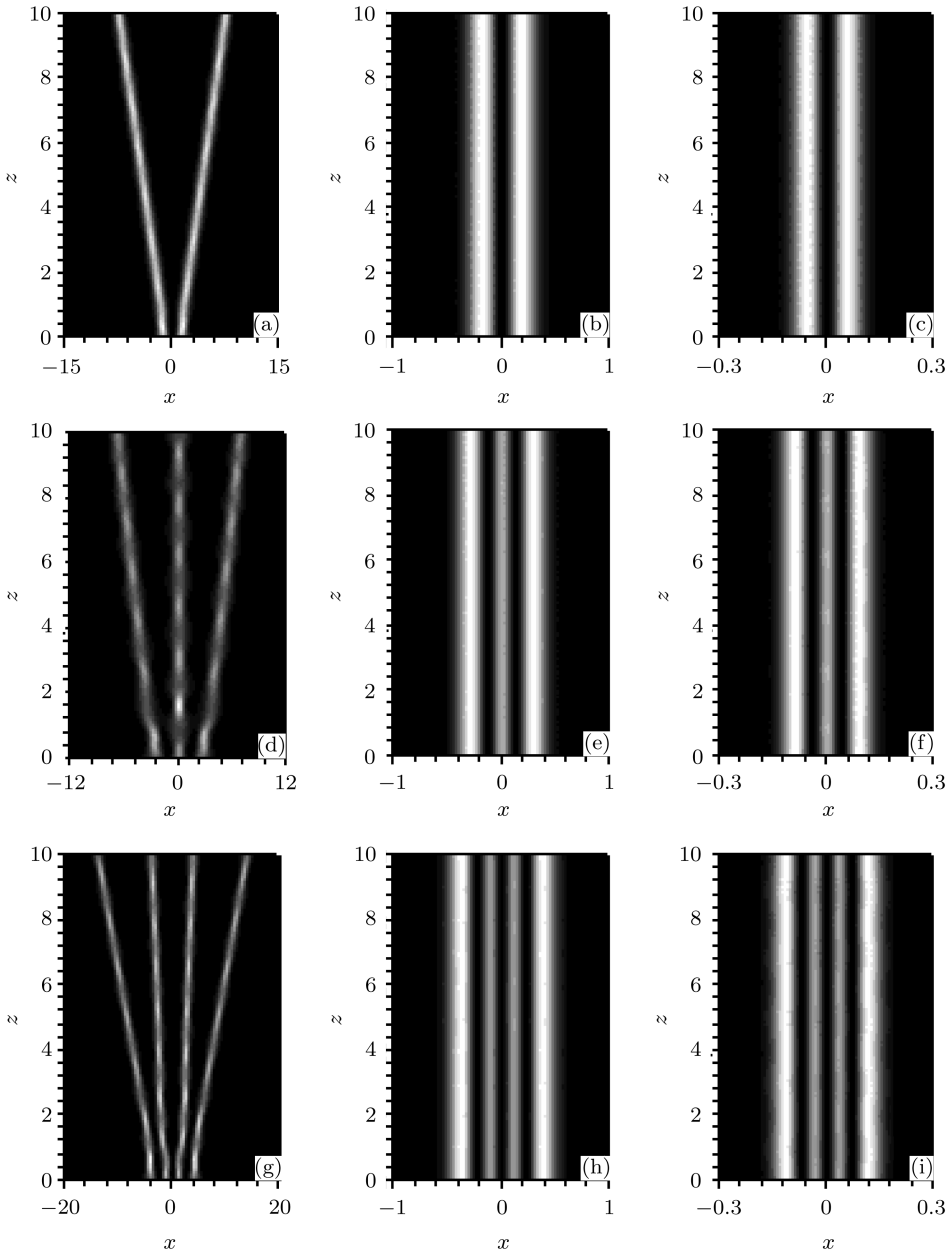


Fig.1. Propagation of (1+1)-dimensional (a)–(c)first-, (d)–(f)second-, and (g)–(i)third-order-mode HG beams up to a distance of $z = 10$ in nonlocal medium. (a) $n = 1, \lambda = 5$, (b) $n = 1, \lambda = 1000$, (c) $n = 1, \lambda = 100000$, (d) $n = 2, \lambda = 5$, (e) $n = 2, \lambda = 1000$, (f) $n = 2, \lambda = 100000$, (g) $n = 3, \lambda = 5$, (h) $n = 3, \lambda = 1000$, (i) $n = 3, \lambda = 100000$. (b), (c), (e), (f), (h) and (i) corresponding to solitons propagation. In (a)–(i) without noise added. (a),(d) and (g) corresponding to the weak nonlocality case; (b),(e) and (h) corresponding to the general nonlocality case; and (c),(f) and (i) corresponding to the strong nonlocality case.

High-order HG function can be expressed by a linear superposition of Gaussian functions with out-of-phase between adjacent functions under appropriate condition in nonlocal nonlinear media.[20] With increasing width of the response function $R(X)$ the electric field distribution in the vicinity of the point X also contributes to the nonlinear response at that point. When the nonlocality is not strong enough, the attractive force between the adjacent Gaussian beams is weak, and the beams cannot form stable bound states, so they break up into several individual beams with propagation. When α is strong enough, diffraction spreading can balance nonlinearity and stable spatial optical solitons can form.

4. Conclusion

The physical features exhibited by HG beams propagating in nonlocal nonlinear media with Gaussian-shaped response have been discussed with an approximate variational method. Using direct numerical simulations, we find that the beam properties in the normalized system become different with the change of the degree of nonlocality. It is shown that initial HG profiles break up into several individual beams with propagation when the degree of nonlocality α is small. HG beams can propagate stably when α is large enough.

References

[1] Snyder A W and Mitchell D J 1997 *Science* **276** 1538

[2] Krolikowski W, Bang O, Rasmussen J J and Wyller J 2001 *Phys. Rev.* E **64** 016612

[3] Yakimenko A I, Lashkin V M and Prikhodko O O 2006 *Phys. Rev.* E **73** 066605

[4] Guo Q, Luo B, Yi F, Chi S and Xie Y Q 2004 *Phys. Rev.* E **69** 016602

[5] Guo Q, Luo B and Chi S 2006 *Opt. Commun.* **259** 336

[6] Huang Y, Guo Q and Cao J N 2006 *Opt. Commun.* **261** 175

[7] Conti C, Peccianti M and Assanto G 2003 *Phys. Rev. Lett.* **91** 073901

[8] Conti C, Peccianti M and Assanto G 2004 *Phys. Rev. Lett.* **92** 113902

[9] Assanto G, Peccianti M and Conti C 2003 *Optics Photonics News* 45

[10] Peccianti M, Conti C and Assanto G 2003 *Phys. Rev.* E **68** 025602

[11] Peccianti M, Brzdakiewicz K A and Assanto G 2002 *Opt. Lett.* **27** 1460

[12] Peccianti M, Conti C and Assanto G 2002 *Appl. Phys. Lett.* **81** 3335

[13] Hu W, Zhang T, Guo Q, Xuan L and Lan S 2006 *Appl. Phys. Lett.* **89** 071111

[14] Ouyang S G, Guo Q and Hu W 2006 *Phys. Rev.* E **74** 036622

[15] Ren H Y, Ouyang S G, Guo Q and Wu L J 2007 *Opt. Commun.* **275** 245

[16] Hutsebaut X, Cambournac C, Haelterman M, Adamski A and Neyts K 2004 *Opt. Commun.* **233** 211

[17] Dreischuh A, Neshev D, Petersen D E, Bang O, Rasmussen J J and Christiansen P L 2006 *Phys. Rev. Lett.* **96** 043901

[18] Buccoliero D, Desyatnikov A S, Krolikowski W and Kivshar Y S 2007 *Phys. Rev. Lett.* **98** 053901

[19] Bai D F, Guo Q and Hu W 2008 *Acta Phys. Sin.* **57** 5684 (in Chinese)

[20] Deng D M, Zhao X, Guo Q and Lan S 2007 *J. Opt. Soc. Am.* B **24** 2537

[21] Anderson D 1983 *Phys. Rev.* A **27** 3135

[22] Xu Z Y, Kartashov Y V and Torner L 2005 *Opt. Lett.* **30** 3171