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Relativistic thermodynamic properties of a weakly interacting Fermi gas in a weak magnetic field

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This paper derives the analytical expression of free energy for a weakly interacting Fermi gas in a weak magnetic field, by using the methods of quantum statistics as well as considering the relativistic effect. Based on the derived expression, the thermodynamic properties of the system at both high and low temperatures are given and the relativistic effect on the properties of the system is discussed. It shows that, in comparison with a nonrelativistic situation, the relativistic effect changes the influence of temperature on the thermodynamic properties of the system at high temperatures, and changes the influence of particle-number density on them at extremely low temperature. But the relativistic effect does not change the influence of the magnetic field and inter-particle interactions on the thermodynamic properties of the system at both high and extremely low temperatures.

Keywords: Fermi gas, relativistic effect, thermodynamic property
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1. Introduction

It is well known that many systems in the universe consist of fermions which have very small rest mass. For example, the rest mass of a neutrino is one of the million less than that of an electron. For a system which consists of these small particles, the rest energies of particles are comparable with their kinetic energies, thus, the relativistic effect on the thermodynamic properties of the system needs to be considered even if under the condition of low temperatures (low energies). On the other hand, although the relativistic effect is weak for general particles, this effect should also be theoretically considered in many cases since the methods and measurements of the related experiments are extremely precise. For example, the relativistic effect is involved in the fine structures of hydrogen atoms, and for the hyperfine structured system of ^{6}Li atoms, where a ^{6}Li atom has six hyperfine states, considering the relativistic effect is quite important, just as considering the weakly inter-particle interactions (s-wave scattering, $a \approx -2160a_{\rm B}$).[1] Comprehensive consideration of the relativistic effect is necessary to understand both the fine structures of thermodynamic properties of the ^{6}Li atom system and the motion state rules of ^{6}Li. Until now, research on the properties of trapped relativistic Fermi gas has been little, in comparison with that on the properties of relativistic Bose gas.[2−17] Recently, characteristic research of the Bose system in the magnetic field has begun to gradually increase. For example, Ref.[18] is a study of the magnetic quantum phase transition of cold atoms in an optical lattice, Ref.[19] is a study of the nonlinear dynamics of a dipolar Bose–Einstein condensate in an optical lattice, and Ref.[20] is a study of the magnetic solitons of spinor Bose–Einstein condensates in an optical lattice. However, we believe that by taking the external magnetic field, relativistic effect and inter-particle interactions into total consideration, research on Fermi systems will undoubtedly have more significance in theory and practice.

In the present paper, by using the methods of quantum statistics, thermodynamic properties of a weakly interacting Fermi gas in a weak magnetic filed are investigated when the relativistic effect is considered, analytical expressions of thermodynamic quantities are given, and the influences of the relativistic effect on the thermodynamic properties of the system are analysed.

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2. Relativistic free energy of the system

We consider a weakly interacting Fermi gas of spin-1/2 fermions, with volume V and particle number N, confined in an even weak magnetic field $B = B_Z$. It has been derived that, if we only consider s-wave as well as spin, the energy eigenvalue of the system can be written as[21]

$$E = \sum_p (n_p^+ + n_p^-)\varepsilon_p + \frac{\alpha}{V} N^+ N^- - (N^+ - N^-)\mu B, \tag{1}$$

where μ is the magnetic moment $\mu_{\rm B}$ of fermions, $\alpha = 4\pi a\hbar^2/m$ is the parameter of interactions, a is the s-wave scattering length, $n_p^+(n_p^-)$ is the number of particles in the state of momentum p with spin-up (spin-down), $N^+(N^-)$ is the total number of particles with spin-up (spin-down), and ε_p is the relativistic energy of a single particle,

$$\varepsilon_p = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} - mc^2, \tag{2}$$

where we only consider the kinetic energy of particles. The partition function of the system is

$$\begin{aligned} Q &= \sum_{\{n_p^+, n_p^-\}} \exp(-\beta E) \\ &= \sum_{N^+} \exp\left[\beta\mu B(2N^+ - N) - \beta\frac{\alpha}{V} N^+(N - N^+)\right] \\ &\quad \times \sum_{N^+} \exp\left(-\beta \sum_p n_p^+ \varepsilon_p\right) \\ &\quad \times \sum_{N^-} \exp\left(-\beta \sum_p n_p^- \varepsilon_p\right), \end{aligned} \tag{3}$$

where $\beta = 1/(kT)$. Based on the method which deals with nonideal Bose gas, proposed by Huang, Yang *et al*,[22,23] we introduce $A_0(\xi)$ and regard it as the free energy of the relativistic system of ξ spinless and non-interacting fermions without external potential in volume V.

$$A_0(\xi) = -\frac{1}{\beta} \ln \sum_{\{\xi_p\}} \exp\left(-\beta \sum_p \xi_p \varepsilon_p\right), \tag{4}$$

then the free energy of the system can be expressed as

$$\begin{aligned} F &= -\frac{1}{\beta} \ln Q \\ &= N\mu B - \frac{1}{\beta} \ln \sum_{N^+} \exp\Big\{ -\beta\Big[A_0(N^+) \\ &\quad + A_0(N - N^+) + \frac{\alpha}{V} N^+(N - N^+) \\ &\quad - 2\mu B N^+\Big]\Big\}. \end{aligned} \tag{5}$$

The logarithm of the sum in Eq.(5) can be expressed by the logarithm of the largest term,

$$\begin{aligned} F &\approx F(\bar{N}^+) \\ &= \mu B(N - 2\bar{N}^+) + A_0(\bar{N}^+) + A_0(N - \bar{N}^+) \\ &\quad + \frac{\alpha}{V} \bar{N}^+(N - \bar{N}^+), \end{aligned} \tag{6}$$

where $\bar{N}^+$ can be decided by $\left.\dfrac{\partial F(N^+)}{\partial N^+}\right|_{N^+ = \bar{N}^+} = 0$, then we obtain

$$u_0(\bar{N}^+) - u_0(N - \bar{N}^+) = 2\mu B - \frac{\alpha}{V}(N - 2\bar{N}^+), \tag{7}$$

where u_0 is the corresponding chemical potential of A_0. Introducing the parameter r and letting $\bar{N}^+ = N(1 + r)/2$, we can express Eq.(7) as

$$u_0\left(\frac{1 + r}{2} N\right) - u_0\left(\frac{1 - r}{2} N\right) = 2\mu B + \alpha n r. \tag{8}$$

Since r is very small, the left side of Eq.(8) can be expanded at $r = 0$ by the Taylor series, and only the terms up to the first power in r are kept. From Eq.(8), we can obtain

$$r = \frac{2\mu B}{\left.\dfrac{\partial u_0(xN)}{\partial x}\right|_{x=1/2} - \alpha n} \approx r_0\left[1 + \frac{\alpha n}{D}\right], \tag{9}$$

where

$$r_0 = \left.\frac{2\mu B}{\partial u_0(xN)/\partial x}\right|_{x=1/2}, \qquad D = \left.\partial u_0(xN)/\partial x\right|_{x=1/2}.$$

Then the free energy of the system can be expressed as

$$\begin{aligned} F &= -N\mu B r + A_0\left(\frac{1 + r}{2} N\right) \\ &\quad + A_0\left(\frac{1 - r}{2} N\right) + \frac{\alpha}{4} n N. \end{aligned} \tag{10}$$

Similarly, we expand A_0 at $r = 0$ by the Taylor series, and keep only the terms up to the first power in r, thus we can get

$$\begin{aligned} F &\approx 2A_0\left(\frac{N}{2}\right) + \frac{\alpha}{4} n N - N\mu B r_0 \\ &\quad - N\mu B \alpha n \frac{r_0}{D}, \end{aligned} \tag{11}$$

where $A_0(N/2)$ is the free energy of the $N/2$ fermion system, in which the relativistic effect is taken into account while inter-particle interactions, external potential and spin are not considered.

3. Relativistic thermodynamic properties of the system

From the canonical ensemble theory, the chemical potential, total energy and heat capacity of the system, respectively, are

$$\begin{aligned} u &= \left(\frac{\partial F}{\partial N}\right)_{V,T}, \\ U &= \frac{\partial}{\partial \beta}[\beta F]_{V,N}, \\ C &= (\frac{\partial U}{\partial T})_{V,N}. \end{aligned} \tag{12}$$

Then, the corresponding relativistic thermodynamic quantities above of the system can be expressed, respectively, as follows:

$$u = u_0 + \frac{1}{2}\alpha n - \mu B \frac{\partial}{\partial N}\left[N r_0 + \frac{N\alpha n r_0}{D}\right], \tag{13}$$

$$\begin{aligned} U = U_0 &+ \frac{1}{4}\alpha n N - N\mu B r_0\left(1 + \frac{\alpha n}{D}\right) \\ &- N\beta\mu B \frac{\partial}{\partial \beta}\left[r_0 + \alpha n \frac{r_0}{D}\right], \end{aligned} \tag{14}$$

$$C = C_0 + N\mu B T \frac{\partial^2}{\partial T^2}\left[r_0 + \alpha n\left(\frac{r_0}{D}\right)\right], \tag{15}$$

where u_0, U_0, C_0 are, respectively, the chemical potential, total energy and heat capacity of the system, in which the relativistic effect is taken into account while the inter-particle interactions and external potential are not considered. Equations (13), (14) and (15) indicate that the thermodynamic properties are not only influenced by the relativistic effect, but also by the magnetic field and inter-particle interactions.

(1) The case of $T \ggg T_{\mathrm{F}}$

When the temperature is much higher than the nonrelativistic Fermi temperature of the noninteracting system without external potential, according to Ref.[24], we can obtain the expressions as follows:

$$\begin{aligned} u_0 &= u_0(T, V, N/2) \\ &= -kT \ln\left\{\frac{8\pi}{n}\left(\frac{mc}{h}\right)^3 \frac{K_2(\phi)}{\phi}\right\} - mc^2, \end{aligned} \tag{16}$$

$$U_0 = U_0(T, V, N) = Nmc^2\left[\frac{K_1(\phi)}{K_2(\phi)} + \frac{3}{\phi} - 1\right], \tag{17}$$

$$\begin{aligned} C_0 &= C_0(T, V, N) \\ &= \frac{Nmc^2}{T}\left\{\phi + \frac{3}{\phi} - \frac{K_1(\phi)}{K_2(\phi)}\left[3 + \phi\frac{K_1(\phi)}{K_2(\phi)}\right]\right\}, \end{aligned} \tag{18}$$

where $K_n(\phi)$ is the correctional Bessel function, $\phi = \beta mc^2$, k is the Boltzmann constant, and h is the Planck constant. Based on Eqs.(9), (13), (14) and (15), we can calculate, respectively, the relativistic chemical potential, total energy and heat capacity of the system,

$$u = u_0 - \frac{(\mu B)^2}{kT} + \frac{1}{2}\alpha n\left[1 - 2(\frac{\mu B}{kT})^2\right], \tag{19}$$

$$U = U_0 - 2N\frac{(\mu B)^2}{kT} + \frac{1}{4}N\alpha n\left[1 - 6\left(\frac{\mu B}{kT}\right)^2\right], \tag{20}$$

$$C = C_0 + 2Nk\left(\frac{\mu B}{kT}\right)^2 + 3Nk\frac{\alpha n}{kT}\left(\frac{\mu B}{kT}\right)^2. \tag{21}$$

When $\phi \ll 1$, that is, under the ultrarelativistic condition, we have

$$\begin{aligned} u_0 &\approx kT \ln\left[\frac{n}{16\pi}\left(\frac{hc}{kT}\right)^3\right], \\ U_0 &\approx 3NkT, \\ C_0 &= 3Nk. \end{aligned} \tag{22}$$

When $\phi \gg 1$, we obtain nonrelativistic results as follows:

$$\begin{aligned} u_0 &\approx kT \ln\left(\frac{n\lambda^3}{2}\right), \\ U_0 &\approx \frac{3}{2}NkT, \\ C_0 &= \frac{3}{2}Nk. \end{aligned} \tag{23}$$

Where $\lambda = [h^2/(2\pi mkT)]^{1/2}$ is the thermal wavelength of the system.

It should be pointed out that the nonrelativistic Fermi temperature T_{F} is very low when the particle-number density is very low. Thus, even if $T \gg T_{\mathrm{F}}$, the energy of particles may not be very large, and the weakly interacting conditions, $|a|/\lambda \ll 1$ as well as $|a|\, n^{1/3} \ll 1$, may also be satisfied.

Through the comparison between Eqs.(19), (20), (21) and Ref.[25], we find that when $T \gg T_{\mathrm{F}}$, the thermodynamic properties of the system will have corresponding changes due to the consideration of the relativistic effect. In particular, the characteristic that the thermodynamic properties change with temperature has corresponding changes, but the effects of the magnetic field as well as inter-particle interactions on

the thermodynamic properties of the system do not change. Namely, no matter whether the relativistic effect is considered or not, the effects of the magnetic field and inter-particle interactions on the thermodynamic properties of the system have no differences. Moreover, when $mc^2/(kT) \gg 1$, the thermodynamic properties of the system, which can be represented in Eqs.(19), (20) and (21), regress to the conclusions of the nonrelativistic situation again.

(2) The case of $T \to 0$

When temperature $T \to 0$, according to Ref.[24], we get the following expressions:

$$u_0 = \varepsilon_{\rm F} = mc^2\sqrt{1+\left(\frac{3h^3N}{8\pi V}\right)^{2/3}\frac{1}{(mc)^2}} - mc^2, \quad (24)$$

$$\begin{aligned} U_0 = {} & \frac{2\pi m^4c^5}{6h^3}V\{8Y_{\rm F}^3[\sqrt{1+Y_{\rm F}^2}-1] \\ & -\sqrt{1+Y_{\rm F}^2}(2Y_{\rm F}^3-3Y_{\rm F}) \\ & -3\ln[Y_{\rm F}+\sqrt{1+Y_{\rm F}^2}]\}, \end{aligned} \quad (25)$$

where $Y_{\rm F} = P_{\rm F}/(mc) = [3n/(8\pi)]^{1/3}h/(mc)$. In the case of the ultrarelativistic situation (this can be realized by a very large particle-number density), we have $Y_{\rm F} \gg 1$. Thus, Eqs.(24), (25) can be expressed as follows:

$$u_0 = \varepsilon_{\rm F}^0 \approx cP_{\rm F} = hc\left(\frac{3}{8\pi}n\right)^{1/3}, \quad (26)$$

$$U_0 \approx \frac{2\pi c}{h^3}VP_{\rm F}^4 = \frac{3}{8}Nhc\left(\frac{3}{8\pi}n\right)^{1/3}. \quad (27)$$

Therefore, under the ultrarelativistic condition, the chemical potential (Fermi energy), total energy (ground state energy) and heat capacity of the system can be expressed, respectively, as follows:

$$u = u_0 - 2\frac{(\mu B)^2}{\varepsilon_{\rm F}^0} + \frac{1}{2}\alpha n\left[1-12\left(\frac{\mu B}{\varepsilon_{\rm F}^0}\right)^2\right], \quad (28)$$

$$U = U_0 - 3N\frac{(\mu B)^2}{\varepsilon_{\rm F}^0} + \frac{1}{4}N\alpha n\left[1-18\left(\frac{\mu B}{\varepsilon_{\rm F}^0}\right)^2\right], \quad (29)$$

$$C = 0. \quad (30)$$

When $Y_{\rm F} \ll 1$, we have

$$\begin{aligned} u_0 = \varepsilon_{\rm F} & \approx mc^2\left[1+\frac{1}{2}h^2(\frac{3}{8\pi}n)^{2/3}\frac{1}{(mc)^2}\right] - mc^2 \\ & = \frac{h^2}{2m}\left(\frac{3}{8\pi}n\right)^{2/3}, \end{aligned} \quad (31)$$

$$U_0 \approx \frac{4\pi V}{5mh^3}P_{\rm F}^5 = \frac{3h^2}{10m}N\left(\frac{3}{8\pi}n\right)^{2/3}. \quad (32)$$

Via similar calculations, we can obtain the chemical potential, total energy and heat capacity of the system, respectively,

$$u = u_0 - \frac{(\mu B)^2}{2\varepsilon_{\rm F}} + \frac{1}{2}\alpha n\left[1-\frac{3}{2}\left(\frac{\mu B}{\varepsilon_{\rm F}}\right)^2\right], \quad (33)$$

$$U = U_0 - \frac{3}{2}N\frac{(\mu B)^2}{\varepsilon_{\rm F}} + \frac{1}{4}N\alpha n\left[1-\frac{9}{2}\left(\frac{\mu B}{\varepsilon_{\rm F}}\right)^2\right], \quad (34)$$

$$C = 0. \quad (35)$$

Compared with Ref.[25], it is found that these are just nonrelativistic resluts.

Thus, under the condition of extremely low temperatures ($T \to 0$) and the ultrarelativistic case ($P_{\rm F}/(mc) \gg 1$), the thermodynamic properties of the system will also change when the relativistic effect is considered, compared with those of the nonrelativistic situation. In particular, the influences of particle-number density on the thermodynamic properties of the system change. However, the property of the influences of both the magnetic field and inter-particle interactions on the thermodynamic quantities does not change, and only the means of influence of the particle-number density on the system changes. In the case of $P_{\rm F}/(mc) \ll 1$, the results completely regress to the theoretical results of the nonrelativistic situation.

4. Conclusions

In the present paper, by using the method of quantum statistics, we have studied relativistic thermodynamic properties of a weakly interacting Fermi gas in a weak magnetic field and have gained the relativistic analytical expressions of chemical potential, total energy and heat capacity. In addition, we have analysed the relativistic effect on the thermodynamic properties of the system. It shows that, compared with the nonrelativistic situation, the relativistic effect changes the influence of temperature on the thermodynamic properties of the system at high temperatures, and changes the influence of particle-number density on the thermodynamic properties of the system at extremely low temperatures ($T \to 0$). But the relativistic effect does not change the influences of the magnetic field and inter-particle interactions on the thermodynamic properties of the system at both high and extremely low temperatures. Under a certain condition, the expressions of this paper can regress to the conclusions of the nonrelativistic situation, and these show that the results of this paper have general meanings.

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