Vol 18 No 7, July 2009
1674-1056/2009/18(07)/2654-05

Chinese Physics B

© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

Thermodynamic stability of interacting fermions system with matrix eigenvalue method*

Qin Fang(覃 昉) and Chen Ji-Sheng(陈继胜)†

Physics Department and Institute of Particle Physics, Central China Normal University, Wuhan 430079, *China*

(Received 26 March 2008; revised manuscript received 9 September 2008)

A matrix eigenvalue method is applied to analyse the thermodynamic stability of two-component interacting fermions. The non-relativistic and ultra-relativistic $d = 1, 2, 3$ dimensions have been discussed in detail, respectively. The corresponding stability region has been given according to the two-body interaction strength and the particle number density ratio.

Keywords: thermodynamic stability, fermions mixing, matrix eigenvalue method
PACC: 0530F, 6410, 0570C

1. Introduction

With the realization of artificial quantum degeneracy, ultra-cold atomic physics provides a playground for studying the interacting quantum many-body physics. It is generally expected that the explored physics is closely associated with a variety of many-body topics.[1−5] The interacting fermion ground state is still a challenging topic in the fundamental Fermi–Dirac statistical physics.[6−9]

Through tuning the external-magnetic field, one can control the scattering length $a_{\rm s}$ or interaction strength U_0 between two different fermions. Adjusting the interaction strength between the atoms is crucial for the realization of the fermions superfluidity and the relevant physics has attracted great attention in contemporary physics. The realized temperature can be extremely low, ultra-cold, while the particle number density can be dense. So we can treat the fermions as strong degeneracy in order to study the quantum collective effects. Compared with the ideal non-interacting case, the interaction between fermions can cause essential changes of the thermodynamic properties, such as chemical potential and internal energy.[10]

From the viewpoint of many-body physics, or more basically from the viewpoint of thermodynamics, the basic problem is how to study the system stability according to the interaction strength and particle number density, and so on.[1]

According to the general many-body theory, the system stability is determined by the sign of the second partial derivative of the energy density based on corresponding chemical potential. If the second partial derivative is positive, the thermodynamic system can be stable; otherwise, the thermodynamic system is unstable or metastable. The problem of how to consider the effects of the interaction between the particles on the system stability needs to be clarified, especially for the two-component fermions mixture (with spin up and spin down). The purpose of this paper is to simplify the problems including interaction.

Generally, the total energy density ϵ is controlled by the inter-particle interaction strength. Therefore, one can calculate the second partial derivatives of the system total energy density with respect to particle number density (n_1, n_2), $\partial^2\epsilon/\partial n_1^2, \partial^2\epsilon/\partial n_1\partial n_2, \partial^2\epsilon/\partial n_2\partial n_1, \partial^2\epsilon/\partial n_2^2$, which give the respective susceptibility of the chemical potential versus the particle number density.[1]

Mathematically, we can construct a hermitian 2×2 matrix by taking the susceptibility as the corresponding matrix element. The remaining task is solving the secular equation to calculate the eigenvalues and the corresponding eigenvectors; i.e., this matrix can be transformed into a diagonal one. The relations among the eigenvalue λ and the number density (n_1, n_2) as well as the interaction strength U_0 can be easily obtained.

As discussed in Refs.[11] and [12], the eigenvalue

*Project supported by the Scientific Starting Research Fund of Central China Normal University of China, the National Natural Science Foundation of China (Grant Nos 10675052 and 10875050) and Ministry of Education of China (Grant No IRT0624).
†E-mail: chenjs@mail.ccnu.edu.cn

method of the susceptibility matrix can contribute to the understanding of the stability problem more concisely. As we will see below, the analytical formulae for the stability criteria can be given explicitly and the explored thermodynamic properties of phase separation can be also indicated diagrammatically.

2. Matrix method and system stability criterion

Now we discuss the thermodynamic stability of the zero-temperature interacting fermion mixing system in the non-relativistic and the extreme relativistic situations for the $d = 1$, 2, 3 dimensional cases separately.

2.1. Three-dimensional cases

Firstly, we derive the system stability condition in detail under the three-dimensional non-relativistic situation. The discussions for other cases are in line with this method and the corresponding details will be omitted for brevity.

2.1.1. Non-relativistic situation

In ultra-cold atomic physics, extensive experimental studies of the Fermi–Fermi mixture of ^{40}K–^{40}K and ^{6}Li–^{6}Li in different hyperfine states have been performed.[13,14] For this kind of non-relativistic system, the internal energy density is[1]

$$\epsilon = \frac{3}{5}n_1\mu_1 + \frac{3}{5}n_2\mu_2 + n_1n_2U_0. \tag{1}$$

The chemical potential is $\mu_i = (6\pi^2)^{2/3}\hbar^2 n_i^{2/3}/(2m_i)$, with m_i being the fermion mass and the number density being denoted by n_i. In the above formula, $m_{1(2)}$ represent the mass of the fermion with $n_{1(2)}$ being the corresponding density. And U_0 denotes the interaction strength between the two different fermions.

In order to discuss the system stability conveniently, we construct the matrix as below

$$\hat{\chi} = \begin{pmatrix} \dfrac{\partial^2\epsilon}{\partial n_1^2} & \dfrac{\partial^2\epsilon}{\partial n_1\partial n_2} \\ \dfrac{\partial^2\epsilon}{\partial n_2\partial n_1} & \dfrac{\partial^2\epsilon}{\partial n_2^2} \end{pmatrix}. \tag{2}$$

From the standard thermodynamic relation, the chemical potential is given by the internal energy density and particle number density according to $\mu = \partial\epsilon/\partial n$. Furthermore, one can have $\partial^2\epsilon/\partial n^2 = \partial\mu/\partial n$. The eigenvalues of $\hat{\chi}$ can be used as the system stability criterion in addition to the conventional discussions.[1]

The eigenvalues λ of $\hat{\chi}$ can be derived readily

$$\begin{aligned}\lambda_1 = & \frac{(6\pi^2)^{2/3}\hbar^2}{6}\left(\frac{n_1^{-1/3}}{m_1} + \frac{n_2^{-1/3}}{m_2}\right) \\ & -\frac{1}{2}\sqrt{\frac{(6\pi^2)^{4/3}\hbar^4}{9}\left(\frac{n_1^{-1/3}}{m_1} - \frac{n_2^{-1/3}}{m_2}\right)^2 + 4U_0^2}\end{aligned} \tag{3}$$

and

$$\lambda_2 = \lambda_1\left(-\frac{1}{2} \to \frac{1}{2}\right) \tag{4}$$

If the entire system is in the stable state, the essential condition must be $\det\hat{\chi} > 0$, i.e., $\lambda_1\lambda_2 > 0$. As Eqs.(3) and (4) indicate, the second value λ_2 is definitely positive $\lambda_2 > 0$ and trivial, while λ_1 is the valid one. Consequently, the criterion is $\lambda_1 > 0$, and the stability condition is given by

$$U_0^2 < \frac{(6\pi^2)^{4/3}\hbar^4}{9m_1m_2}n_1^{-1/3}n_2^{-1/3}. \tag{5}$$

From the inequality (5), we can see that the system stability depends both on the interaction strength and the particle number densities. The mixture will be unstable for large enough fermion number densities. This conclusion is consistent with the recent discussion.[15]

Furthermore, when the smaller eigenvalue λ_2 tends to zero (from the positive side), the related eigenvector χ characterizes the instability more efficiently, as generally analyzed in Refs.[11, 12]. One can plot the eigenvector χ in the plane (U_0, t), then the instability of the Fermi system is uniquely characterized by the angle θ between χ and t-axis. This kind of discussion contributes to the understanding of the multi-component mixing or the thermodynamic properties of phase separation.

2.1.2. Extreme relativistic situation

Analogous to the non-relativistic discussion,[1,15] the relativistic energy density of the interacting system in the weak contact interaction occasion can be generalized as

$$\epsilon = \frac{3}{4}n_1\mu_1 + \frac{3}{4}n_2\mu_2 + n_1n_2U_0, \tag{6}$$

while the chemical potential is $\mu_i = (6\pi^2)^{1/3}\hbar c n_i^{1/3}$. The corresponding eigenvalues are

$$\lambda = \frac{1}{6}(6\pi^2)^{1/3}\hbar c(n_1^{-2/3} + n_2^{-2/3}) \pm \frac{1}{2}\sqrt{\frac{1}{9}(6\pi^2)^{2/3}\hbar^2c^2(n_1^{-2/3} - n_2^{-2/3})^2 + 4U_0^2}. \quad (7)$$

The stability condition is

$$U_0^2 < \frac{1}{9}(6\pi^2)^{2/3}\hbar^2c^2n_1^{-2/3}n_2^{-2/3}, \quad (8)$$

which is similar to inequality (5).

2.2. Two-dimensional cases

2.2.1. Non-relativistic situation

The internal energy density is

$$\epsilon = \frac{1}{2}n_1\mu_1 + \frac{1}{2}n_2\mu_2 + n_1n_2U_0, \quad (9)$$

and the chemical potential is $\mu_i = 2\pi\hbar^2 n_i/m_i$.

The result obtained is

$$\lambda = \pi\hbar^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm \sqrt{\pi^2\hbar^4\left(\frac{1}{m_1} - \frac{1}{m_2}\right)^2 + U_0^2}, \quad (10)$$

where λ is also a constant.

The system stability condition is only determined by the interaction strength between the two distinct fermions and does not rely on the particle number density

$$U_0^2 < \frac{4\pi^2\hbar^4}{m_1m_2}. \quad (11)$$

The inequality indicates that the interaction strength for the stability region does not depend on the particle number density directly. Of course, in the non-relativistic case the stability condition, inequality (11), is closely related to the respective intrinsic atomic mass. It is very interesting that the atomic mass difference can affect the thermodynamic stability for the given inter-particle interaction strength.

2.2.2. Extreme relativistic situation

With the internal energy density

$$\epsilon = \frac{2}{3}n_1\mu_1 + \frac{2}{3}n_2\mu_2 + n_1n_2U_0, \quad (12)$$

and chemical potential $\mu_i = 2\sqrt{\pi}\hbar c n_i^{1/2}$, the result obtained is

$$\lambda = \frac{\sqrt{\pi}}{2}\hbar c(n_1^{-1/2} + n_2^{-1/2}) \pm \frac{1}{2}\sqrt{\pi\hbar^2c^2(n_1^{-1/2} - n_2^{-1/2})^2 + 4U_0^2}. \quad (13)$$

The corresponding stability condition is

$$U_0^2 < \pi\hbar^2c^2n_1^{-1/2}n_2^{-1/2}. \quad (14)$$

Compared with inequality (11) for the non-relativistic situation, the stability region given by inequality (14) depends not only on the interaction strength but also the particle number densities. And inequality (14) is similar to the inequalities (5) and (8). The stability criteria for the ultra-relativistic two-dimensional, non-relativistic and extreme relativistic three-dimensional cases have the similar analytical properties.

2.3. One-dimensional situation

2.3.1. Non-relativistic situation

Consider the mixture of two fermions, the internal energy density of which is

$$\epsilon = \frac{1}{3}n_1\mu_1 + \frac{1}{3}n_2\mu_2 + n_1n_2U_0 \quad (15)$$

with the chemical potential $\mu_i = \pi^2\hbar^2n_i^2/(8m_i)$.

The eigenvalues of the mixing matrix are

$$\lambda = \frac{\pi^2\hbar^2}{8}\left(\frac{n_1}{m_1} + \frac{n_2}{m_2}\right) \pm \frac{1}{2}\sqrt{\frac{\pi^4\hbar^4}{16}\left(\frac{n_1}{m_1} - \frac{n_2}{m_2}\right)^2 + 4U_0^2}. \quad (16)$$

The stability condition is given by

$$U_0^2 < \frac{\pi^4\hbar^4n_1n_2}{16m_1m_2}. \quad (17)$$

As shown in the inequality (17), the smaller the density, the less stable the mixed system.

2.3.2. Extreme relativistic situation

The internal energy density is

$$\epsilon = \frac{1}{2}n_1\mu_1 + \frac{1}{2}n_2\mu_2 + n_1n_2U_0. \quad (18)$$

The chemical potential is $\mu_i = \pi\hbar c n_i$ with c being the light velocity.

The corresponding eigenvalues for the ultra-relativistic occasion are

$$\lambda = \pi\hbar c \pm |U_0|. \quad (19)$$

We can see that λ is a constant which has nothing to do with the particle number density. This means that the system stability is only determined by the interaction strength between the fermions.

The stability condition is $\lambda_2 = \pi\hbar c - |U_0| > 0$, i.e.

$$U_0^2 < \pi^2\hbar^2c^2. \tag{20}$$

The inequality (20) is similar to (11). From inequality (20), we can find that the upper bound of the interaction strength for the stability region is a constant.

3. Discussion

If interaction exists between two fermions, i.e. $U_0 \neq 0$, then we have the following conclusions:

1) For the one-dimensional non-relativistic occasion the eigenvalue λ is a function of n_1, n_2, m_1, m_2, and the interaction strength. Therefore, the thermodynamic stable condition of the system is determined by interaction strength, fermion number densities and the fermion mass ratio. This characteristic can be shown by Fig.1. An arbitrary point in the ellipse indicates a stable state.

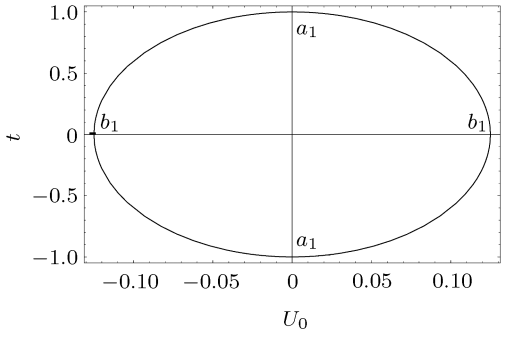


Fig.1. The stability phase diagram indicated by the particle number ratio $t = (n_1 - n_2)/(n_1 + n_2)$ with $n_1 + n_2 = n = 1$ versus the interaction strength U_0. The two points at which the function curve intersects t-axis are respectively $(-a_1, 0)$ and $(a_1, 0)$ $(a_1 > 0)$, and the other two points at which the function curve intersects U_0-axis are respectively $(-b_1, 0)$ and $(b_1, 0)$ $(b_1 > 0)$. For the one-dimensional non-relativistic situation, $a_1 = 1$; $b_1 = \dfrac{\pi^2\hbar^2}{8}/\sqrt{m_1 m_2}$.

2) The extreme relativistic one-dimensional and non-relativistic two-dimensional situations possess similar analytical properties.

3) The stability criteria of the remaining situations have similar analytical properties. In the two-dimensional extreme relativistic, the three-dimensional non-relativistic and the three-dimensional extreme relativistic situations, the eigenvalue is a function of n_1, n_2 and the interaction strength.

Figure 2 shows that the curves (solid lines) are symmetric about t-axis and U_0-axis. They have $t = 1$ and $t = -1$ as asymptotes (the dashed lines). The regions which are encircled by the curves and the two asymptotes (labeled by the 3 regions) are the stable regions, and an arbitrary point in these regions is a stable state. The Fermi–Fermi system is easily to approach stable when the interaction strength is weaker and the ratio of the two distinct fermi number densities becomes smaller. In essence, Fig.2 is equivalent to the Figs.2, 4 and 8 of Ref.[11].

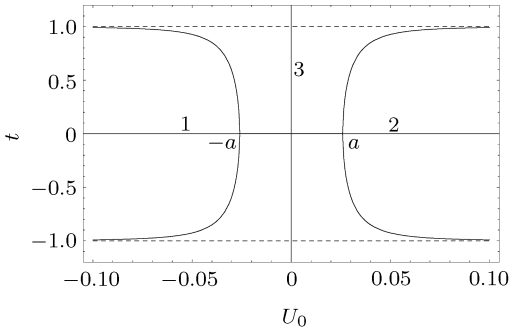


Fig.2. The stability region indicated by $t = (n_1 - n_2)/(n_1 + n_2)$ versus U_0 with scaled $n_1 + n_2 = n = 1$. The two points at which the function curve intersects U_0-axis are respectively $(-a, 0)$ and $(a, 0)$ $(a > 0)$. For the two-dimensional extreme relativistic situation, $a = \sqrt{2\pi}\hbar c$; for the three-dimensional non-relativistic situation, $a = 2\left(\dfrac{\pi^4}{3}\right)^{1/3}\hbar^2/\sqrt{m_1 m_2}$; for the three-dimensional extreme relativistic situation, $a = 2\left(\dfrac{\pi}{3}\right)^{2/3}\hbar c$.

4. Conclusion

In summary, we have analyzed the eigenvalues of the matrix composed of the susceptibility elements obtained by the partial derivative of the chemical potential with respect to the particle number density. Through these eigenvalues, the thermodynamic stability condition of the fermion-mixing system has been discussed. This method can simplify the stability discussion and express the stability regime diagrammatically, which contributes to the addressing of the phase separation analysis in the interacting fermion physics.

References

[1] Pethick C J and Smith H 2002 *Bose-Einstein Condensation in Dilute Gases* (Cambridge, England: Cambridge University Press)

[2] Cohen T D 2005 *Phys. Rev. Lett.* **95** 120403

[3] Chevy F 2006 *Phys. Rev. Lett.* **96** 130401

[4] Zwierlein M W, Abo-Shaeer J R, Schirotzek A, Schunck C H and Ketterle W 2005 *Nature* **435** 1047

[5] Partridge G B, Li W, Kamar R I, Liao Y A and Hulet R G 2006 *Science* **311** 503

[6] Chen J S 2007 *Chin. Phys. Lett.* **24** 1825
Chen J S 2007 *Commun. Theor. Phys.* **48** 99

[7] Chen J S, Cheng C M, Li J R and Wang Y P 2007 *Phys. Rev.* A **76** 033617

[8] Chen J S, Li J R, Wang Y P and Xia X J 2008 *J. Stat. Mech.* P12008

[9] Wang Y P and Chen J S 2008 *Chin. Phys.* B **17** 4401

[10] Adhikari S K 2006 *New J. Phys.* **8** 258

[11] Chen X S and Forstmann F 1992 *J. Chem. Phys.* **97** 3696

[12] Chen X S and Forstmann F 1992 *Mol. Phys.* **76** 1203

[13] DeMarco B and Jin D S 1999 *Science* **285** 1703

[14] O'Hara K M, Hemmer S L, Gehm M E, Granade S R and Thomas J E 2002 *Science* **298** 2179

[15] Adhikari S K 2007 *Phys. Rev.* A **76** 053609