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Effects of localized impurity on a dark soliton in a Bose–Einstein condensate with an external magnetic trap*

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The dynamics of a dark soliton has been investigated in a Bose–Einstein condensate with an external magnetic trap, and the effects of localized impurity on the dynamics are discussed by the variational approach based on the renormalized integrals of motion. The reciprocal movement of the dark soliton is discussed by performing a standard linear analysis, and it is found that the effects of the localized impurity depend strictly on the positive or negative value of the impurity strength corresponding to the repulsive or attractive impurity. The numerical results confirm the theoretical analysis, and show that the effects also depend on the effective nonlinear coefficient and the harmonic frequency.

Keywords: Bose–Einstein condensate, dark soliton, the variational approach, impurity
PACC: 0530J, 0290, 1110L

1. Introduction

The current experimental and theoretical studies of Bose–Einstein condensates (BECs) have attracted a great deal of interest, and the behaviour of the BECs crucially depends on the sign of the atomic interactions. The mean-field description for the macroscopic BEC wavefunction is constructed using the Hartree–Fock approximation, and results in the Gross–Pitaevskii equation (GPE), and it is well known that the GPE supports solitonic solutions.[1−5] Dark solitons (with positive scattering length) of Bose-condensed atoms were experimentally observed a few years ago, while bright solitons (with negative scattering length) have been detected only very recently using an optical red-detuned laser beam along the axial direction of the sample to impose a transverse (radial) confinement, and studied in one-dimension models. A dark soliton in a BEC is a macroscopic excitation with a corresponding positive scattering length, and is characterized by a local density minimum and a phase gradient of the wave function at the position of the minimum. The stability of the dark solitons in a nonlinear system is a critical property, and the dynamics of the dark solitons in the presence of the external magnetic trap has been extensively studied, including thermal and dynamical instabilities.[6−8]

The matter wave dynamics can be effectively manipulated by external fields. The harmonic trap induces a dynamical instability of the soliton, culminating in sound emission. Controlling the generation of dark soliton trains from periodic waves can be achieved by the variation of the scattering length. Recent experiments have demonstrated that tuning of the s-wave scattering length can be achieved due to the Feshbach resonance. It offers the possibility to vary the interaction strength in ultracold atomic gases simply by applying an external magnetic field. The strength of the interaction can be adjusted experimentally from large negative values to large positive values. As a result, the experiments with magnetically trapped ultra cold atomic gases, where the s-wave scattering length fully determines the interaction effects, have an unprecedented high level of control over the interatomic interactions.[9−11] A relevant interesting issue is to learn how to control the motion of BEC solitons, and the question then arises how one could affect or even guide their motion.

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Impurities could be used as elements of a matter-wave device that controls the motion of a network of nonlinear excitations, the interaction of solitons trapping in a confining potential with a localized inhomogeneity or an impurity and their dynamics has been studied theoretically. For example, the effect of a potential step or impurity on the BEC mean field is investigated, and the full set of stationary states are presented. The dynamics of BECs in the presence of impurities have recently become a subject of growing interest, and have been studied in the framework of almost all nonlinear evolution equations possessing soliton solutions. The dynamics of solitons in the presence of various trap potentials, such as the external magnetic trap, have been extensively studied, including dynamical instabilities.[12−14] But the dynamics of the dark soliton for a BEC in trap potential with a localized impurity and effects of the impurity have not been studied in detail yet. In this paper, the effects of the localized impurity on the dynamics of the dark soliton in a BEC are investigated by means of the variational approach based on the renormalized integrals of motion and numerical simulation, and some interesting results are obtained.

2. Theoretical analysis

In this work, a dark soliton is considered in a BEC with repulsive interaction, the particle number of the BEC is not very large in the cigar-shaped trap, and the descriptive equation is the one-dimensional Schrödinger equation[12]

$$\mathrm{j}\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial z^2}-\frac{2a_\mathrm{s}}{a_\mathrm{B}}|u|^2u-V(z)u=0, \tag{1}$$

where u is the condensate wavefunction, a_s is the asymptotic value of the scattering length far from the Feshbach resonance, a_B is the Bohr radius, and

$$a_\mathrm{s}=a\left[1+\frac{\sigma}{B_0-B}\right], \tag{2}$$

where a_s is the scattering length, B is the externally applied magnetic field, σ is the width of resonance and B_0 is the Feshbach resonant value of the magnetic field.

The external magnetic trap has the normalized potential $V(z)$ in the longitudinal direction (z direction). In order to study the dynamics of a BEC dark soliton in the external magnetic trap with a localized impurity in the framework of Eq.(1), it is convenient to decompose the external potential $V(z)$ as

$$V(z)=U(z)+b\delta(z), \tag{3}$$

where $U(z)$ is the conventional time-independent trapping potential, which is assumed to be smooth and slowly varying on the soliton scale, and the additional sharp potential $b\delta(z)$ accounts for a localized impurity which is described by a Dirac function $\delta(z)$. The value of parameter b in Eq.(3), which measures the impurity strength, is assumed to be small and may take either positive or negative value corresponding to the repulsive or attractive impurity, respectively.

The conventional time-independent trap potential is assumed to be the external magnetic trap (MT) potential, and its form is

$$U(z)=\frac{1}{2}\Omega^2z^2, \tag{4}$$

where Ω is the harmonic frequency (the trap coefficient) in the longitudinal direction (z direction).

A perturbation theory for the dark soliton can be used on the renormalized integrals of motion.[15,16] The solution to Eq.(1) is given by

$$u(z,t)=\phi(z)\mathrm{e}^{\mathrm{j}\alpha t}v(z,t), \tag{5}$$

where $\phi(z)$ is the stationary background wavefunction, $\alpha=2a_\mathrm{s}/a_\mathrm{B}$ is the effective nonlinear coefficient , and the solution of the dark soliton is

$$v(z,t)=\mathrm{j}\cos\theta+\sin\theta\tanh[\sin\theta(z-\Delta)], \tag{6}$$

where $\Delta(t)$ is the centre position of the dark soliton, $\theta(t)$ is the distributing angle, $\cos\theta$ is the soliton speed related to the speed of sound, and $\sin\theta$ is the depth of the dark soliton.

Substituting Eqs.(3) and (5) into Eq.(1) leads to the following equation

$$\begin{aligned}&\mathrm{j}\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial^2 v}{\partial z^2}-\alpha\phi^2(|v|^2-1)v\\&+\frac{\partial\ln\phi}{\partial z}\frac{\partial v}{\partial z}-b\delta(z)v=0.\end{aligned} \tag{7}$$

The stationary background wavefunction $\phi(z)$ can be approximated in the Thomas–Fermi (TF) framework as[17]

$$\begin{aligned}\phi(z)&=\sqrt{1-\frac{U(z)}{\alpha}}=\sqrt{1-\frac{1}{2\alpha}\Omega^2z^2}\\&=\sqrt{1-\frac{a_\mathrm{B}}{4a_\mathrm{s}}\Omega^2z^2}\end{aligned} \tag{8}$$

in the region where $U(z)/\alpha < 1$ and $\phi(z) = 0$ outside.

The atom density can be given in the TF framework as

$$n_{\rm d}(\alpha, \Omega) = \int_{-\infty}^{\infty} |\phi(z)|^2 \, {\rm d}z. \tag{9}$$

The BEC dark soliton is considered in the TF regime described by Eqs.(7), (8) and (9). As we know, the semi-classical methods of TF type are a powerful tool for the study of nuclear properties that smoothly vary with the number of particles. We can give a prescription in a TF framework for a given atom density $n_{\rm d}(\alpha, \Omega)$, and there are several values of harmonic frequency Ω and the effective nonlinear coefficient α. For example, the number density of atoms in the BEC can be changed for the different harmonic frequency and the effective nonlinear coefficient by applying the external magnetic field via Feshbach resonance in a BEC.

The logarithmic derivative of the background function $\phi(z)$ can be expressed as

$$\frac{\partial \ln \phi}{\partial z} = \frac{1}{\sqrt{1-U/\alpha}} \frac{\partial \sqrt{1-U/\alpha}}{\partial z} = -\frac{[1 + U/\alpha + 2(U/\alpha)^2 + 3(U/\alpha)^3 + \ldots)}{2\alpha} \frac{\partial U}{\partial z}. \tag{10}$$

The logarithmic derivative is approximated as $\dfrac{\partial \ln \phi}{\partial z} \approx -\dfrac{1}{2\alpha}\dfrac{\partial U}{\partial z}$ due to the harmonic frequency $\Omega \ll 1$ (such as $\Omega = 0.2$) in this paper, and the renormalized equation can be expressed in the following form

$$\mathrm{j}\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial z^2} - \alpha\phi^2(|v|^2 - 1)v - \frac{1}{2\alpha}\frac{\partial U}{\partial z}\frac{\partial v}{\partial z} - b\delta(z)v = 0. \tag{11}$$

The averaged Lagrangian of Eq.(11) can be obtained with the variational approach based on the renormalized integrals of motion

$$\begin{aligned} L(t) = \int_{-\infty}^{\infty} & \left[\frac{\mathrm{j}}{2}\left(v^*\frac{\partial v}{\partial t} - v\frac{\partial v^*}{\partial t}\right)\left(1 - \frac{1}{|v|^2}\right) - \frac{1}{2}\left|\frac{\partial v}{\partial z}\right|^2 - \frac{\alpha\phi^2}{2}(|v|^2 - 1)^2 \right. \\ & \left. - \frac{1}{2\alpha}\frac{\partial U}{\partial z}\left(v^*\frac{\partial v}{\partial z} + v\frac{\partial v^*}{\partial z}\right) - b\delta(z)\,|v|^2\right]{\rm d}z \\ = & \frac{{\rm d}\Delta}{{\rm d}t}[\sin(2\theta) - 2\theta] - \frac{2(1+\alpha)\sin^3\theta}{3} + \frac{A\Omega^2\sin\theta}{4} - \frac{\Omega^2\Delta^2\sin^3\theta}{2} - \frac{B\Omega^2\sin\theta}{\alpha} \\ & - b[\cos^2\theta + \sin^2\theta\tanh^2(-\Delta\sin\theta)], \end{aligned} \tag{12}$$

where $A = \int_{-\infty}^{\infty} x^2 \operatorname{sech}^4 x {\rm d}x$, and $B = \int_{-\infty}^{\infty} x \tanh x \operatorname{sech}^2 x {\rm d}x$.

The motion equations for the distributing angle and the centre position are obtained from the averaged Lagrangian by using ${\rm d}L(t)/{\rm d}\sigma - {\rm d}[{\rm d}L(t)/{\rm d}({\rm d}\sigma/{\rm d}t)]/{\rm d}t = 0$ $(\sigma = \theta, \Delta)$

$$\begin{aligned} \frac{{\rm d}\theta}{{\rm d}t} &= \frac{\Delta\Omega^2\sin\theta}{4} - b\sin\theta\tanh(-\Delta\sin\theta)sech^2(-\Delta\sin\theta) = f_1(\theta, \Delta), \\ \frac{{\rm d}\Delta}{{\rm d}t} &= \frac{(1+\alpha)}{2}\cos\theta - \frac{A\Omega^2\cos\theta}{8\sin^2\theta} + \frac{3\Omega^2\Delta^2\cos\theta}{8} + \frac{B\Omega^2\cos\theta}{2\alpha\sin^2\theta} \\ & \quad - b\,\mathrm{ctg}\,\theta[1 - \tanh^2(-\Delta\sin\theta)] + \frac{b\sin(2\theta)}{4}\tanh(-\Delta\sin\theta)\operatorname{sech}^2(-\Delta\sin\theta) = f_2(\theta, \Delta). \end{aligned} \tag{13}$$

According to the equation $H = [\partial\theta/\partial t * \partial L/\partial(\dot{\theta}) + \partial\Delta/\partial t * \partial L/\partial(\dot{\Delta})] - L$, the Hamiltonian of the BEC system is obtained

$$H = \frac{2(1+\alpha)\sin^3\theta}{3} - \frac{A\Omega^2\sin\theta}{4} + \frac{\Omega^2\Delta^2\sin^3\theta}{2} + \frac{B\Omega^2\sin\theta}{\alpha} + b[\cos^2\theta + \sin\theta^2\tanh^2(-\Delta\sin\theta)]. \tag{14}$$

Equations (13) and (14) determine the dynamics of the dark soliton in a BEC with an external magnetic trap and a localized impurity.

3. Effects of the localized impurity on the dynamics of the dark soliton

From the motion equations (13), we can see that the motion equation for the center position is $\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \cos\theta$ for the effective nonlinear coefficient of $\alpha = 1.0$ in the absence of the external magnetic trap and the localized impurity. Namely, the velocity of the center position is the soliton speed relative to the speed of sound. The special case coincides with the optical dark soliton solution in the framework of Eq.(1) without perturbations (the external magnetic trap and the localized impurity).

The stationary states can be obtained by setting the time derivatives in Eqs.(13) to zero,

$$\theta_{\mathrm{s}} = 0 \quad \text{and} \quad \Delta_{\mathrm{s}} \text{ of arbitrary value}, \tag{15a}$$

and

$$\theta_s = \pi/2 \text{ and } \Delta_s = 0. \tag{15b}$$

The stationary state (15a) corresponds to $v(z,t) = 1$, and this situation denotes the submergence of the dark soliton. The stationary state (15b) corresponds to $v(z,t) = \tanh(z)$, namely the dark soliton is the black soliton. The special kink case (both the speed and the centre position of the dark soliton are zero) coincides with the optical vector dark soliton solution. The stationary state (15b) is stable from a standard linear stability analysis based on the Eqs.(13) and (14).[10,13]

From the motion equations (13), we can obtain

$$\begin{aligned}\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}f_2(\theta,\Delta)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= \frac{\mathrm{d}f_2(\theta,\Delta)}{\mathrm{d}\theta} f_1(\theta,\Delta). \end{aligned} \tag{16}$$

Linearizing Eq.(15) in θ and Δ around the stable stationary state of $\theta_{\mathrm{s}} = \pi/2$ and $\Delta_{\mathrm{s}} = 0$, Eq.(16) reduces approximately to the very simple form

$$\begin{aligned}&\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + \left[\frac{(1+\alpha)}{2} - b - \frac{A\Omega^2}{8} + \frac{B\Omega^2}{2\alpha}\right]\left(\frac{1}{4}\Omega^2 + b\right)\Delta \\ =&\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2} + \omega^2\Delta = 0, \end{aligned} \tag{17}$$

where

$$\omega = \left\{\left[\frac{(1+\alpha)}{2} - b - \frac{A\Omega^2}{8} + \frac{B\Omega^2}{2\alpha}\right]\left(\frac{1}{4}\Omega^2 + b\right)\right\}^{1/2}$$

is the effective frequency of the reciprocal movement for the centre position of the dark soliton . The period of time of the reciprocal movement is

$$T_{\mathrm{p}} = \frac{2\pi}{\omega} = \frac{2\pi}{\left\{\left[\frac{(1+\alpha)}{2} - b - \frac{A\Omega^2}{8} + \frac{B\Omega^2}{2\alpha}\right]\left(\frac{1}{4}\Omega^2 + b\right)\right]\right\}^{1/2}}. \tag{18}$$

The above result means that the dark soliton moves reciprocally around the centre of the potential as time changes, and the period of time depends strictly on the harmonic frequency, the effective nonlinear coefficient and the impurity strength. From Eq.(18), we can see that the period of time of the reciprocal movement goes smaller when the harmonic frequency, or the effective nonlinear coefficient becomes larger, however, it gets larger when the impurity strength becomes larger. The attractive impurity may lead to the distortion and submergence of the dark soliton in the evolution, and the reciprocal movement of the dark soliton may disappear around the centre ($z = 0$) of the magnetic trap as $b < -\frac{1}{4}\Omega^2$.

4. Numerical simulation

We can perform a series of direct numerical simulations of the Schrödinger equation (1) to discuss the reciprocal movement of the dark soliton, and the effects of the localized impurity on the different effective nonlinear coefficient, the harmonic frequency, and the impurity strengths. The initial input soliton is $u(z, t = 0) = \sqrt{1 - \Omega^2 z^2/2} \times \{\, \mathrm{j}\cos\theta + \sin\theta \tanh[\sin\theta(z - \Delta)]\, \}$, where the initial distributing angle is $\theta = \pi/2$ and the initial centre position is $\Delta = 2$. The numerical calculations of the Dirac $\delta(z)$ function for the localized impurity have been replaced by the steep function $30 \times \mathrm{sech}(z/0.02)$ for the repulsive or attractive impurities in the simulations below.[12]

In Fig.1 we show the intensity evolutions of the dark solitons with the same harmonic frequency of $\Omega = 0.20$ and the effective nonlinear coefficient of $\alpha = 1.0$ for the impurity strengths of $b = 0.00$, 0.05, and 0.10. We can see that the dark soliton moves periodically around the centre ($z = 0$) of the magnetic potential as time changes, and the evolution depends strictly on the impurity strength. For example, the period of time becomes slightly longer when the impurity strength gets larger. We have systematically compared the evolutions of the dark solitons, and the numerical results show that there are obvious self-trapping effects of the dark soliton around the stationary state Eq.(15b), and the black soliton is stable when it has zero velocity and its centre position is at the centre ($z = 0$) of the magnetic potential if the impurity strength is small enough. These are in agreement with previous theoretical analysis.

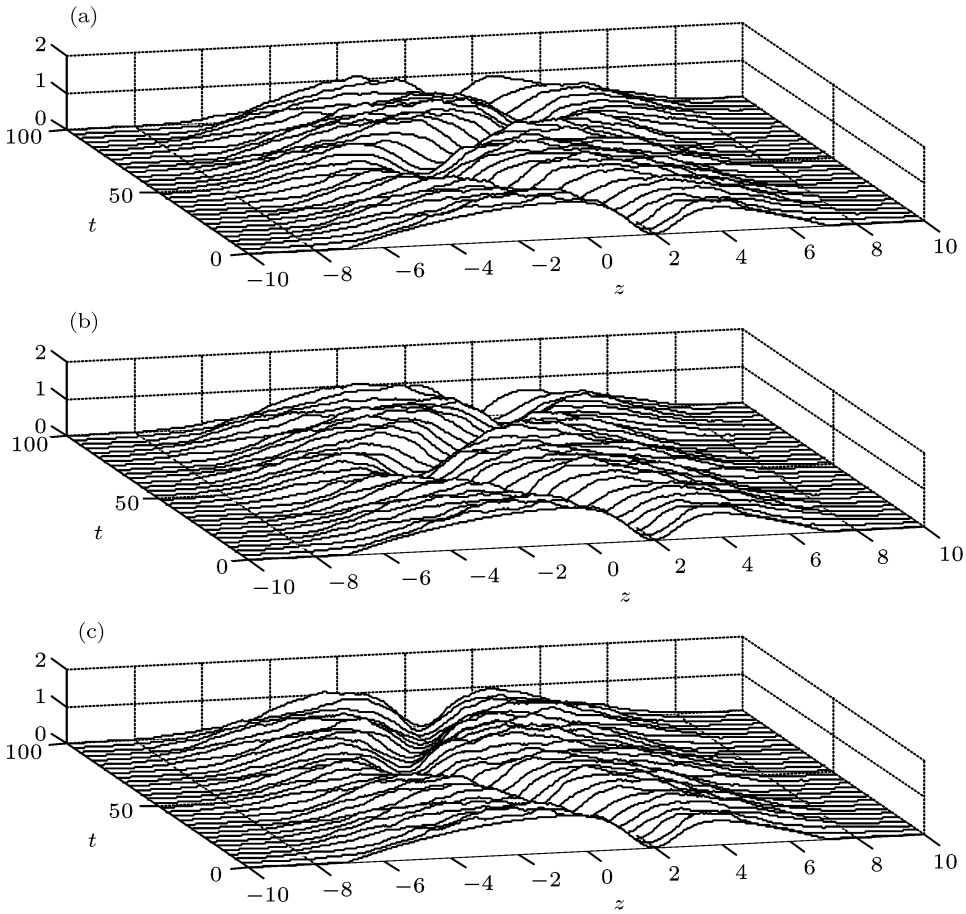


Fig.1. The intensity evolution of the dark soliton with the same harmonic frequency and the effective nonlinear coefficient for different impurity strengths. (a) $b = 0.00$; (b) $b = 0.05$; (c) $b = 0.10$.

In Fig.2 we refer to the same harmonic frequency of $\Omega = 0.20$, and the effective nonlinear coefficient of $\alpha = 1.0$ for the impurity strengths of $b = 0.00$, –0.05, or –0.10. We can see that the effects of the impurity with the negative parameter value b (corresponding to the attractive impurity) are different from those of the impurity with positive b (corresponding to the repulsive impurity), which means that the effects of the localized impurity depend strictly on the repulsive or attractive impurity. For example, the attractive impurity may lead to the distortion and disintegration of the dark soliton, and the reciprocal movement of the dark soliton may disappear around the centre ($z = 0$) of the magnetic potential when the negative parameter value b (corresponding to the attractive impurity) is small enough.

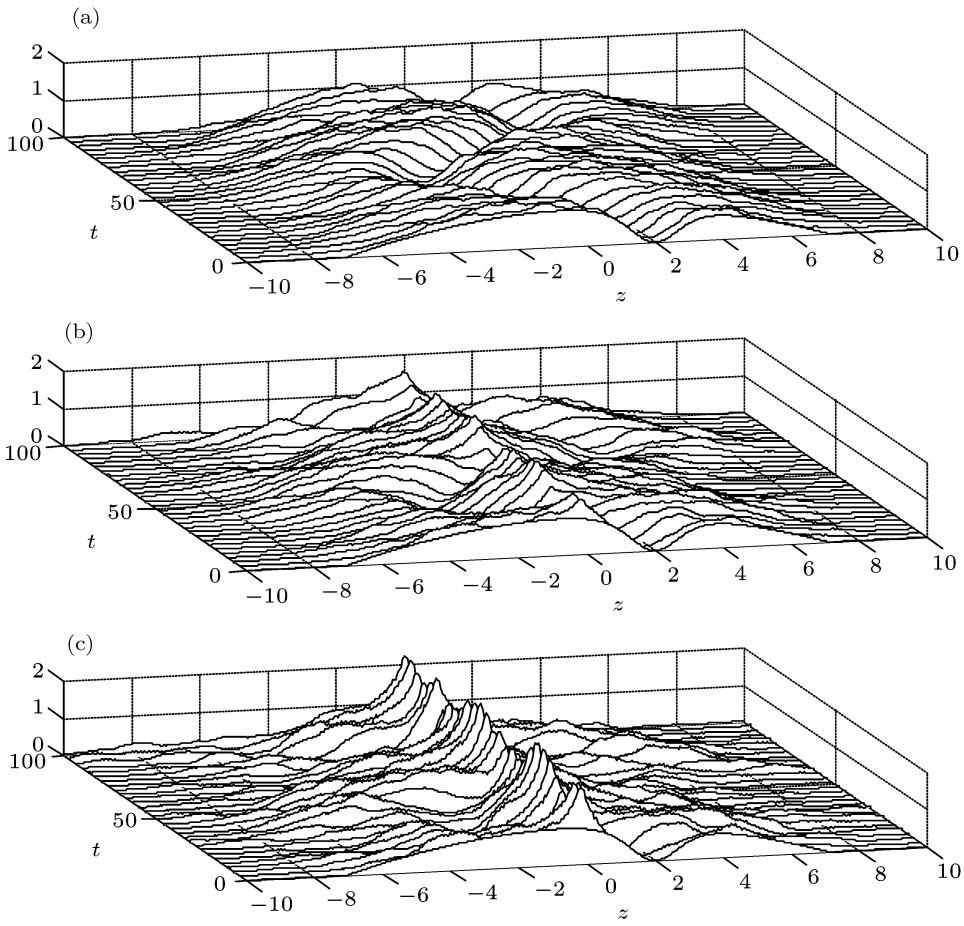
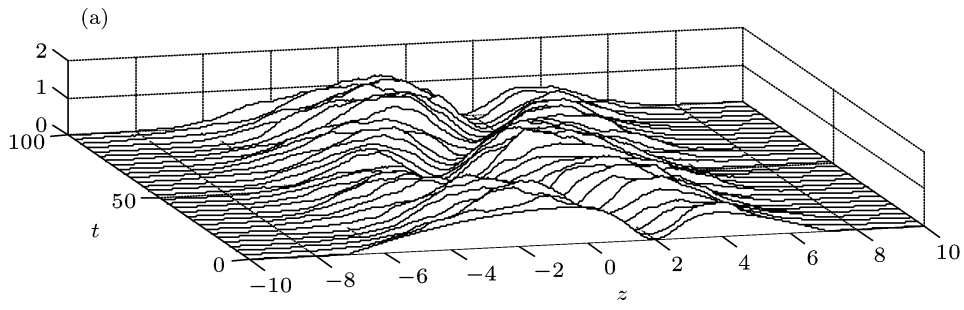


Fig.2. The intensity evolution of the dark soliton with the same harmonic frequency and the effective nonlinear coefficient for the different impurity strengths. (a) $b = 0.00$; (b) –0.05; (c) –0.10.

We then show the intensity evolution of the dark solitons with the same harmonic frequency of $\Omega = 0.20$, and the impurity strength of $b = 0.05$ for the different effective nonlinear coefficients of $\alpha = 0.7$, 1.0, or 1.4 in Fig.3. The periodical movement depends strictly on the effective nonlinear coefficient. For example, the period of time gets smaller when the effective nonlinear coefficient becomes larger. In Fig.4 we refer to the same effective nonlinear coefficient of $\alpha = 1.0$, and the impurity strength of $b = 0.05$ for the harmonic frequencies of $\Omega = 0.17$, 0.20, or 0.25. We know that the dark soliton can be controlled by the number of atoms continuously injected into BEC from the reservoir on applying the external magnetic field via Feshbach resonance in a BEC.[17] The period of time depends strictly on the harmonic frequency, and the number density of atoms in the BEC can be changed for the different harmonic frequency by Eq.(9). For example, the period of time gets smaller when the harmonic frequency becomes larger, and the numerical results confirm the theoretical analysis. The results mean that the dark soliton may maintain tanh-shape if the atoms are continuously injected into the BEC.



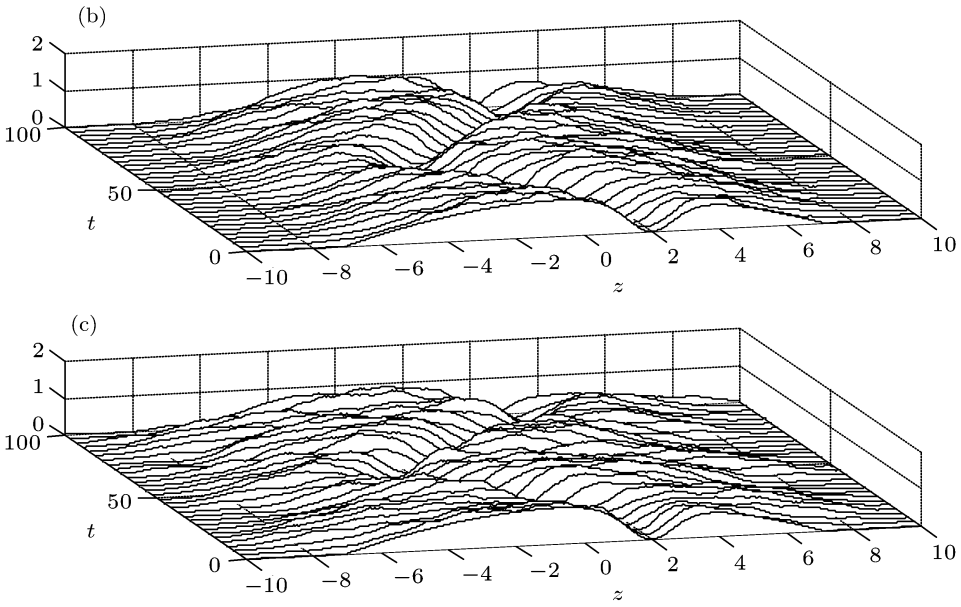


Fig.3. The intensity evolution of the dark soliton with the same harmonic frequency and the impurity strength for the different effective nonlinear coefficients. (a) $\alpha = 0.7$; (b) 1.0; (c) 1.4.

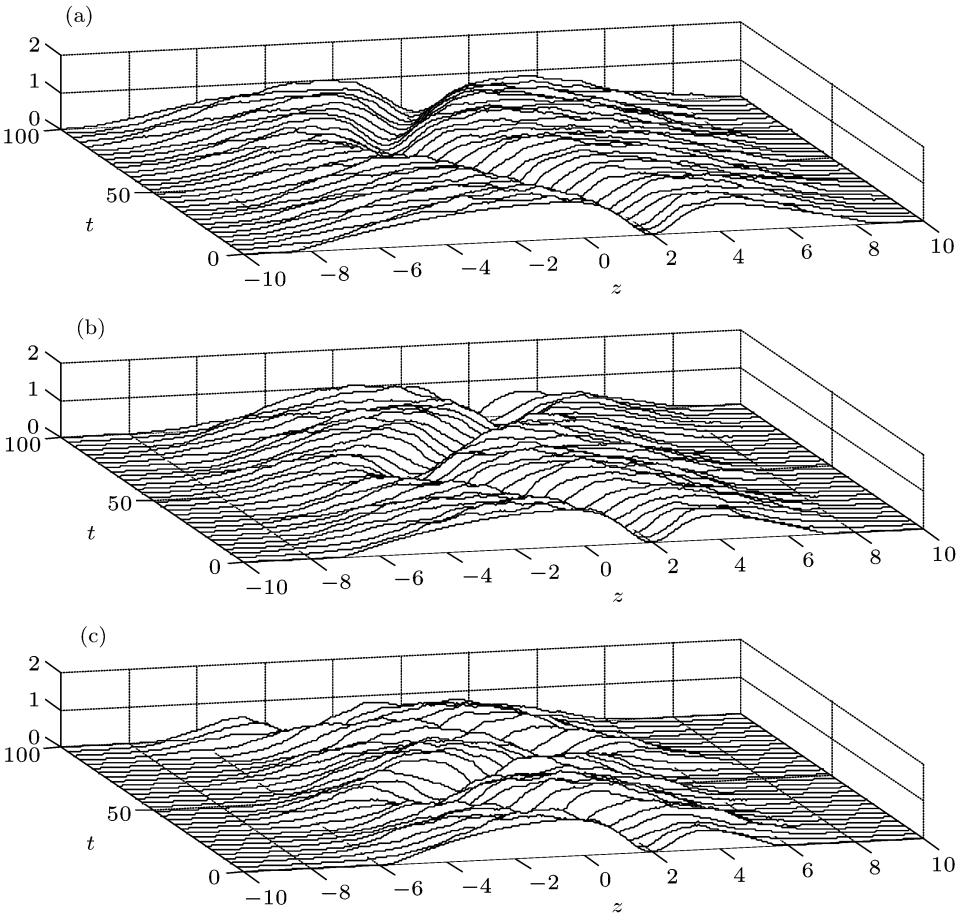


Fig.4. The intensity evolution of the dark soliton with the same effective nonlinear coefficient and the impurity strength for the different harmonic frequencies. (a) $\Omega = 0.17$; (b) 0.20; (c) 0.25.

5. Conclusion

Dynamics of the dark soliton in a BEC with an external magnetic trap has been investigated by the variational approach based on the renormalized integrals of motion, and the stationary states as physical solutions to the descriptive equation are obtained. The reciprocal movement of the dark soliton was discussed by performing a standard linear analysis, and it is found that the effects of the localized impurity depend strictly on the impurity strength. The dynamics was investigated by direct numerical simulations, and it is found that there are obvious self-trapping effects on the dark soliton around the stable stationary state, and the black soliton is stable when it has zero velocity and its centre position is at the centre of the magnetic potential. The dark soliton moves reciprocally around the centre of the potential as time changes, and the evolution depends strictly on the positive or negative value of the impurity strength, which corresponds to the repulsive or attractive impurity. The attractive impurity may lead to the distortion and disintegration of the dark soliton, and the reciprocal movement of the dark soliton may disappear. Also the theoretical and numerical results show that the effects also depend on the effective nonlinear coefficient and the harmonic frequency.

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