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A novel fractional-order hyperchaotic system and its synchronization*

Zhou Ping(周 平)[a)b)†], Wei Li-Jia(危丽佳)[b)], and Cheng Xue-Feng(程雪峰)[b)]

[a)] *Key Laboratory of Network Control & Intelligent Instrument of Ministry of Education, Chongqing University of Posts and Telecommunications, Chongqing* 400065, *China*

[b)] *Institute of Applied Physics, Chongqing University of Posts and Telecommunications, Chongqing* 400065, *China*

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In this paper, a novel hyperchaotic system with one nonlinear term and its fractional order system are proposed. Furthermore, synchronization between two fractional-order systems with different fractional-order values is achieved. The proposed synchronization scheme is simple and theoretically rigorous. Numerical simulations are in agreement with the theoretical analysis.

Keywords: hyperchaotic system, fractional-order hyperchaotic system, nonlinear term, synchronization

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1. Introduction

There has been increasing interest in exploiting chaotic dynamics in engineering applications, such as electrical engineering, telecommunications, information processing, material engineering, etc. This is due to the features of chaos desired in some engineering and technological applications, where chaos can provide certain unique features for special needs. But, in secure communication, there exists a possibility that the messages, masked by a simple chaotic system with only a single positive Lyapunov exponent, can sometimes be easily extracted when they are intercepted. An example of how to extract messages that are masked by a chaotic signal in a system of two Lorenz systems was shown in Ref.[1]. Because hyperchaos systems have more than one positive Lyapunov exponent, and more complex dynamic behavior, they have the characteristics of high capacity, high security and high efficiency. Typical examples are the four-dimensional (4D) hyperchaotic Rossler system, 4D hyperchaotic Lorenz–Haken system, 4D hyperchaos Chua's circuit, and 4D hyperchaos Chen system.[2−4]

In recent years, the study of the dynamics of fractional-order differential systems has attracted increasing interest from many researchers. It has been demonstrated that some fractional-order differential systems behave chaotically or hyperchaotically, such as the fractional-order Chua's system, the fractional Rossler system, the fractional modified Duffing system and the Chen system.[5−10] On the other hand, chaos synchronization for the fractional-order systems are is beginning to attract some attention due to its potential applications in secure communication and control processing.[11−16] But in Refs.[11]–[16], the synchronization among fractional-order systems is only investigated through numerical simulations.

In this paper, a novel hyperchaotic system with one nonlinear term is proposed, and its fractional-order system is studied numerically. We also investigate the synchronization of the novel fractional-order hyperchaotic systems with different fractional-order values.

2. A novel hyperchaotic system with one nonlinear term

Consider the following new dynamical system

$$\begin{aligned} \dot{x} &= 0.56x - y, \\ \dot{y} &= x - 0.1yz^2, \\ \dot{z} &= 4y - z - 6w, \\ \dot{w} &= 0.5z + 0.8w. \end{aligned} \tag{1}$$

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†Corresponding author. E-mail: zhouping@cqupt.edu.cn

Here, x, y, z and w are state variables. The new system (1) is symmetric with respect to the origin, which is invariant for the coordinate transformation (x, y, z, w) $\rightarrow$ ($-x$, $-y$, $-z$, $-w$).

The Lyapunov exponents of system (1) are $\lambda_1 = 0.074726$, $\lambda_2 = 0.021509$, $\lambda_3 = 0$, $\lambda_4 = -0.43386$. So system (1) is a hyperchaotic system only with one nonlinear term. The Lyapunov dimension is $D_{\mathrm{L}} = 3 + (\lambda_1 + \lambda_2)/|\lambda_4| = 3.2218$. Figure 1 shows the hyperchaos phase portrait.

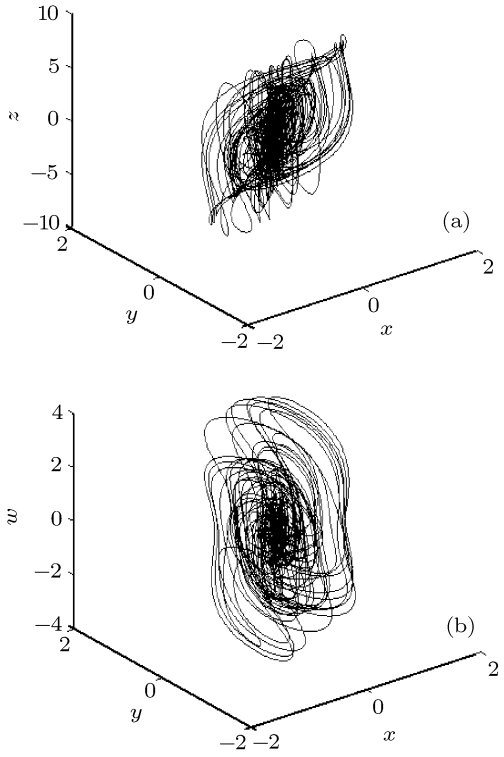


Fig.1. Hyperchaotic phase portraits of system (1): (a) $x - y - z$ plane; (b) $x - y - w$ plane.

3. Numerical solution of fractional differential equations

Based on the above hyperchaotic system (1), we can obtain the fractional-order hyperchaotic system (2).

$$\begin{aligned}
\frac{\mathrm{d}^{q_1}x}{\mathrm{d}t^{q_1}} &= 0.56x - y,\\
\frac{\mathrm{d}^{q_2}y}{\mathrm{d}t^{q_2}} &= x - 0.1yz^2,\\
\frac{\mathrm{d}^{q_3}z}{\mathrm{d}t^{q_3}} &= 4y - z - 6w,\\
\frac{\mathrm{d}^{q_4}w}{\mathrm{d}t^{q_4}} &= 0.5z + 0.8w,
\end{aligned} \tag{2}$$

where q is fractional order, $0 < q_1, q_2, q_3, q_4 \leq 1$.

There are two approximation methods for solving fractional differential equations. The first method is an improved version of the Adams–Bashforth–Moulton algorithm.[7,17,18] The second method is the frequency domain approximation. However, the frequency domain approximation in the numerical simulations of fractional systems may result in wrong consequences.[19] For example, this approximation can numerically demonstrate chaos in non-chaotic fractional-order systems.[19]

Now, we use the first method for solving fractional differential equation (2). Similar to that in Ref.[7], the predictor–corrector scheme[20] for system (2) is derived. This scheme is the generalization of the Adams–Bashforth–Moulton one.

Let $h = T/N$ and $t_n = nh (n = 0, 1, 2 \ldots, N)$. So, Eq.(2) with initial condition (x_0, y_0, z_0, w_0) can be discretized as follows:

$$\begin{aligned}
x_{n+1} &= x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}\left[0.56x^p_{n+1} - y^p_{n+1} + \sum_{j=0}^{n}\alpha_{1,j,n+1}(0.56x_j - y_j)\right],\\
y_{n+1} &= y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}\left[x^p_{n+1} - 0.1y^p_{n+1}(z^p_{n+1})^2 + \sum_{j=0}^{n}\alpha_{2,j,n+1}(x_j - 0.1y_jz_j^2)\right],\\
z_{n+1} &= z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}\left[4y^p_{n+1} - z^p_{n+1} - 6w^p_{n+1} + \sum_{j=0}^{n}\alpha_{3,j,n+1}(4y_j - z_j - 6w_j)\right],\\
w_{n+1} &= w_0 + \frac{h^{q_4}}{\Gamma(q_4+2)}\left[0.5z^p_{n+1} + 0.8w^p_{n+1} + \sum_{j=0}^{n}\alpha_{4,j,n+1}(0.5z_j + 0.8w_j)\right],
\end{aligned}$$

where

$$x_{n+1}^p = x_0 + \frac{1}{\Gamma(q_1)}\sum_{j=0}^{n}\beta_{1,j,n+1}(0.56x_j - y_j),$$

$$y_{n+1}^p = y_0 + \frac{1}{\Gamma(q_2)}\sum_{j=0}^{n}\beta_{2,j,n+1}(x_j - 0.1y_j z_j^2),$$

$$z_{n+1}^p = z_0 + \frac{1}{\Gamma(q_3)}\sum_{j=0}^{n}\beta_{3,j,n+1}(4y_j - z_j - 6w_j),$$

$$w_{n+1}^p = w_0 + \frac{1}{\Gamma(q_4)}\sum_{j=0}^{n}\beta_{4,j,n+1}(0.5z_j + 0.8w_j),$$

$$\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i}, & j = 0; \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, & 1 \le j \le n; \\ 1 & j = n+1. \end{cases}$$

$$\beta_{i,j,n+1} = \frac{h^{q_i}}{q_i}[(n-j+1)^{q_i} - (n-j)^{q_i}], \quad 0 \le j \le n, \quad i = 1,2,3,4.$$

By simulations we have obtained the phase portraits of the fractional-order system (2), as shown in Figs.2 and 3.

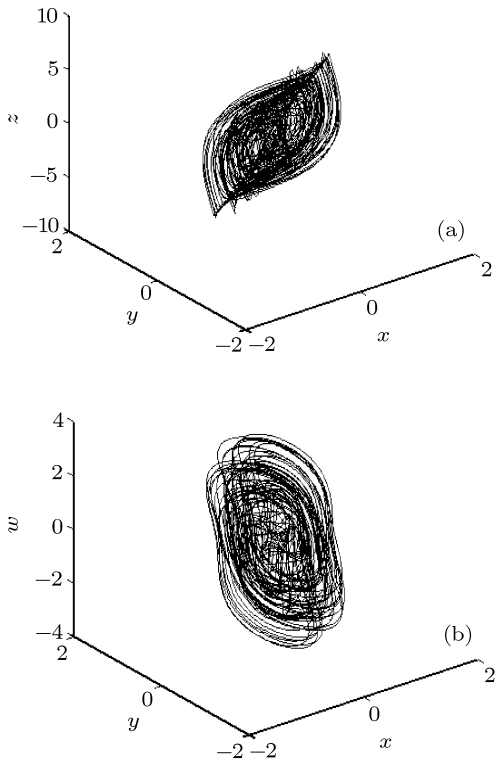


Fig.2. Hyperchaotic phase portraits of the fractional-order system (2) for $q_1 = q_2 = q_3 = q_4 = 0.95$: (a) $x-y-z$ plane; (b) $x-y-w$ plane.

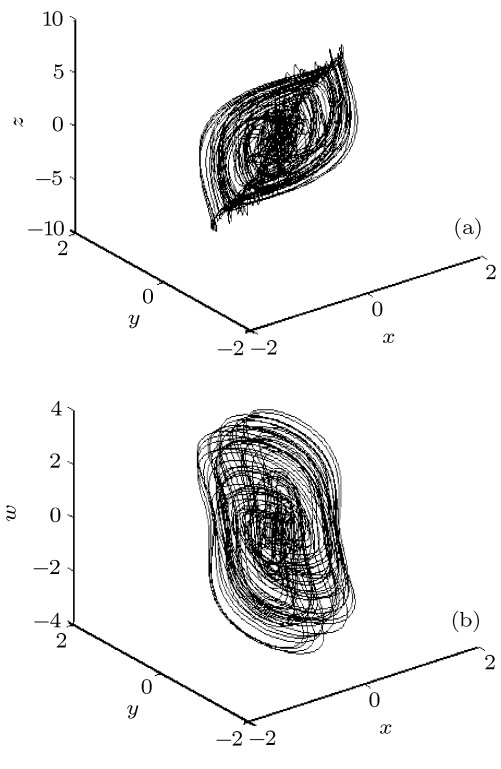


Fig.3. Hyperchaotic phase portraits of the fractional-order system (2) for $q_1 = q_2 = 0.98$, $q_3 = q_4 = 0.95$: (a) $x-y-z$ plane; (b) $x-y-w$ plane.

4. Synchronization between two fractional-order hyperchaotic systems

Now, we will focus on the synchronization of two fractional-order hyperchaotic systems (2) for $q_1 = q_2 = 0.98$, and $q_3 = q_4 = 0.95$.

Consider another fractional-order hyperchaotic system, which is described by

$$\begin{aligned}
\frac{\mathrm{d}^{q_1}x'}{\mathrm{d}t^{q_1}} &= 0.56x' - y', \\
\frac{\mathrm{d}^{q_2}y'}{\mathrm{d}t^{q_2}} &= x' - 0.1y'z'^2, \\
\frac{\mathrm{d}^{q_3}z'}{\mathrm{d}t^{q_3}} &= 4y' - z' - 6w', \\
\frac{\mathrm{d}^{q_4}w'}{\mathrm{d}t^{q_4}} &= 0.5z' + 0.8w'.
\end{aligned} \tag{3}$$

System (2) is regarded as the driving system. The coupled slave fractional-order hyperchaotic system has the following form

$$\begin{aligned}
\frac{\mathrm{d}^{q_1}x'}{\mathrm{d}t^{q_1}} &= 0.56x' - y' + k(x' - x), \\
\frac{\mathrm{d}^{q_2}y'}{\mathrm{d}t^{q_2}} &= x' - 0.1y'z'^2 + 0.1z^2(y' - y) - 0.2yz(z - z'), \\
\frac{\mathrm{d}^{q_3}z'}{\mathrm{d}t^{q_3}} &= 4y' - z' - 6w', \\
\frac{\mathrm{d}^{q_4}w'}{\mathrm{d}t^{q_4}} &= 0.5z' + 0.8w',
\end{aligned} \tag{4}$$

where k is a constant to be fixed. To investigate the synchronization of systems (2) and (4), we define the error states $e_1 = x' - x$, $e_2 = y' - y$, $e_3 = z' - z$, and $e_4 = w' - w$. Then, the corresponding error dynamics system can be obtained by subtracting Eq.(2) from Eq.(4),

$$\begin{pmatrix} \mathrm{d}^{q_1}e_1/\mathrm{d}t^{q_1} \\ \mathrm{d}^{q_2}e_2/\mathrm{d}t^{q_2} \\ \mathrm{d}^{q_3}e_3/\mathrm{d}t^{q_3} \\ \mathrm{d}^{q_4}e_4/\mathrm{d}t^{q_4} \end{pmatrix} = \begin{pmatrix} 0.56 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & -1 & -6 \\ 0 & 0 & 0.5 & 0.8 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} + \begin{pmatrix} ke_1 \\ -0.1e_2e_3^2 - 0.1ye_3^2 - 0.2e_2e_3z \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

If the zero solution $e_1 = e_2 = e_3 = e_4 = 0$ of error dynamics system (5) is globally asymptotically stable for a suitable constant k, the two systems (2) and (4) will synchronize with each other.

Theorem 1 If $(k + 0.56) < 2\cos(0.5\pi q)$, then systems (2) and (4) can be brought into synchronization, where $q = \max(q_1, q_2, q_3, q_4)$, $0 < 0.5\pi q < 0.5\pi$.

Proof

Form error dynamics system (5), we can obtain that $e_1 = e_2 = e_3 = e_4 = 0$ is an equilibrium point of system (5). Linearizing system (5) at this equilibrium point yields the Jacobian matrix

$$\boldsymbol{J} = \begin{bmatrix} 0.56 + k & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & -1 & -6 \\ 0 & 0 & 0.5 & 0.8 \end{bmatrix}$$

and its characteristic equation is

$$[\lambda^2 - (k + 0.56)\lambda + 1][\lambda^2 + 0.2\lambda + 2.2] = 0.$$

The eigenvalues of matrix J are $\lambda_\pm = 0.5(k + 0.56) \pm 0.5\sqrt{(k + 0.56)^2 - 4}$, $\lambda'_\pm = -0.1 \pm \mathrm{i}\sqrt{2.19}$.

Because $(k + 0.56) < 2\cos(0.5\pi q)$, the following three cases should be satisfied,

Case 1 If $k + 0.56 < 0$, then all the eigenvalues of matrix $\boldsymbol{J}$ have negative real values. Therefore, the equilibrium point $e_1 = e_2 = e_3 = e_4 = 0$ of the error dynamics system (5) is globally asymptotically stable, which implies that the two systems (2) and (4) are synchronized.

Case 2 If $0 < (k + 0.56) < 2\cos(0.5\pi q)$, then the argument of eigenvalues $\lambda_\pm$ is

$$\begin{aligned}
|\arg\lambda_\pm| &= \left|\arctan\frac{\sqrt{4 - (k + 0.56)^2}}{k + 0.56}\right| \\
&> \left|\arctan\frac{\sqrt{4 - (k + 0.56)^2}}{2\cos(0.5\pi q)}\right|;
\end{aligned}$$

that is,

$$|\arg\lambda_{\pm}| > \left|\arctan\frac{\sqrt{4-(k+0.56)^2}}{2\cos(0.5\pi q)}\right| > \left|\arctan\frac{\sqrt{4-(2\cos(0.5\pi q))^2}}{2\cos(0.5\pi q)}\right|,$$

so

$$|\arg\lambda_{\pm}| > \left|\arctan\frac{\sqrt{4-(2\cos(0.5\pi q))^2}}{2\cos(0.5\pi q)}\right| = 0.5\pi q,$$

and we have $|\arg\lambda_i| > 0.5\pi q (i = 1, 2, 3, 4)$.

According to Ref.[21], the equilibrium point $e_1 = e_2 = e_3 = e_4 = 0$ of error dynamics system (5) is globally asymptotically stable, which implies that the synchronization between the systems (2) and (4) can be achieved.

Case 3 If $k + 0.56 = 0$, then $|\arg\lambda_{\pm}| = 0.5\pi$, and $\left|\arg\lambda'_{\pm}\right| > 0.5\pi$. We have $|\arg\lambda_i| > 0.5\pi q$ $(i = 1, 2, 3, 4)$.

Similar to **Case 2**, the equilibrium point $e_1 = e_2 = e_3 = e_4 = 0$ of system (5) is globally asymptotically stable. Namely, the synchronization between the systems (2) and (4) can be achieved. Thus the proof is completed.

The above results indicate that the fractional-order hyperchaotic systems (2) and (4) are synchronized under a suitable constant k. Some simulation results are described below.

Now, we choose $k = -2$ for numerical simulations. The initial states of the drive system (2) and the response system (4) are taken to be $x(0) = -1$, $y(0) = 1.5$, $z(0) = -1$, $w(0) = -2$ and $x'(0) = 0$, $y(0) = 1$, $z'(0) = 1$, $w'(0) = -1$. The simulation results are depicted in Fig.4.

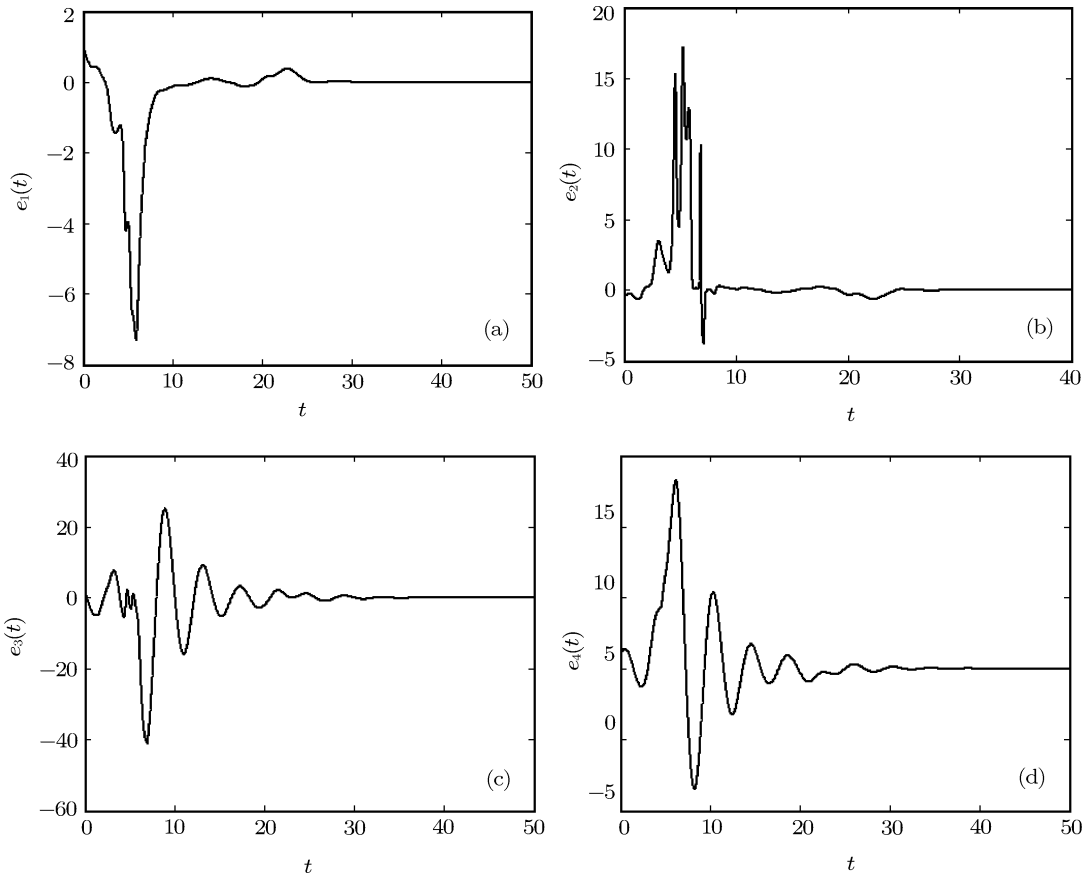


Fig.4. The synchronization error functions of four state variables versus time t: (a) $x' - x$; (b) $y' - y$; (c) $z' - z$; (d) $w' - w$.

5. Conclusion

In this paper, a novel hyperchaotic system with one nonlinear term and its fractional order system are proposed. Its dynamical behaviours are studied. Moreover, synchronization between two such fractional-order hyperchaotic systems with different fractional-order values has been achieved via feedback control. The proposed synchronization approach is simple and theoretically rigorous. Finally, numerical simulation is given to verify the effectiveness of the proposed synchronization scheme.

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