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Fractional Fourier transform of Lorentz beams*

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This paper introduces Lorentz beams to describe certain laser sources that produce highly divergent fields. The fractional Fourier transform (FRFT) is applied to treat the propagation of Lorentz beams. Based on the definition of convolution and the convolution theorem of the Fourier transform, an analytical expression for a Lorentz beam passing through a FRFT system has been derived. By using the derived formula, the properties of a Lorentz beam in the FRFT plane are illustrated numerically.

Keywords: Lorentz beam, fractional Fourier transform, propagation properties
PACC: 4225B, 4230K

1. Introduction

As a new kind of realizable beams, Lorentz beams have been recently introduced by Gawhary and Severini.[1] With the same spatial extension, the angular spreading of a Lorentzian distribution is higher than that of a Gaussian description.[2] Therefore, Lorentz beams provide appropriate models to describe some certain laser sources, e.g. double heterojunction $Ga_{1-x}Al_xAs$ lasers, which produce highly divergent fields.[2,3] The propagation of a Lorentz beam under Fresnel approximation has been investigated in free space, and the corresponding closed-form scalar representation on a general transverse plane has been presented.[1] The characteristic of a Lorentz beam in the far field has been examined from the vectorial structure, and the vectorial structure of a Lorentz beam has been compared with that of a Gaussian beam.[4] At the paraxial case, the beam propagation factor of a Lorentz beam has been verified to be $\sqrt{2}$.[5] A finite linear combination of Lorentz beams has been proven to be a super-Lorentzian beam.[6] Early in 1993, the fractional Fourier transform (FRFT) was first introduced to optics.[7–9] Since then, a series of works have been done on its implementation, properties and applications.[10,11] The FRFT has been widely used in beam analysis, beam shape and signal processing. The FRFTs of various laser beams have been widely investigated.[12–24] However, the research in the FRFT of Lorentz beams has not been reported elsewhere. In the rest of this paper, therefore, the FRFT is applied to treat the propagation of Lorentz beams. An analytical expression for the FRFT of a Lorentz beam is derived, and the properties of a Lorentz beam in the FRFT plane are graphically illustrated with numerical examples.

2. Fractional Fourier transform of Lorentz beams

In the Cartesian coordinates system, the z-axis is taken to be the propagation axis. The Lorentz beam in the source plane $z = 0$ takes the form as[1]

$$E(x_0, y_0) = \frac{E_0}{w_{0x} w_{0y} [1 + (x_0/w_{0x})^2][1 + (y_0/w_{0y})^2]}, \quad (1)$$

where E_0 is a constant, w_{0x} and w_{0y} are the parameters related to the beam widths in the x- and y-directions, respectively. The time dependent factor $\exp(-i\omega t)$ is omitted in the Eq.(1), and ω is the circular frequency. As the Lohmann I optical system and the Lohmann II optical system are equivalent, the Lohmann I optical system is used to implement the FRFT of the Lorentz beam, which is shown in Fig.1. In Fig.1, f_1 is the standard focal length, $\varphi = p\pi/2$, p is the order of FRFT, and d is the distance. P_1 and P_2 are the input and output planes, respectively. The focus of the lens is $f = f_1/\sin\varphi$. The distance between the input and output planes is $2d = 2f_1 \tan(\varphi/2)$.

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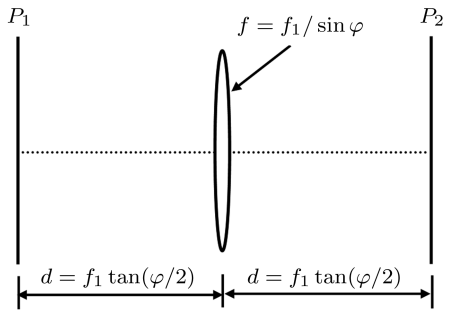


Fig.1. The Lohmann I optical system for performing the fractional Fourier transform.

The Lohmann I optical system is described by the following transfer matrix:

$$\begin{aligned}\boldsymbol{R}_1 &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\varphi & f_1\sin\varphi \\ -\sin\varphi/f_1 & \cos\varphi \end{pmatrix}. \end{aligned} \tag{2}$$

The Lorentz beam passing through the Lohmann I optical system obeys the well-known Collins integral formula

$$E(x,y) = \frac{\exp(\mathrm{i}kz)}{\mathrm{i}\lambda B}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x_0,y_0)\exp\left\{\frac{\mathrm{i}k}{2B}[A(x_0^2+y_0^2) - 2(x_0x+yy_0) + D(x^2+y^2)]\right\}\mathrm{d}x_0\mathrm{d}y_0, \tag{3}$$

where $k = 2\pi/\lambda$, λ is the wavelength. Substituting Eqs.(1) and (2) into Eq.(3), one can obtain the output field distribution in the FRFT plane

$$E(x,y) = \frac{E_0}{\mathrm{i}\lambda f_1\sin\varphi}\exp(\mathrm{i}kz)E(x)E(y), \tag{4}$$

with $E(x)$ and $E(y)$ given by

$$\begin{aligned} &E(s) \\ &= \int_{-\infty}^{\infty}\frac{1}{w_{0s}[1+(s_0/w_{0s})^2]} \\ &\quad\times\exp\left[\frac{\mathrm{i}k}{2f_1\sin\varphi}(s_0^2\cos\varphi - 2s_0s + s^2\cos\varphi)\right]\mathrm{d}s_0, \end{aligned} \tag{5}$$

where $s = x$ or y (hereafter). If $\sin\varphi \neq 0$, Eq.(5) can be simplified to be

$$\begin{aligned} E(s) &= w_{0s}\exp\left(\frac{ks^2\tan\varphi}{2\mathrm{i}f_1}\right)\int_{-\infty}^{\infty}\frac{1}{w_{0s}^2+s_0^2} \\ &\quad\times\exp\left[\frac{\mathrm{i}k\cot\varphi}{2f_1}\left(\frac{s}{\cos\varphi} - s_0\right)^2\right]\mathrm{d}s_0. \end{aligned} \tag{6}$$

Equation (6) is difficult to directly perform the integral. However, one may note that the convolution of two functions $F_1(\tau)$ and $F_2(\tau)$ is defined by

$$F_1(\tau) * F_2(\tau) = \int_{-\infty}^{\infty} F_1(\eta)F_2(\tau-\eta)\mathrm{d}\eta, \tag{7}$$

where the asterisk denotes the convolution. Comparing Eq.(6) with Eq.(7), we can rewrite Eq.(6) in the form of the convolution:

$$\begin{aligned} &E(s) \\ &= w_{0s}\exp\left(\frac{ks^2\tan\varphi}{2\mathrm{i}f_1}\right)\left[F_1\left(\frac{s}{\cos\varphi}\right) * F_2\left(\frac{s}{\cos\varphi}\right)\right], \end{aligned} \tag{8}$$

where the functions $F_1(\tau)$ and $F_2(\tau)$ are given by

$$F_1(\tau) = \frac{1}{w_{0s}^2+\tau^2}, \tag{9}$$

$$F_2(\tau) = \exp\left(\frac{\mathrm{i}k\cot\varphi}{2f_1}\tau^2\right). \tag{10}$$

The convolution theorem of the Fourier transform is[25]

$$F_1(\tau) * F_2(\tau) = \int_{-\infty}^{\infty} F_1(\xi)F_2(\xi)\exp(-\mathrm{i}\xi\tau)\mathrm{d}\xi, \tag{11}$$

where $F_1(\xi)$ and $F_2(\xi)$ are the Fourier transforms of $F_1(\tau)$ and $F_2(\tau)$, respectively. Therefore, Eq.(8) is found to be

$$\begin{aligned} E(s) &= \sqrt{\frac{\pi\mathrm{i}f_1\tan\varphi}{2k}}\exp\left(\frac{ks^2\tan\varphi}{2\mathrm{i}f_1}\right) \\ &\quad\times\int_{-\infty}^{\infty}\exp\left(-w_{0s}|\xi|\right)\exp\left(-\frac{\mathrm{i}f_1\tan\varphi}{2k}\xi^2\right) \\ &\quad\times\exp\left(-\mathrm{i}\frac{s}{\cos\varphi}\xi\right)\mathrm{d}\xi. \end{aligned} \tag{12}$$

Equation (12) can be divided into two parts:

$$E(s) = \sqrt{\frac{\pi \mathrm{i} f_1 \tan\varphi}{2k}} \exp\left(\frac{k s^2 \tan\varphi}{2\mathrm{i} f_1}\right) \left\{ \exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0s} + \mathrm{i}\frac{s}{\cos\varphi}\right)^2\right] \times \int_0^\infty \exp\left\{-\frac{\mathrm{i}f_1\tan\varphi}{2k}\left[\xi + \frac{k\cot\varphi}{\mathrm{i}f_1}\left(w_{0s} + \mathrm{i}\frac{s}{\cos\varphi}\right)\right]^2\right\}\mathrm{d}\xi + \exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0s} - \mathrm{i}\frac{s}{\cos\varphi}\right)^2\right] \int_0^\infty \exp\left\{-\frac{\mathrm{i}f_1\tan\varphi}{2k}\left[\xi + \frac{k\cot\varphi}{\mathrm{i}f_1}\left(w_{0s} - \mathrm{i}\frac{s}{\cos\varphi}\right)\right]^2\right\}\mathrm{d}\xi \right\}. \quad (13)$$

Equation (13) can be further rewritten as

$$E(s) = \sqrt{\pi}\exp\left(\frac{k s^2\tan\varphi}{2\mathrm{i}f_1}\right)\left\{\exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0s} + \mathrm{i}\frac{s}{\cos\varphi}\right)^2\right]\left[\int_0^\infty \exp(-\xi'^2)\mathrm{d}\xi' - \int_0^{T_1}\exp(-\xi'^2)\mathrm{d}\xi'\right] + \exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0s} - \mathrm{i}\frac{s}{\cos\varphi}\right)^2\right]\left[\int_0^\infty \exp(-\xi'^2)\mathrm{d}\xi' - \int_0^{T_2}\exp(-\xi'^2)\mathrm{d}\xi'\right]\right\}, \quad (14)$$

where

$$T_1 = \sqrt{\frac{k\cot\varphi}{2\mathrm{i}f_1}}\left(w_{0s} + \mathrm{i}\frac{s}{\cos\varphi}\right), \quad T_2 = \sqrt{\frac{k\cot\varphi}{2\mathrm{i}f_1}}\left(w_{0s} - \mathrm{i}\frac{s}{\cos\varphi}\right). \quad (15)$$

We note that

$$\mathrm{erf}(T) = \frac{2}{\sqrt{\pi}}\int_0^T \exp(-s^2)\mathrm{d}s \quad (16)$$

is the usual error function.[26] Therefore, Eq.(14) yields

$$E(s) = \frac{\pi}{2}\exp\left(\frac{k\tan\varphi}{2\mathrm{i}f_1}s^2\right)[E_s^+(s) + E_s^-(s)], \quad (17)$$

where

$$E_s^\pm(s) = \exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0s} \pm \mathrm{i}\frac{s}{\cos\varphi}\right)^2\right] \times \left\{1 - \mathrm{erf}\left[\sqrt{\frac{k\cot\varphi}{2\mathrm{i}f_1}}\left(w_{0s} \pm \mathrm{i}\frac{s}{\cos\varphi}\right)\right]\right\}. \quad (18)$$

An analytical expression for the output field distribution in the FRFT plane turns out to be

$$E(x,y) = \frac{E_0\pi^2}{\mathrm{i}4\lambda f_1\sin\varphi}\exp(\mathrm{i}kz)\exp\left(\frac{k(x^2+y^2)}{2\mathrm{i}f_1\cot\varphi}\right)\left\{\exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0x} + \mathrm{i}\frac{x}{\cos\varphi}\right)^2\right] \times \left\{1 - \mathrm{erf}\left[\sqrt{\frac{k\cot\varphi}{2\mathrm{i}f_1}}\left(w_{0x} + \mathrm{i}\frac{x}{\cos\varphi}\right)\right]\right\} + \exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0x} - \mathrm{i}\frac{x}{\cos\varphi}\right)^2\right] \times \left\{1 - \mathrm{erf}\left[\sqrt{\frac{k\cot\varphi}{2\mathrm{i}f_1}}\left(w_{0x} - \mathrm{i}\frac{x}{\cos\varphi}\right)\right]\right\}\right\}\left\{\exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0y} + \mathrm{i}\frac{y}{\cos\varphi}\right)^2\right] \times \left\{1 - \mathrm{erf}\left[\sqrt{\frac{k\cot\varphi}{2\mathrm{i}f_1}}\left(w_{0y} + \mathrm{i}\frac{y}{\cos\varphi}\right)\right]\right\} + \exp\left[\frac{k\cot\varphi}{2\mathrm{i}f_1}\left(w_{0y} - \mathrm{i}\frac{y}{\cos\varphi}\right)^2\right] \times \left\{1 - \mathrm{erf}\left[\sqrt{\frac{k\cot\varphi}{2\mathrm{i}f_1}}\left(w_{0y} - \mathrm{i}\frac{y}{\cos\varphi}\right)\right]\right\}\right\}. \quad (19)$$

If $\sin\varphi = 0$, it means $p = 2n(n = 0, 1, 2, \ldots)$. In this case, the output field distribution in the FRFT plane yields

$$E(x,y) = (-1)^n \frac{E_0 \exp(\mathrm{i}kz)}{w_{0x}w_{0y}} \frac{1}{(1+x^2/w_{0x}^2)(1+y^2/w_{0y}^2)}. \quad (20)$$

3. Numerical calculations

According to the obtained analytical expression, the properties of a Lorentz beam in the FRFT plane are investigated numerically. The parameters are set to be as follows: $\lambda = 0.8$ μm, $f_1 = 1250$ mm, $w_{0x} = w_{0y} = 2$ mm, and $E_0 = w_{0x}w_{0y}$. Figure 2 represents the normalized intensity distribution of a Lorentz beam with different fractional order p in the FRFT plane. The normalized intensity is given by $|E(x,y)|^2 / |E(x,y)|^2_{\max}$, where $|E(x,y)|^2_{\max}$ denotes the maximum of intensity $|E(x,y)|^2$. The variation of normalized intensity distribution with p is periodic, and the period is 2. The case of $p = 1$ corresponds to the ordinary Fourier transform case. When a Lorentz

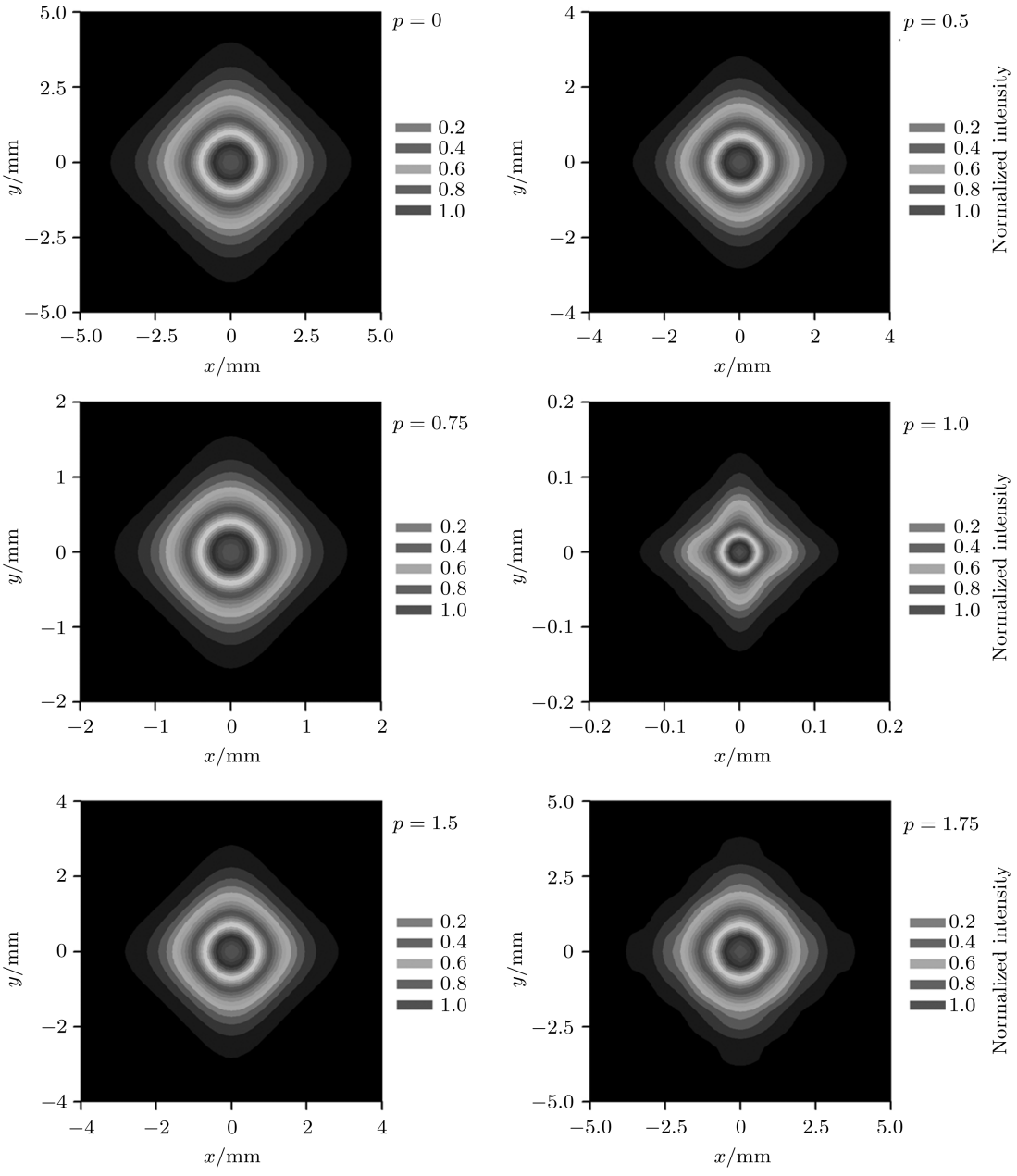


Fig.2. Normalized intensity of a Lorentz beam with different fractional order p in the FRFT plane.

beam passes through FRFT systems with different fractional orders, the beam shape in the FRFT plane keeps invariant. When $p = 2n + 1(n = 0, 1, 2, \ldots)$, the beam spot reaches the minimum size. While for $p = 2n(n = 0, 1, 2, \ldots)$, the beam spot reaches the maximum size. Therefore, the requisite size of the beam spot can be realized by choosing an appropriate value of p. In Fig.2, w_{0x} is set to be equal to w_{0y}, which results in the symmetrical beam shape. However, this is a special case. To describe the inequality of two transverse beam widths, a new parameter is introduced as follows:

$$e = \frac{w_{0y}}{w_{0x}}. \tag{21}$$

Figure 3 shows the normalized intensity of a Lorentz beam with different parameter e in the FRFT plane. For simplicity, w_{0x} is still 2 mm and keeps invariable, whereas w_{0y} is varied. The $p = 1.25$ and the rest of the parameters are the same as those in Fig.2. When $e = 1$, the beam spot in the FRFT plane is symmetric. Otherwise, the beam spot is asymmetric. If $e > 1$, the long axis of the beam spot is located at the y-direction. When $e < 1$, the long axis of the beam spot is located at the x-direction. Laser beams generated by highly divergent laser sources are usually nonsymmetrical. Accordingly, these highly divergent nonsymmetrical beams can be described by choosing an appropriate value of e.

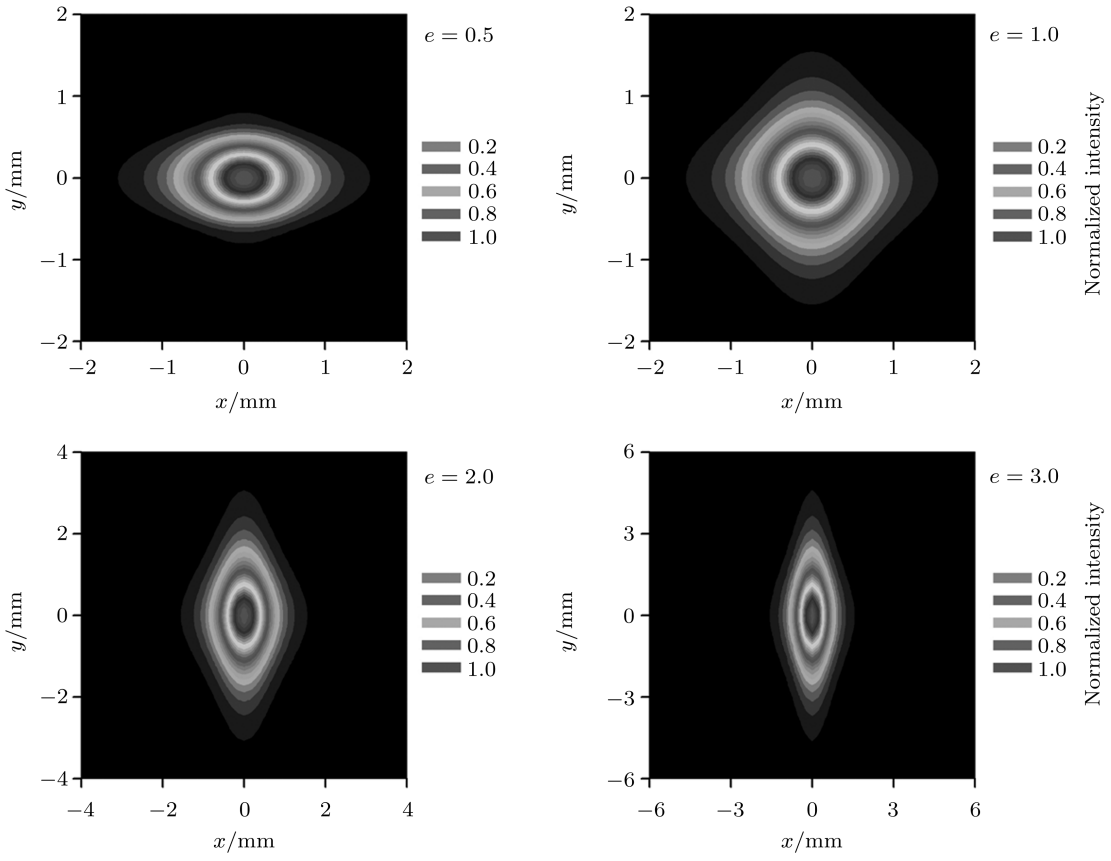


Fig.3. Normalized intensity of a Lorentz beam with different parameter e in the FRFT plane.

4. Conclusions

In conclusion, the FRFT of a Lorentz beam has been investigated. Based on the definition of convolution and the convolution theorem of the Fourier transform, an analytical expression for a Lorentz beam passing through a FRFT system has been derived. According to the derived formula, the properties of

a Lorentz beam in the FRFT plane are illustrated by virtue of numerical simulations. The variation of normalized intensity distribution with the fractional order p is periodic, and the period is 2. The size and the non-symmetry of the beam spot can be controlled by choosing appropriate values of p and e. The FRFT optical system provides a convenient way for controlling the properties of the Lorentz beam.

References

[1] Gawhary O E and Severini S 2006 *J. Opt.* A: *Pure Appl. Opt.* **8** 409
[2] Naqwi A and Durst F 1990 *Appl. Opt.* **29** 1780
[3] Dumke W P 1975 *J. Quantum Electron.* **QE-11** 400
[4] Zhou G Q, Zheng J and Xu Y Q 2008 *J. Mod. Opt.* **55** 993
[5] Zhou G Q 2008 *Acta Phys. Sin.* **57** 3494 (in Chinese)
[6] Gawhary O E and Severini S 2007 *Opt. Commun.* **269** 274
[7] Mendlovic D and Ozaktas H M 1993 *J. Opt. Soc. Am.* A **10** 1875
[8] Ozaktas H M and Mendlovic D 1993 *J. Opt. Soc. Am.* A **10** 2522
[9] Lohmann A W 1993 *J. Opt. Soc. Am.* A **10** 2181
[10] Lohmann A W, Mendlovic D and Zalevsky Z 1998 *Prog. Opt.* **38** 263
[11] Ozaktas H M Zalevsky Z and Kutay M A 2001 *The Fractional Fourier Transform with Applications in Optics and Signal Processing* (New York: Wiley)
[12] Zhao D M and Wang S M 2001 *Acta Phys. Sin.* **50** 1935 (in Chinese)
[13] Du X Y and Zhao D M 2006 *Appl. Opt.* **45** 9049
[14] Du X Y and Zhao D M 2007 *Phys. Lett.* A **366** 271
[15] Du X Yand Zhao D M 2008 *Optik* **119** 379
[16] Cai Y J and Lin Q 2004 *Chin Phys.* **13** 1025
[17] Lin Q and Cai Y J 2002 *Opt. Lett.* **27** 1672
[18] Yuan L 2008 *Chin. Phys.* B **17** 170
[19] Cai Y J and Lin Q 2003 *J. Opt. Soc. Am.* A **20** 1528
[20] Cai Y J and Lin Q 2004 *J. Opt.* A: *Pure Appl. Opt.* **6** 307
[21] Zhao D M, Mao H D, Liu H J, Wang S M, Jing F and Wei X F 2004 *Opt. Commun.* **236** 225
[22] Zheng C W 2006 *Opt. Commun.* **259** 445
[23] Zheng C W 2006 *Phys. Lett.* A **355** 156
[24] Zheng C W 2008 *Opt. Laser Technol.* **40** 632
[25] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic Press)
[26] Abramowitz M andStegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)