Vol 18 No 7, July 2009
1674-1056/2009/18(07)/2634-08

Chinese Physics B

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Noether conserved quantities and Lie point symmetries of difference Lagrange–Maxwell equations and lattices*

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(Received 26 September 2008; revised manuscript received 4 December 2008)

This paper presents a method to find Noether-type conserved quantities and Lie point symmetries for discrete mechanico-electrical dynamical systems, which leave invariant the set of solutions of the corresponding difference scheme. This approach makes it possible to devise techniques for solving the Lagrange–Maxwell equations in differences which correspond to mechanico-electrical systems, by adapting existing differential equations. In particular, it obtains a new systematic method to determine both the one-parameter Lie groups and the discrete Noether conserved quantities of Lie point symmetries for mechanico-electrical systems. As an application, it obtains the Lie point symmetries and the conserved quantities for the difference equation of a model that represents a capacitor microphone.

Keywords: Lagrange–Maxwell equation, Lie point symmetry, discrete mechanico-electrical system, conserved quantity

PACC: 0320, 0300, 0420M

1. Introduction

Symmetries have always played a very important role in the study of differential equations.[1] Sophus Lie presented his symmetry theory as a tool to solve ordinary differential equations.[2] These techniques have been used also to classify the equations and to establish the properties of their corresponding space of solutions.[3−7] A few years ago, Lie symmetry and Noether symmetries theory was extended successfully to discrete equations, such as finite difference equations and differential-difference equations.[8−20]

A mechanico-electrical dynamical system couples a mechanical process with an electromagnetic one. Because of their applications the study of these systems is interesting, but the corresponding equations are generally difficult to solve due to the presence of nonlinear terms. In recent works on mechanico-electrical dynamical systems, authors have studied Lie symmetries and conserved quantities,[21−23] non-Noether symmetries and Lutzky conserved quantities,[24] momentum-dependent symmetries and non-Noether conserved quantities,[25] and obtained the algebraic structure and Poisson's theory for these systems.[26] Besides this theoretical frame, it is interesting to build tools which can provide the numerical solutions and we have introduced difference techniques to obtain the variational principles and first-integrals for discrete mechanico-electrical dynamical systems.[27]

The purpose of this paper is, on the one hand,

*Project supported by the National Natural Science Foundation of China (Grants Nos 10672143 and 60575055) and State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences. Tang Yi-Fa acknowledges the support under Sabbatical Program (SAB2006-0070) of the Spanish Ministry of Education and Science. Jiménez S and Vázquez L acknowledge support of the Spanish Ministry of Education and Science (Grant No MTM2005-05573).
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to extend the technique of Lie symmetries to discrete mechanico-electrical dynamical systems and, on the other hand, to obtain the corresponding discrete Noether-type conserved quantities. We will consider Lagrange–Maxwell equations in differences for mechanico-electrical dynamical systems on a uniform lattice. The lattice itself is determined by a difference equation. First, we will look for Lie point symmetries of the difference Lagrange–Maxwell equations and lattice equation while leaving the common solution sets of these equations invariants. Second, we will state the Noether theorem for the discrete mechanico-electrical systems.

The structure of the paper is as follows. In Section 2, we obtain the formalism necessary to compute the Lie point symmetries for the discrete Lagrange–Maxwell equations and the lattice equation. In Section 3, we obtain the determining equations introducing a vector field of the infinitesimal transformations with respect to time, generalized coordinates and electric quantities, and we state the Noether theorem by using difference operators on the discrete Lagrange–Maxwell equations. In Section 4, we consider an actual example which consists in a capacitor microphone. The conclusions are drawn in Section 5.

2. Lagrange–Maxwell equations

We will consider a mechanico-electrical dynamical system consisting in the mechanical part of n particles, described by n generalized coordinates q^l $(l = 1, \dots, n)$, and, on the electrodynamical part, of m electric circuits with linear conductors and capacitors. For circuit j $(j = 1, \dots, m)$, we denote by i^j the current, by u^j the electric potential, by e^j (such that $\dot{e}^j = i^j$) the charge in the capacitor, by R_i the resistance, and by C_j the capacitance.

The Lagrange function for such a mechanico-electrical system is

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q}) + W_m(\boldsymbol{q}, \dot{\boldsymbol{e}}) - W_e(\boldsymbol{q}, \boldsymbol{e}), \quad (1)$$

where T and V are, respectively, the kinetic and the potential energy. The electric and the magnetic field energies are defined, respectively, by

$$W_e = \frac{1}{2}\frac{\left(e^j\right)^2}{C_j},$$
$$W_m = \frac{1}{2}L_{j\beta}i^j i^\beta, \ (j, \beta = 1, \dots, m), \quad (2)$$

where $L_{j\beta}$ $(j \neq \beta)$ is the mutual inductance of circuit j due to circuit β and L_{jj} is self inductance of circuit j.

The motion of the system is governed by the Lagrange–Maxwell system of equations:[22,23]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^l} - \frac{\partial L}{\partial q^l} + \frac{\partial F}{\partial \dot{q}^l} = Q^l, \quad (l = 1, \dots, n), \quad (3)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{e}^j} - \frac{\partial L}{\partial e^j} + \frac{\partial F}{\partial \dot{e}^j} = u^j, \quad (j = 1, \dots, m), \quad (4)$$

where Q^l $(l = 1, \dots, n)$ are generalized, non-potential forces. This is a system of $n + m$ ordinary differential equations with respect to the n generalized coordinates q^l $(l = 1, \dots, n)$ and the m generalized electric quantities $e^j (j{=}1, \dots, m)$. The dissipative function F of the system is given by

$$F = F_e(\dot{\boldsymbol{e}}) + F_m(\boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (5)$$

where the electric dissipative function is

$$F_e = \frac{1}{2}R_j\left(i^j\right)^2 = \frac{1}{2}R_j\left(\dot{e}^j\right)^2, \quad (6)$$

and F_m is the function for the viscous frictional damping forces, and thus, $-\partial F/\partial \dot{q}^l$ $(l = 1, \dots, n)$ correspond to dissipative forces.

The system of Lagrange–Maxwell equations (3) and (4) for the mechanico-electrical systems can be written in compact form as

$$\ddot{q}^l = \alpha_l(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{e}, \dot{\boldsymbol{e}}, t) \ (l = 1, \dots, n), \quad (7)$$

$$\ddot{e}^j = \beta_j(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{e}, \dot{\boldsymbol{e}}, t) \quad (j = 1, \dots, m). \quad (8)$$

Equations (7) and (8) can be discretized to give

$$E_1\left(t, t_-, t_+, t_{++}, q^l, q^l_-, q^l_+, q^l_{++}, e^j, e^j_-, e^j_+, e^j_{++}\right) = 0, \quad (9)$$

$$E_2\left(t, t_-, t_+, t_{++}, q^l, q^l_-, q^l_+, q^l_{++}, e^j, e^j_-, e^j_+, e^j_{++}\right) = 0, \quad (10)$$

$$\Omega\left(t, t_-, t_+, t_{++}, q^l, q^l_-, q^l_+, q^l_{++}, e^j, e^j_-, e^j_+, e^j_{++}\right) = 0, \quad (11)$$

if we consider a four-point scheme on an equally-spaced temporal mesh with time-step h. We denote by $t = t_k$ the current time, and define $t_- = t_{k-1}$, $t_+ = t_{k+1}$ and $t_{++} = t_{k+2}$. Similarly for q^l and e^j. Whenever E_1, E_2 and Ω are independent of t_{++}, q^l_{++} and $\boldsymbol{e}^j_{++}$, this scheme reduces to a three-point scheme.

3. Noether conserved quantities and Lie point symmetries

3.1. Lie point symmetries of difference Lagrange–Maxwell equations

Let us consider that a one-parameter group of local point transformations with respect to the generalized coordinates, the electric charges and the time,

has the following form:

$$\begin{aligned}\tilde{t} &= \boldsymbol{\Gamma}_{\varepsilon}\left(t, \boldsymbol{q}, \boldsymbol{e}\right),\\ \tilde{q} &= \tilde{\boldsymbol{\Xi}}_{\varepsilon}\left(t, \boldsymbol{q}, \boldsymbol{e}\right),\\ \tilde{e} &= \boldsymbol{\Phi}_{\varepsilon}\left(t, \boldsymbol{q}, \boldsymbol{e}\right),\end{aligned} \tag{12}$$

where $\boldsymbol{q} = \left\{q^1, q^2, \ldots, q^n\right\}$, $\boldsymbol{e} = \left\{e^1, e^2, \ldots, e^m\right\}$. We shall require that they leave the system of Eqs.(9), (10) and (11) invariant on the solution set of this system. Since we are interested in continuous transformations for the discrete Lagrange–Maxwell dynamical system, we use an infinitesimal approach and write the transformations up to order ε as

$$\tilde{t} = t + \varepsilon\xi_0\left(t, q^l, e^j\right), \tag{13}$$

$$\tilde{q}^l = q^l + \varepsilon\xi_l\left(t, q^l, e^j\right), \tag{14}$$

$$\tilde{e}^j = e^j + \varepsilon\eta_j\left(t, q^l, e^j\right), \tag{15}$$

where ε is a small parameter, and ξ_0, ξ_l and η_j are the corresponding infinitesimal generators.

It is convenient to express the invariance condition for the Lagrange–Maxwell dynamical system given by Eqs.(9), (10) and (11) in terms of a formalism involving the transformation operator and their prolongations. The transformation operator itself has the form

$$\begin{aligned}\tilde{X} = {}& \xi_0\left(t, q^l, e^j\right)\frac{\partial}{\partial t} + \xi_l\left(t, q^l, e^j\right)\frac{\partial}{\partial q^l}\\ &+ \eta_j\left(t, q^l, e^j\right)\frac{\partial}{\partial e^j},\end{aligned} \tag{16}$$

where ξ_0, ξ_l and η_j are the same as those in Eqs.(13), (14) and (15). The prolongation of the transformation operator (16) acting on the Eqs.(9), (10) and (11), for a three-point scheme is

$$\mathrm{pr}\tilde{\boldsymbol{X}} = \tilde{X} + \tilde{X}_- + \tilde{X}_+, \tag{17}$$

and

$$\begin{aligned}\tilde{X}_- = {}& \xi_0\left(t_-, q^l_-, e^j_-\right)\frac{\partial}{\partial t_-} + \xi_l\left(t_-, q^l_-, e^j_-\right)\frac{\partial}{\partial q^l_-}\\ &+ \eta_j\left(t_-, q^l_-, e^j_-\right)\frac{\partial}{\partial e^j_-},\end{aligned} \tag{18}$$

$$\begin{aligned}\tilde{X}_+ = {}& \xi_0\left(t_+, q^l_+, e^j_+\right)\frac{\partial}{\partial t_+} + \xi_l\left(t_+, q^l_+, e^j_+\right)\frac{\partial}{\partial q^l_+}\\ &+ \eta_j\left(t_+, q^l_+, e^j_+\right)\frac{\partial}{\partial e^j_+}.\end{aligned} \tag{19}$$

The requirement that Eqs.(9), (10) and (11) are invariant under the considered one-parameter group translates into the requirement

$$\begin{aligned}&\mathrm{pr}\tilde{X}E_1|_{E_1=0,E_2=0,\Omega=0} = 0,\\ &\mathrm{pr}\tilde{X}E_2|_{E_1=0,E_2=0,\Omega=0} = 0,\\ &\mathrm{pr}\tilde{X}\Omega|_{E_1=0,E_2=0,\Omega=0} = 0.\end{aligned} \tag{20}$$

In Eq.(20), once Eqs.(9), (10) and (11) are taken into account, all involved variables are considered to be independent. Eqs.(20) are thus the determining equation for the infinitesimal generators ξ_0, ξ_l and η_j.

In the case of a four-point scheme, the transformation operator is again Eq.(16) and its prolongation is

$$\mathrm{pr}\tilde{X} = \tilde{X} + \tilde{X}_- + \tilde{X}_+ + \tilde{X}_{++} \tag{21}$$

and

$$\begin{aligned}\tilde{X}_{++} = {}& \xi_0\left(t_{++}, q^l_{++}, e^j_{++}\right)\frac{\partial}{\partial t_{++}}\\ &+ \xi_l\left(t_{++}, q^l_{++}, e^j_{++}\right)\frac{\partial}{\partial q^l_{++}}\\ &+ \eta_j\left(t_{++}, q^l_{++}, e^j_{++}\right)\frac{\partial}{\partial e^j_{++}}.\end{aligned} \tag{22}$$

The infinitesimal generators ξ_0, ξ_l and η_j are obtained by solving the determining Eqs.(20).

3.2. Discrete conserved quantities associated with the Lie point symmetries

We now turn our attention to construct an invariant difference scheme with all of the appropriate conservation laws.

Using a mesh with a constant time-step h (see Fig.1), we define the linear difference operators $\underset{-h}{S}$ and $\underset{+h}{S}$ as

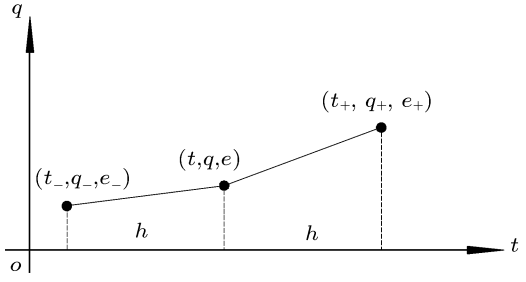


Fig.1. A three-point scheme with the invariant mesh h in the t-direction.

$$\underset{+h}{S}\, q^l = q^l_+, \quad \underset{-h}{S}\, q^l = q^l_-, \quad \underset{+h}{S}\, e^j = e^j_+, \quad \underset{-h}{S}\, e^j = e^j_-, \tag{23}$$

where $\underset{-h}{S}$ and $\underset{+h}{S}$ are, respectively, the backward and forward shift operators (or left and right discrete translation operators). From $\underset{-h}{S}$ and $\underset{+h}{S}$ we can define the finite-difference operators which correspond to the forward and backward (or right and left) discrete differentiation

$$\begin{aligned} \underset{+h}{D} &= \frac{\underset{+h}{S}-1}{h} = \sum_{s=1}^{\infty} \frac{h^{(s-1)}}{s!} D^s, \\ \underset{-h}{D} &= \frac{1-\underset{-h}{S}}{h} = \sum_{s=1}^{\infty} \frac{(-h)^{(s-1)}}{s!} D^s. \end{aligned} \tag{24}$$

The operators $\underset{+h}{S}$, $\underset{-h}{S}$, $\underset{+h}{D}$ and $\underset{-h}{D}$ commute in any combination, while

$$\underset{+h}{D} = \underset{-h}{D}\,\underset{+h}{S}, \quad \underset{-h}{D} = \underset{+h}{D}\,\underset{-h}{S}. \tag{25}$$

On this mesh we can consider the discrete Hamilton action of the system given by

$$M_{\mathrm{d}} = \sum_k L_{\mathrm{d}} \left(t, q^l, e^j, q^l_k, e^j_k\right) h, \tag{26}$$

where

$$\begin{aligned} q^l_k &= \underset{+h}{D}\left(q^l\right) = \frac{q^l_+ - q^l}{t_+ - t} = \frac{1}{h}\left(q^l_+ - q^l\right), \\ e^j_k &= \underset{+h}{D}\left(e^j\right) = \frac{e^j_+ - e^j}{t_+ - t} = \frac{1}{h}\left(e^j_+ - e^j\right) \end{aligned} \tag{27}$$

are the corresponding right discrete derivatives. Our motivation is now to use this to derive a discretization scheme with the conservation laws. In a recent paper,[27] we have given the first integral and the discrete analogous of the Noether theorem for a mechanico-electrical system as follows:

Theorem 1 If a discrete gauge function $G = G\left(t,\ q^l,\ e^j,\ q^l_k,\ e^j_k\right)$ exists such that the infinitesimal transformation generators ξ_0, ξ_l and η_j satisfy the discrete Noether identity

$$\begin{aligned} &\xi_0 \frac{\partial L_{\mathrm{d}}}{\partial t} + \xi_l \frac{\partial L_{\mathrm{d}}}{\partial q^l} + \eta_j \frac{\partial L_{\mathrm{d}}}{\partial e^j} + \left(\underset{+h}{D}\left(\xi_l\right) - q^l_k \underset{+h}{D}\left(\xi_0\right)\right) \frac{\partial L_{\mathrm{d}}}{\partial q^l_k} + \left(\underset{+h}{D}\left(\eta_j\right) - e^j_k \underset{+h}{D}\left(\xi_0\right)\right) \frac{\partial L_{\mathrm{d}}}{\partial e^j_k} \\ &+ \frac{1}{2}\left(Q^l_{\mathrm{d}} - \frac{\partial F_{\mathrm{d}}}{\partial q^l_k}\right)^{-} \left(\xi_l - q^l_{\bar{k}}\xi_0\right) + \frac{1}{2}\left(Q^l_{\mathrm{d}} - \frac{\partial F_{\mathrm{d}}}{\partial q^l_k}\right)^{+} \left(\xi_l - q^l_k\xi_0\right) + \frac{1}{2}\left(u^j_{\mathrm{d}} - \frac{\partial F_{\mathrm{d}}}{\partial e^j_k}\right)^{-} \left(\eta_j - e^j_{\bar{k}}\xi_0\right) \\ &+ \frac{1}{2}\left(u^j_{\mathrm{d}} - \frac{\partial F_{\mathrm{d}}}{\partial e^j_k}\right)^{+} \left(\eta_j - e^j_k\xi_0\right) + L_{\mathrm{d}} \underset{+h}{D}\left(\xi_0\right) + \underset{+h}{D}\left(G\right) = 0, \end{aligned} \tag{28}$$

then the mechanico-electrical system (Eqs.(9) and (10)) possesses a discrete conserved quantity given by

$$I_{\mathrm{d}} = \xi_0 L^-_{\mathrm{d}} + \left(\xi_l - q^l_{\bar{k}}\xi_0\right) \underset{-h}{S}\left(\frac{\partial L_{\mathrm{d}}}{\partial q^l_k}\right) + \left(\eta_j - e^j_{\bar{k}}\xi_0\right) \underset{-h}{S}\left(\frac{\partial L_{\mathrm{d}}}{\partial e^j_k}\right) + G = \mathrm{const}, \tag{29}$$

where $L^-_{\mathrm{d}} = S_- L_{\mathrm{d}}\left(t, q^l, e^j, q^l_k, e^j_k\right) = L_{\mathrm{d}}\left(t_-, q^l_-, e^i_-, q^l_{-k}, e^j_{-k}\right)$, and $q^l_{\bar{k}} = \dfrac{q^l - q^l_-}{h}, e^j_{\bar{k}} = \dfrac{e^j - e^j_-}{h}$.

Now we state the discrete analogues of the Lie theorem associated with the Lie point symmetry for the mechanico-electrical system:

Theorem 2 If the infinitesimal generators ξ_0, ξ_l and η_j satisfy determining equation (20), the discrete gauge function $G = G\left(t, q^l, e^j\right)$ exists, and the Noether identity (28) is satisfied, then the mechanico-electrical system (Eqs.(9)–(11)) possesses a conserved quantity (29).

Theorem 1 and Theorem 2 are, respectively, the discrete analogues of the generalized Noether theorem and the generalized Lie theorem for the mechanico-electrical system.

4. An example

Lie point symmetries of difference equations and lattices were proved in a recent paper.[13]

Here we present the Lie point symmetries of difference Lagrange–Maxwell equations for a mechanico-electrical system.

A model of the microphone of the capacitor is presented in Fig.2. (Where m denotes now the mass of the capacitor plate and k the spring constant). The sound waves are transformed into the vibration of the capacitor plate and induce a current in the circuit. The circuit consists in a capacitor C, a resistance R and an electromotive force E_0. The signal is taken from the resistance R. We denote by q the coordinate that represents the displacement of the diaphragm, by $F(t)$ the force acting on it, and we represent by e the charge in the capacitor. The kinetic energy and the potential energy of the diaphragm are, respectively, $T=\frac{1}{2}m\dot{q}^2$, $V=\frac{1}{2}k(q-q_0)^2$, while the electric energy in the capacitor is $W_{\mathrm{e}}=\frac{1}{2C(q)}e^2$.

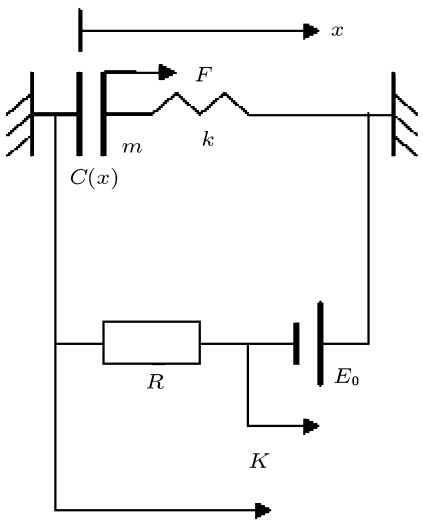


Fig.2. A mode of the microphone of the capacitor.

The Lagrangian for such a system is

$$L=\frac{1}{2}m\dot{q}^2-\frac{1}{2}k(q-q_0)^2-\frac{1}{2C(q)}e^2, \tag{30}$$

where q_0 is the resting position of the diaphragm.

$$\delta A=(E_0-R\dot{e})\,\delta e+(F-h_1\dot{q})\,\delta q, \tag{31}$$

here h_1 is the viscous damp coefficient. The Lagrange–Maxwell equations of the capacitor microphone are

$$m\ddot{q}+h_1\dot{q}+k(q-q_0)+\frac{1}{2}e^2\frac{d}{dq}\left(\frac{1}{C(q)}\right)=F, \tag{32}$$

$$R\dot{e}+\frac{e}{a}q=E_0. \tag{33}$$

If the capacitance is of the form $C(q)=\frac{a}{q}$, then Eqs.(32) and (33) lead to

$$m\ddot{q}+h_1\dot{q}+k(q-q_1)+\frac{e^2}{2a}=F, \tag{34}$$

$$R\dot{e}+\frac{e}{a}q=E_0. \tag{35}$$

If we suppose $F(t)$ to be constant, Eqs.(34) and (35) reduce to

$$\ddot{q}+a_1\dot{q}+a_2q+a_3e^2-a_4=0, \tag{36}$$

$$\dot{e}+b_1eq-b_2=0, \tag{37}$$

where $a_1=\frac{h_1}{m}$, $a_2=\frac{k}{m}$, $a_3=\frac{1}{2am}$, $a_4=a_2q_0-\frac{F}{m}$, and $b_1=\frac{1}{Ra}$, $b_2=\frac{E_0}{R}$.

A natural way to discretize Eqs.(36) and (37) is to use a uniform lattice and write

$$E_1=\frac{q_+-2q+q_-}{(t_+-t)^2}+a_1\frac{q-q_-}{t-t_-}+a_2q+a_3e^2-a_4=0, \tag{38}$$

$$E_2=\frac{e_+-e}{t-t_-}+b_1eq-b_2=0, \tag{39}$$

$$t_+-2t+t_-=0. \tag{40}$$

Let us now apply the symmetry algorithm (20). The condition $\mathrm{pr}\Omega=0$ at $E=0$ and $\Omega=0$, implies

$$\xi_0(t_+,q_+,e_+)-2\xi_0(t,q,e)+\xi_0(t_-,q_-,e_-)=0,$$

i.e.

$$\begin{aligned}&\xi_0(2t-t_-,\,2q-q_--a_1(q-q_-)(t-t_-)\\&-a_2q(t-t_-)^2-\left(a_3e^2-a_4\right)(t-t_-)^2,\\&e-b_1eq(t-t_-)+b_2(t-t_-))-2\xi_0(t,q,e)\\&=-\xi_0(t_-,q_-,e_-).\end{aligned} \tag{41}$$

Equation (41) reduces to

$$\begin{aligned}&\xi_0(2t-t_-,2q-q_--a_1(q-q_-)(t-t_-)\\&-a_2q(t-t_-)^2-(a_3e^2-a_4)(t-t_-)^2)\\&-2\xi_0(t,q)+\xi_0(t_-,q_-)=0.\end{aligned} \tag{42}$$

Differentiating first by ∂_{q_-}, then by ∂_q, we obtain

$$\begin{aligned}&-[1+a_1(t-t_-)]\,\xi_{0q_+}(2t-t_-,2q-q_--a_1(q-q_-)\\&\times(t-t_-)-a_2q(t-t_-)^2-(a_3e^2-a_4)(t-t_-)^2)\\&+\xi_{0q_-}(t_-,q_-)=0,\end{aligned} \tag{43}$$

$$\begin{aligned}&-[1+a_1(t-t_-)]\left[2-a_1(t-t_-)-a_2(t-t_-)^2\right]\\&\times\xi_{0q_+q_+}(2t-t_-,2q-q_--a_1(q-q_-)(t-t_-)\\&-a_2q(t-t_-)^2-(a_3e^2-a_4)(t-t_-)^2)=0.\end{aligned} \tag{44}$$

Equation (44) implies that ξ is linear in q

$$\xi_0(t,q) = f(t)\,q + f_0(t). \tag{45}$$

With this, Eq.(31) reduces to

$$-\left[1 - a_1(t-t_-)\right] f(2t-t_-) + f(t_-) = 0, \tag{46}$$

whose solution is

$$f(t_+) = f(t_-) = 0, \tag{47}$$

and thus $\xi_0(t) = f_0(t)$. Substituting this result into Eq.(36) we obtain

$$f_0(t_+) - 2f_0(t) + f_0(t_-) = 0. \tag{48}$$

We differentiate this equation, now with respect to t and t_-, and we have

$$f_0(t) = d_1 t + d_0, \tag{49}$$

where d_1 and d_0 are constants. Thus, the invariance of equation $t_+ - 2t + t_- = 0$ in Eq.(34) implies

$$\xi_0 = d_1 t + d_0. \tag{50}$$

The infinitesimal generators $\xi(t,q,e)$ and $\eta(t,q,e)$ are restricted by the requirement $\mathrm{pr}XE_1 = 0$, $\mathrm{pr}XE_2 = 0$ for $E_1 = 0$, $E_2 = 0$, $\Omega = 0$. The invariance conditions are given by

$$\begin{aligned}
&\xi\left(2t-t_-\,, 2q-q_- - a_1(q-q_-)(t-t_-) - a_2 q(t-t_-)^2 - \left(a_3e^2 - a_4\right)(t-t_-)^2, e - b_1 eq(t-t_-)\right.\\
&\left.+b_2(t-t_-)\right) - 2\xi(t,q,e) + \xi(t_-,q_-,e_-) + a_1(t-t_-)\,\xi(t,q,e) - a_1(t-t_-)\,\xi(t_-,q_-,e_-)\\
&+ a_2(t-t_-)^2\,\xi(t,q,e) + a_1 b_1(q-q_-)(t-t_-) + 2a_2 b_1 q(t-t_-)^2 + 2a_3 e(t-t_-)^2\,\eta(t,q,e)\\
&+ 2b_1\left(a_3e^2 - a_4\right)(t-t_-)^2 = 0,
\end{aligned} \tag{51}$$

$$\begin{aligned}
&\eta\left(2t-t_-\,, 2q-q_- - a_1(q-q_-)(t-t_-) - a_2 q(t-t_-)^2 - \left(a_3e^2 - a_4\right)(t-t_-)^2, e - b_1 eq(t-t_-)\right.\\
&\left.+b_2(t-t_-)\right) - \eta(t,q,e) + b_1 q(t-t_-)\,\eta(t,q,e) + b_1 e(t-t_-)\,\xi(t,q,e)\\
&+ b_1^2 eq(t-t_-) - b_1 b_2(t-t_-) = 0.
\end{aligned} \tag{52}$$

We differentiate Eqs.(51) and (52) with respect to q_- and obtain

$$\begin{aligned}
&-(1 - a_1(t-t_-))\,\xi_{q_+}\left(2t-t_-\,, 2q-q_- - a_1(q-q_-)(t-t_-) - a_2 q(t-t_-)^2 - \left(a_3e^2 - a_4\right)(t-t_-)^2,\right.\\
&\left.e - b_1 eq(t-t_-) + b_2(t-t_-)\right) + (1 - a_1(t-t_-))\,\xi_{q_-}(t_-,q_-,e_-) - a_1 b_1(t-t_-) = 0,
\end{aligned} \tag{53}$$

and

$$-(1 - a_1(t-t_-))\eta_{q_+} = 0, \tag{54}$$

respectively. Equation (54) implies that η is independent of q and we may, thus, write it as

$$\eta(t,e) = g(t,e), \tag{55}$$

further we successively differentiate Eq.(53) with respect to q and obtain

$$\begin{aligned}
&(2 - a_1(t-t_-) - a_2(t-t_-)^2)\xi_{q_+q_+}\left(2t-t_-\,, 2q-q_- - a_1(q-q_-)(t-t_-) - a_2 q(t-t_-)^2\right.\\
&\left.- \left(a_3e^2 - a_4\right)(t-t_-)^2, e - b_1 eq(t-t_-) + b_2(t-t_-)\right) - b_1 e(t-t_-)\\
&\times \xi_{q_+e_+}\left(2t-t_-\,, 2q-q_- - a_1(q-q_-)(t-t_-) - a_2 q(t-t_-)^2 - \left(a_3e^2 - a_4\right)(t-t_-)^2,\right.\\
&\left.e - b_1 eq(t-t_-) + b_2(t-t_-)\right) = 0.
\end{aligned} \tag{56}$$

Substituting Eq.(55) into Eq.(52) we obtain the remaining determining equation

$$\begin{aligned}
&g\left(2t-t_-, q - b_1 eq(t-t_-) + b_2(t-t_-)\right) - g(t,e) + b_1 q(t-t_-)\,g(t,e)\\
&+ b_1 e(t-t_-)\,\xi(t,q,e) + b_1^2 eq(t-t_-) - b_1 b_2(t-t_-) = 0.
\end{aligned} \tag{57}$$

Differentiating this equation with respect to q leads to

$$-b_1 e g_{e_+}\left(2t-t_-, e-b_1 e q\left(t-t_-\right)+b_2\left(t-t_-\right)\right)+b_1 g\left(t, e\right)+b_1 e \xi_q\left(t, q, e\right)+b_1^2 e=0, \tag{58}$$

and we obtain

$$\begin{aligned} &g\left(t, e\right)=-b_1 e, \\ &\xi\left(t, q, e\right)=-b_1 q+F_0\left(t, e\right), \end{aligned} \tag{59}$$

which satisfies Eq.(56). Substituting Eq.(59) into Eq.(57) we obtain

$$F_0\left(t, e\right)=\frac{2b_2}{e}, \tag{60}$$

we finally obtain the Lie group of the infinitesimal transformations of the difference system (38) and (39):

$$\begin{aligned} &\xi_0\left(t\right)=d_1 t+d_0, \\ &\eta\left(t, q, e\right)=-b_1 e=-\frac{1}{Ra} e, \\ &\xi\left(t, q, e\right)=-b_1 q+2\frac{b_2}{e}=-\frac{1}{Ra} q+2\frac{E_0}{R}\frac{1}{e}. \end{aligned} \tag{61}$$

Substituting Eq.(61) into the discrete analogy of Noether identity (28) and difference conserved quantity (29), we have

$$\begin{aligned} &\left(-\frac{1}{Ra} q+2\frac{E_0}{R}\frac{1}{e}\right)\left(-k\left(q-q_0\right)-\frac{1}{2a} e^2\right)+\left(-\frac{1}{Ra} e\right)\left(-\frac{q}{a} e\right) \\ &+\left(-\frac{1}{Ra} q_k-\frac{2E_0}{R}\frac{1}{e^2} e_k-d_1 q_k\right) m q_k+\left(F-h_1 q_k\right)\left(-\frac{1}{Ra} q+2\frac{E_0}{R}\frac{1}{e}-\left(d_1 t+d_0\right) q_k\right) \\ &+\left(E_0-R\, e_k\right)\left(-\frac{1}{Ra} e-\left(d_1 t+d_0\right) e_k\right)+d_1\left(\frac{1}{2} m q_k^2-\frac{1}{2} k\left(q-q_0\right)^2-\frac{q}{2a} e^2\right)=-\underset{+h}{D}\left(G\right), \end{aligned} \tag{62}$$

i.e

$$\begin{aligned} \underset{+h}{D}\left(G\right)=&\left(\frac{1}{2} m d_1+\frac{m}{Ra}-d_1 h_1 t+d_0 h\right) q_k^2+R\left(d_1 t+d_0\right) e_k^2+\frac{2mE_0}{R}\frac{1}{e^2} e_k q_k \\ &+\left[F\left(d_1 t+d_0\right)+\frac{2h_1 E_0}{R}\frac{1}{e}-\frac{h_1}{Ra} q\right] q_k+\left[E_0\left(d_1 t+d_0\right)-\frac{e}{a}\right] e_k+\frac{d_1 k}{2}\left(q-q_0\right)^2 \\ &-\frac{k}{Ra} q\left(q-q_0\right)+\frac{2kE_0}{R}\frac{q-q_0}{e}+\left(\frac{d_1}{2a}-\frac{3}{2Ra^2}\right) q e^2+\frac{2E_0}{Ra} e+\frac{F}{Ra} q-\frac{2FE_0}{R}\frac{1}{e}, \end{aligned} \tag{63}$$

and

$$\begin{aligned} I_{\mathrm{d}}=&\left(d_1 t+d_0\right)\left(\frac{1}{2} m q_{\bar{k}}^2-\frac{1}{2} k\left(q_--q_0\right)-\frac{q_-}{2a}\left(e_-\right)^2\right) \\ &+\left(-\frac{1}{Ra} q+2\frac{E_0}{R}\frac{1}{e}\right) m q_{\bar{k}}+G=\text{const.} \end{aligned} \tag{64}$$

These two equations, although difficult to solve, can be used to obtain the gauge function G and the discrete conserved quantity I_{d}, and from them, the numerical solution.

To illustrate this, we represent in Figs.3, 4 and 5 some numerical results obtained from Eqs.(34) and (35) for the following values of the parameters: $k=1.3$, $a=1$, $d_1=1$, $d_0=0$, $R=1.004$, $m=1$, $E_0=1$, $h=0.1$, $h_1=1$, $d=1$, $F=0.3$, $x_0=1$, $q_0=1.5$, $G_0=2$.

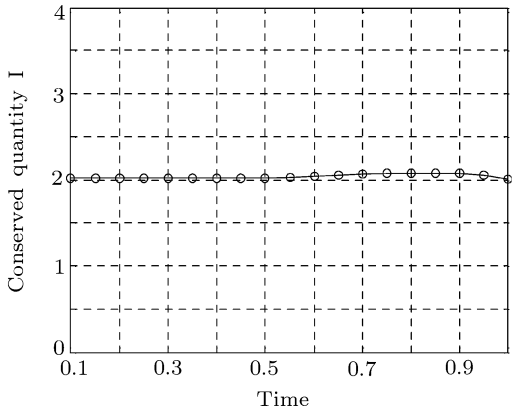


Fig.3. Discrete Noether conserved quantity I relative to time t.

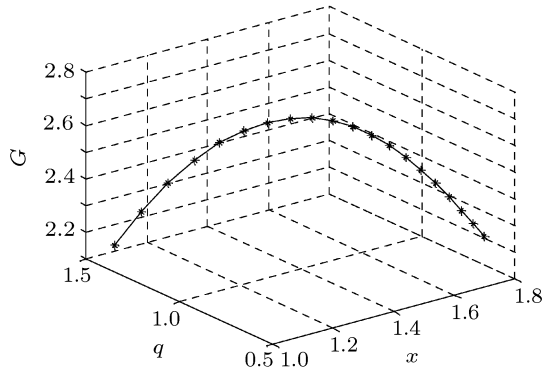


Fig.4. Discrete regular function G relative to generalized coordinate q and electric charge e.

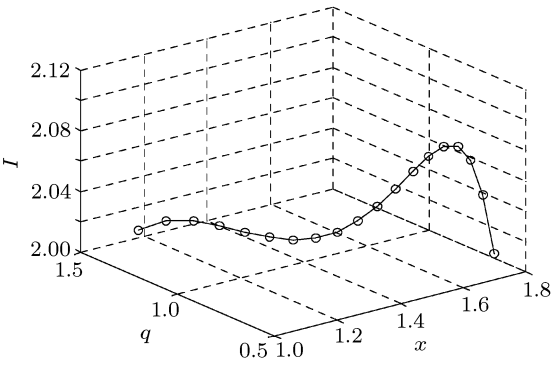


Fig.5. Discrete Noether conserved quantity I relative to generalized coordinate q and electric charge e.

From Fig.3, we can see that the conserved quantity has a small error in the system. However, we obtain the conserved quantity and Lie point symmetry of a mechanico-electrical system in Figs.4 and 5.

5. Conclusion

In this paper, we have extended Lie point symmetries to discrete Lagrange–Maxwell mechanico-electrical systems with nonconservative forces and dissipation functions. Our results represent a significant approach to find both the one-parameter Lie groups and the discrete conserved quantities of Lie point symmetries associated with the discrete mechanico-electrical systems.

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