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Gaussian solitons in nonlocal media: variational analysis*

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Exact solutions of Gaussian solitons in nonlinear media with a Gaussian nonlocal response are obtained. Using the variational approach, we obtain the approximate solutions of such solitons when the degree of the nonlocality is arbitrary. Specifically, we study the conditions for Gaussian solitons that propagate in weakly and highly nonlocal media. We also compare the variational result with the known exact solutions for weakly and highly nonlocal media.

Keywords: Gaussian solitons, nonlocal nonlinearity
PACC: 4265, 4265S, 4265J

1. Introduction

The phenomenon of solitons is a subject of great interest in a wide range of physical settings and has attracted considerable attention in recent years.[1] In the nonlinear optical domain, the nonlinear response of the media can be described in terms of the induced change in the refractive index which is often approximated as a local function of the wave intensity. However, in many real physical systems the nonlinear response is spatially nonlocal, which means that the refractive index depends on the beam intensity in the neighbourhood of each spatial point.[2] This occurs, for instance, when the nonlinearity is associated with some sort of transport process such as heat conduction in media with thermal response,[3,4] diffusion of charge carriers or atoms or molecules in atomic vapours.[5] It is also the case in systems exhibiting long-range interaction of constituent molecules or particles such as those in nematic liquid crystals[6] or dipolar Bose–Einstein condensates.[7]

Nonlocality is thus a feature of a large number of nonlinear systems leading to novel phenomena of a generic nature.[2] It is shown that nonlocal nonlinearity can support a series of novel solitons, such as stable multi-pole solitons,[8,9] or discrete solitons.[10,11] Nonlocality plays an important role both in the interaction[12] of two out-of-phase bright solitons[13,14] and two in-phase dark solitons[15,16] in nonlocal media. Modulation instability of coherent solitons[17,18] and incoherent solitons[19] in nonlocal media have also been studied theoretically and experimentally. Nonlocality also sustains stable, complex soliton structures, such as vortex solitons,[20−23] and incoherent solitons.[24−28] Furthermore, in nonlocal media, the propagation and stability of vector solitons have been studied recently.[29−32]

Gaussian solitons have attracted considerable attention for their propagation stability, and the exact solution of such solitons can be easily obtained. However, to our best knowledge, nonlocal Gaussian solitons have not been studied in the general case that can represent the arbitrary degree of the nonlocality. Up to now, one can only treat the special case of highly nonlocal[33] and weakly nonlocal nonlinearity.[34] So a question arises naturally: can we obtain Gaussian solitons in nonlocal media with an arbitrary degree of nonlocality?

In this paper, we investigate systematically Gaussian solitons in nonlinear media with a Gaussian nonlocal response in the general form. Using the variational approach, we obtain the approximate solutions of such solitons when the degree of the nonlocality is arbitrary. We obtain the existence conditions of such nonlocal Gaussian solitons and discuss the evolution of their properties. Specifically, we also study the prop-

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erties of the solitons when the nonlocality is weak and the linear oscillation behaviours when the nonlocality is high.

2. Model and variational method

Consider a Gaussian beam propagating along the z axis within a nonlocal Kerr-type nonlinear media. The propagation equations of the beam are given by

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2k}\frac{\partial^2 u}{\partial x^2}+u\frac{k}{n_0}\int R(x-x')|u(x',z)|^2\mathrm{d}x'=0, \tag{1}$$

$$R(x-x')=\frac{1}{\sqrt{\pi}\sigma}\exp\left[-\frac{(x-x')^2}{\sigma^2}\right], \tag{2}$$

where u is the amplitude of the beam and k is the wave number. $R(x-x')$ is the normalized nonlocal response function with a characteristic width denoted by σ which determines the degree of the nonlocality. Here we assume the media has a Gaussian nonlocal response.[22,24] We investigate the Gaussian solitons by using the variational approach, and the Lagrange density of Eqs.(1) and (2) is

$$L=\frac{\mathrm{i}}{2}\left(u^*\frac{\partial u}{\partial z}-u\frac{\partial u^*}{\partial z}\right)-\frac{1}{2k}\left|\frac{\partial u}{\partial x}\right|^2+\frac{k}{2n_0}|u|^2\int R(x-x')|u(x',z)|^2\mathrm{d}x'. \tag{3}$$

The variational method is represented by $\delta\int\int L\mathrm{d}x\mathrm{d}z=0$, $\delta\int L_\mathrm{r}\mathrm{d}z=0$, where $L_\mathrm{r}=\int_{-\infty}^{\infty}L\mathrm{d}x$.

We seek the solution to Eqs.(1) in the fundamental Gaussian form

$$u(x,z)=A(z)\exp\left[-\frac{x^2}{2W^2(z)}\right]\exp[\mathrm{i}\theta(z)+\mathrm{i}c(z)x^2], \tag{4}$$

where $A(z)$ and $\theta(z)$ are the amplitude and the phase of the beam, $W(z)$ is the beam width, and $c(z)$ represents the beam phase-front curvature. Then we can obtain the presentation of the Lagrange density

$$L_\mathrm{r}=-\frac{\sqrt{\pi}}{2}A^2W^3\frac{\mathrm{d}c}{\mathrm{d}z}-\sqrt{\pi}A^2W\frac{\mathrm{d}\theta}{\mathrm{d}z}-\frac{\sqrt{\pi}A^2}{4kW}-\frac{\sqrt{\pi}}{k}A^2c^2W^3+\frac{\sqrt{\pi}kA^4W^2}{2n_0\sqrt{\sigma^2+2W^2}}. \tag{5}$$

In deriving the Lagrange density, we have calculated the nonlinear refractive index

$$\delta n(I)=\frac{A^2W}{\sqrt{W^2+\sigma^2}}\exp\left(-\frac{x^2}{W^2+\sigma^2}\right), \tag{6}$$

and three integrals in the following form

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{W^2}}\mathrm{d}x=\sqrt{\pi}W, \tag{7}$$

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{W^2}}x^2\mathrm{d}x=\frac{1}{2}\sqrt{\pi}W^3, \tag{8}$$

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{W^2}}\mathrm{e}^{-\frac{x^2}{W^2+\sigma^2}}\mathrm{d}x=\frac{\sqrt{\pi}}{\sqrt{\dfrac{1}{W^2}+\dfrac{1}{W^2+\sigma^2}}}. \tag{9}$$

Following the standard procedures of the variational approach, we obtain, respectively

$$\frac{\delta L_\mathrm{r}}{\delta\theta}=0\Rightarrow\frac{\mathrm{d}(A^2W)}{\mathrm{d}z}=0, \tag{10}$$

$$\frac{\delta L_\mathrm{r}}{\delta c}=0\Rightarrow 2W\frac{\mathrm{d}A}{\mathrm{d}z}+3A\frac{\mathrm{d}W}{\mathrm{d}z}-\frac{4}{k}AWc=0, \tag{11}$$

$$\frac{\delta L_\mathrm{r}}{\delta W}=0\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}z}+\frac{3W^2}{2}\frac{\mathrm{d}c}{\mathrm{d}z}=\frac{1}{4kW^2}-\frac{3}{k}c^2W^2+\frac{kA^2(W^2+\sigma^2)}{n_0W^5}\left(\frac{\sigma^2+2W^2}{W^4}\right)^{-\frac{3}{2}}, \tag{12}$$

$$\frac{\delta L_\mathrm{r}}{\delta A}=0\Rightarrow W^3\frac{\mathrm{d}c}{\mathrm{d}z}+2W\frac{\mathrm{d}\theta}{\mathrm{d}z}=-\frac{1}{2kW}-\frac{2}{k}c^2W^3+\frac{2kA^2W^2}{n_0\sqrt{\sigma^2+2W^2}}. \tag{13}$$

Equation (10) represents the fact that the beam power is conserved, from which we can obtain

$$\int_{-\infty}^{\infty}|u(x,z)|^2\mathrm{d}x=\sqrt{\pi}A^2W=P. \tag{14}$$

Inserting Eq.(10) into Eq.(11) yields the relation

$$\frac{\mathrm{d}W}{\mathrm{d}z}=\frac{2}{k}Wc. \tag{15}$$

Combining Eqs.(12) and (13) gives two ordinary differential equations for the parameters θ and c

$$\frac{\mathrm{d}\theta}{\mathrm{d}z}=-\frac{1}{2kW^2}+kA^2\left[\frac{2\sigma^2+5W^2}{2n_0W^5}\right]\left(\frac{\sigma^2+2W^2}{W^4}\right)^{-\frac{3}{2}}, \tag{16}$$

$$\frac{\mathrm{d}c}{\mathrm{d}z}=\frac{1}{2kW^4}-\frac{2}{k}c^2-\frac{kA^2}{n_0W^5}\left(\frac{\sigma^2+2W^2}{W^4}\right)^{-\frac{3}{2}}. \tag{17}$$

Combining Eqs.(16) and (17) gives the evolution equation for the beam width

$$\frac{\mathrm{d}^2W}{\mathrm{d}z^2}-\frac{1}{k^2W^3}+\frac{2P}{\sqrt{\pi}n_0W^5}\left(\frac{\sigma^2+2W^2}{W^4}\right)^{-\frac{3}{2}}=0, \tag{18}$$

which describes the evolution dynamic of the width of the Gaussian solitons in nonlocal media and shows competition among diffraction, nonlinearity and nonlocality. Let $W(z)=W(0)=W_0$, which means that the Gaussian beam keeps its initial width. In such a case, a stationary nonlocal Gaussian soliton can be

formed. Then we can obtain the critical power for the Gaussian solitons:

$$P = \frac{\sqrt{\pi} n_0 (\sigma^2 + 2W_0^2)^{\frac{3}{2}}}{2k^2 W_0^4}; \tag{19}$$

when $\sigma = 0$, it is indeed the local case, i.e.,

$$P = \frac{\sqrt{2\pi} n_0}{k^2 W}, \tag{20}$$

which is the critical power for a Gaussian soliton existing in a local Kerr media.

In Fig.1, the critical power is shown as a function of the beam width for different degrees of nonlocality. We can see that the power will decrease when the beam width increases, whereas the power will increase when the degree of nonlocality increases. This is because the nonlinearity depends on the power in nonlocal Kerr media; the larger the power, the larger the nonlinearity. As the degree of the nonlocality increases, the nonlinearity of the media will be weakened, then the diffraction of the beam will be strengthened. To balance the diffraction requires a larger nonlinearity induced by the power.

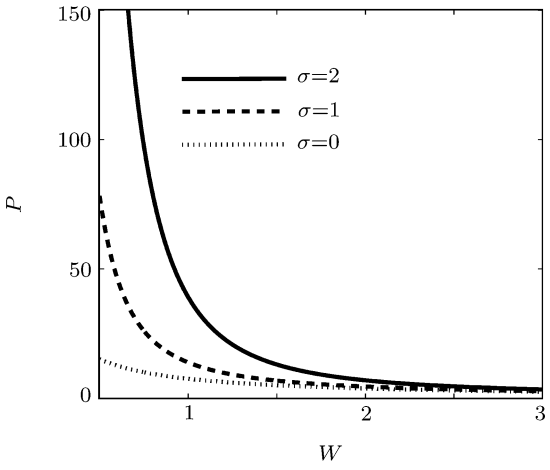


Fig.1. The critical power P versus the beam width W for Gaussian solitons in nonlocal media. The initial parameters are $k = 1$, and $n_0 = 3$.

Equation (18) is the general form for Gaussian solitons in nonlocal media, which can describe the beam width when the degree of the nonlocality is local, weak, general, and high. In some special case, we can obtain the approximate solution of such nonlocal Gaussian solitons. Next, we concentrate on the special case when the degree of the nonlocality is weak or strong.

3. Gaussian solitons in weakly nonlocal media

Firstly, in the limit of weakly nonlocal (the width of the nonlocal response function is much smaller than the beam width), we can expand Eq.(18) in Taylor series with respect to $\sigma = 0$ to the second order, and then we can obtain the following expression:

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} - \frac{1}{k^2 W^3} + \frac{\sqrt{2}P}{2\sqrt{\pi} n_0 W^2} - \frac{3\sqrt{2}P}{8\sqrt{\pi} n_0 W^4}\sigma^2 = 0. \tag{21}$$

From Eq.(21), we can obtain the solution of Gaussian solitons in weakly nonlocal media,

$$W_0 = \frac{\dfrac{1}{k^2} + \sqrt{\dfrac{1}{k^4} + \dfrac{3P^2\sigma^2}{2\pi n_0^2}}}{\dfrac{\sqrt{2}P}{\sqrt{\pi} n_0}}. \tag{22}$$

From Eq.(22) we can see that the nonlocality will effectively weaken the self-focusing of the media, thus leading to expansion of the beam width. In Fig.2, we plot the spatial intensity distribution when the power is fixed, for different degrees of nonlocality. We can see that: as the degree of nonlocality increases, the soliton width will increase, while the maximum of the intensity will decrease. In weakly nonlocal media, the nonlocality can be regarded as a small perturbation to the nonlinearity which will effectively weaken the self-focusing of the beam and lead to the expansion of the beam width.

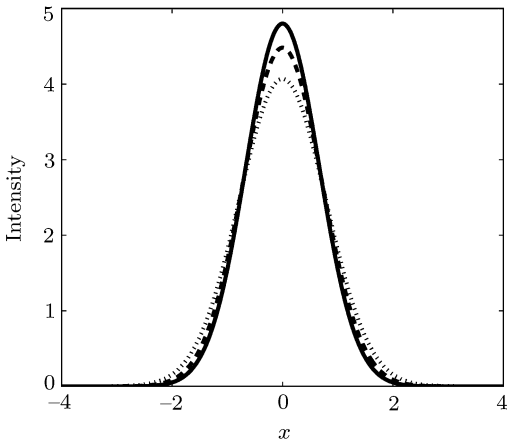


Fig.2. The intensity of Gaussian solitons in weakly nonlocal media. The initial parameters are $P = 8$, $k = 1$, $n_0 = 3$, with $\sigma = 0$ (solid line), $\sigma = 0.3$ (dashed line), and $\sigma = 0.5$ (dotted line).

Next, we compare the variational result with the known solution for weakly nonlocal media. In weakly

nonlocal media, the width of the nonlocal response function is much smaller than the beam width, then the nonlinear refractive index can be expressed as

$$\delta n(I) = |u|^2 + \epsilon \frac{\partial^2 |u|^2}{\partial x^2}, \tag{23}$$

where

$$\epsilon = \frac{1}{2}\int_{-\infty}^{\infty} x^2 R(x)\mathrm{d}x = \frac{1}{4}\sigma^2 \tag{24}$$

is a small parameter. In this case, we can obtain the evolution behaviour of the Gaussian beam (Eq.(4))

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} = \frac{1}{k^2 W^3} - \frac{P}{\sqrt{2\pi} n_0 W^2} + \frac{3\epsilon P}{\sqrt{2\pi} n_0 W^4}. \tag{25}$$

In such a case, the stationary Gaussian solitons can be obtained

$$W_0 = \frac{\dfrac{1}{k^2} + \sqrt{\dfrac{1}{k^4} + \dfrac{6\epsilon P^2}{\pi n_0^2}}}{\dfrac{\sqrt{2}P}{\sqrt{\pi} n_0}}, \tag{26}$$

which is the same result of Eq.(22) obtained using the Lagrangian density method in this paper.

4. Accessible harmonic oscillation of nonlocal Gaussian solitons

In the following, we concentrate on another special case, highly nonlocal nonlinearity (the width of the response function is much larger than the beam width or tends to infinity[35]). In the limit of high nonlocality, similarly we can also obtain the following expression from Eq.(18):

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} - \frac{1}{k^2 W^3} + \frac{2PW}{\sqrt{\pi} n_0 \sigma^3} = 0. \tag{27}$$

In fact, the nonlocal response function $R(x - x')$ can be expanded in a second-order Taylor series with respect to x' about $x = x'$[33,36] in highly nonlocal media, which yields $\delta n(I) = R_0 P - \frac{1}{2}\gamma P x^2$, where $R_0 = R(x)|_{x=0}$ is the maximum of the nonlocal response function, $\gamma = -R^{(2)}(x)|_{x=0} = 2/\sqrt{\pi}\sigma^3 > 0$ and P is the incident power. Inserting Eq.(4) into the Schrödinger equation with highly nonlocal parabolic nonlinearity, we can obtain the evolution of beam width in the highly nonlocal media,

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} - \frac{1}{k^2 W^3} + \frac{\gamma P W}{n_0} = 0, \tag{28}$$

which is the same as the result obtained using the Lagrangian density method (Eq.(27)). Some results for Gaussian solitons in highly nonlocal media have been discussed in Ref.[33].

Assuming that the beam at $z = 0$ has $\mathrm{d}W/\mathrm{d}z|_{z=0} = 0$, and integrating Eq.(27) once, we can obtain

$$\left[\frac{\mathrm{d}W}{\mathrm{d}z}\right]^2 + \frac{1}{k^2 W_0^2}\left[\frac{W_0^2}{W^2} - 1\right] + \frac{2P}{\sqrt{\pi} n_0 \sigma^3}\left[W^2 - W_0^2\right] = 0. \tag{29}$$

If let

$$V(W) = \frac{1}{k^2 W_0^2}\left[\frac{W_0^2}{W^2} - 1\right] + \frac{2P}{\sqrt{\pi} n_0 \sigma^3}\left[W^2 - W_0^2\right]$$

denote the potential of the Gaussian soliton, then Eq.(29) depicts a classical Newton's equation describing an effective particle with kinetic energy $(\mathrm{d}W/\mathrm{d}z)^2$ moving in the potential $V(W)$. The asymmetric potential of the accessible dipole solitons is illustrated in Fig.3 with the initial parameters $n_0 = 3.0$, $k = 1$, $W_0 = 1$, and $\sigma = 100$, which can satisfy the condition that the width of the nonlocal response function should be much larger than the beam width in strongly nonlocal media. From Fig.3 we can see that, as the incident power increases, the minimum and the width of the potential will decrease synchronously.

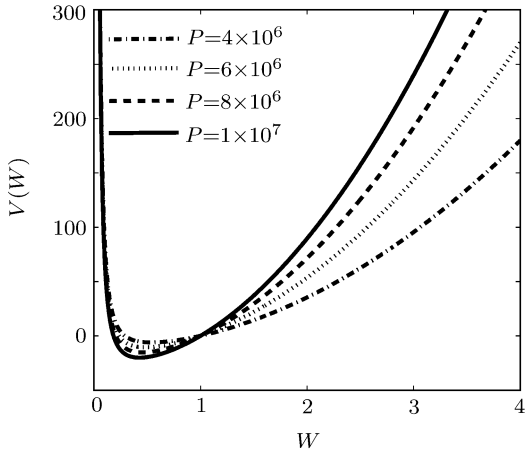


Fig.3. Potential $V(W)$ versus beam width W in strongly nonlocal nonlinear media for different values of incident power P. The initial parameters are $n_0 = 3$, $k = 1$, $W_0 = 1$, and $\sigma = 50$.

Accessible nonlocal Gaussian solitons (corresponding to the effective particles located at the bottom of the potential) can be obtained as $\left[\dfrac{\partial V(W)}{\partial W}\right]|_{W(z)=W_0} = 0$,

$$W_0^4 = \frac{\sqrt{\pi} n_0 \sigma^3}{2k^2 P}. \tag{30}$$

So the critical power for the accessible solitons is obtained

$$P_{\rm c} = \frac{\sqrt{\pi} n_0 \sigma^3}{2k^2 W_0^4}. \tag{31}$$

It is noted that the solitons will be stable when the input power is a restricted value related to the beam width and the nonlocal response function of the media.

When the incident power does not satisfy Eq. (31), the nonlocal accessible Gaussian solitons will undergo linear harmonic oscillations (corresponding to the effective particles oscillating near the bottom of the potential). Let $W(z)/W_0 = Y(z)$ and rewrite Eq.(29) as

$$\left(\frac{{\rm d}Y}{{\rm d}z}\right)^2 = \frac{m(Y^2-1)(\eta - Y^2)}{Y^2 W_0^2}, \tag{32}$$

where $m = 2pW_0^2/\sqrt{\pi} n_0 \sigma^3$, $\eta = P_{\rm c}/P_0 > 0$, $P_{\rm c} = \sqrt{\pi} n_0 \sigma^3 / 2k^2 W_0^4$. From Eq.(32) we can obtain the simple form of Gaussian solitons in strongly nonlocal media

$$W^2(z) = W_0^2 \left[\cos^2(\beta z) + \eta \sin^2(\beta z)\right], \tag{33}$$

where $\beta = \sqrt{m}/W_0 = \sqrt{2P/\sqrt{\pi} n_0 \sigma^3}$ is a constant for the oscillation. The intensity of the beam is obtained from Eq.(4) during the propagation process through

$$I(x,z) = \frac{I_0 W_0^2}{W^2} \exp\left[-\frac{x^2}{W^2}\right], \tag{34}$$

where I_0 is the peak intensity of the incident beam. When $\eta = 1$, $W(z) = W_0$, the accessible nonlocal Gaussian soliton maintains its initial width during the propagation and the soliton is stable, as shown by the solid line in Fig.4. The intensity will also maintain its initial Gaussian statistical form during its propagation process, i.e., a stationary nonlocal Gaussian solitons form, shown in Fig.5. When $\eta \neq 1$, the solitons will undergo linear harmonic oscillation.[36,37]

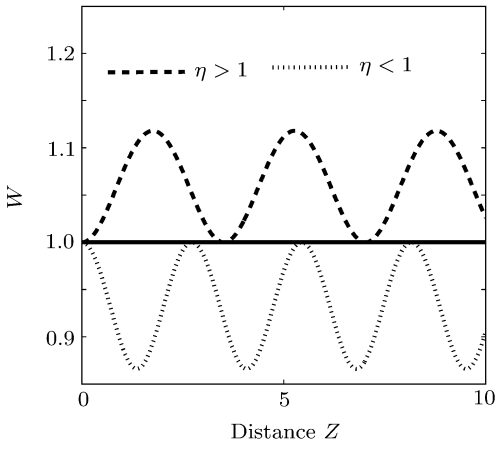


Fig.4. Linear harmonic oscillations of Gaussian solitons in highly nonlocal media. The solid line represents the incident power equal to the critical power ($P = 2.66 \times 10^6$, $\eta = 1$), the dashed line represents the incident power smaller than the critical power ($P = 2.13 \times 10^6$, $\eta > 1$), and the dotted line represents the incident power larger than the critical power ($P = 3.55 \times 10^6$, $\eta < 1$). The initial parameters are $n_0 = 3$, $k = 1$, $W_0 = 1$, and $\sigma = 100$.

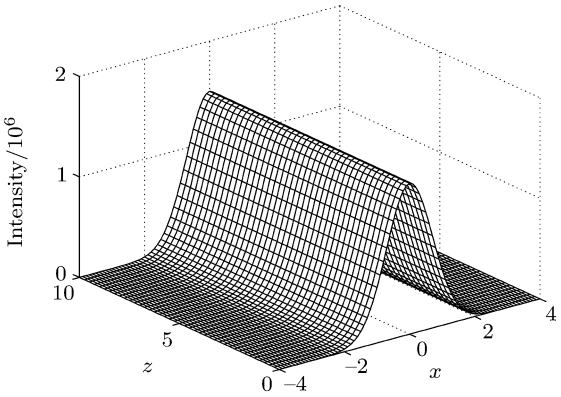


Fig.5. Stationary Gaussian solitons in highly nonlocal media. The initial parameters are the same as the solid line in Fig.4.

This is indeed the case illustrated in Fig.4, where we show the nonstationary propagation of the accessible nonlocal Gaussian solitons in strongly nonlocal media. When the critical power is larger than the incident power ($\eta > 1$), the beam will oscillate periodically with propagation distance between W_0 and some larger values. The beam width will increase at first and decrease when it reaches the maximum, as shown by the dashed line in Fig.4. However, the intensity will initially decrease, and then increase when it reaches the minimum, oscillating between the initial value and a smaller value, as shown in Fig.6.

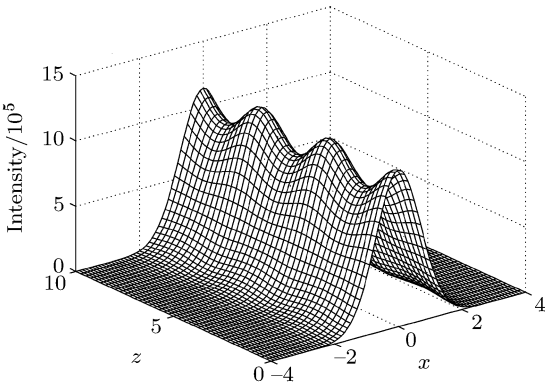


Fig.6. Linear harmonic oscillation of Gaussian solitons in highly nonlocal media. The initial parameters are the same as those for the dashed line in Fig.4.

When the incident power is larger than the critical power ($0 < \eta < 1$), the beam width will oscillate between W_0 and somewhat smaller values, as

shown by the dotted line in Fig.4. The beam width will initially decrease and increase when it reaches the minimum. The intensity initially increases, and then decreases when it reaches the maximum, oscillating between the initial value and a larger value, as shown in Fig.7.

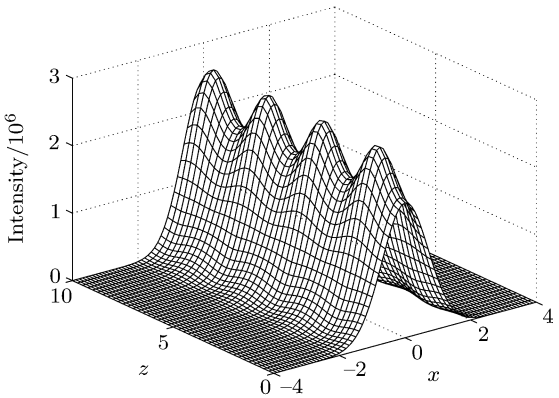


Fig.7. Linear harmonic oscillation of Gaussian solitons in highly nonlocal media. The initial parameters are the same as those for the dotted line in Fig.4.

5. Conclusion

In summary, we have investigated systematically the propagation of Gaussian solitons in nonlocal nonlinear media with arbitrary degree of nonlocality. Using the variational approach, we obtain the exact solutions of such solitons when the degree of the nonlocality is arbitrary. Specifically, we study the conditions for solitons propagating in weakly and highly nonlocal media. We obtain the conditions for solitons that can exist in different nonlocal cases. We also discussed the linear harmonic oscillation of such Gaussian solitons in the limit of highly nonlocal nonlinearity.

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