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A new method to obtain approximate symmetry of nonlinear evolution equation from perturbations

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A novel method for obtaining the approximate symmetry of a partial differential equation with a small parameter is introduced. By expanding the independent variable and the dependent variable in the small parameter series, we obtain more affluent approximate symmetries. The method is applied to two perturbed nonlinear partial differential equations and new approximate solutions are derived.

Keywords: approximate symmetry, approximate solutions, expansion, perturbed equation
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1. Introduction

Group theoretical methods based on symmetry(local and nonlocal) are systematic and effective for constructing group invariant solutions, calculating conservation laws of partial differential equations (PDEs) and linearizing nonlinear PDEs by invertible mappings.[1−9] However, many PDEs arising in mathematical physics depend on a small parameter and do not admit nontrivial exact symmetries, so it is of great importance and interest to find the approximate symmetry.[10−14] In 1989, Fushchich and Shtelen[10] first introduced an effective method to search for the approximate symmetry and the method was later followed by Euler *et al.*[15−17] They firstly expanded the dependent variable in terms of a small parameter (maybe a physical parameter or artificially introduced) as the usual perturbation analysis. Terms are then separated at each order of approximation and a system of equations in a hierarchy is obtained. The system is assumed to be coupled and the approximate symmetry of the original equation is defined to be the exact symmetry of the system of equations obtained from perturbations.

The purpose of this paper is to present a new method to obtain the approximate symmetry for the perturbed PDEs. Apart from expanding the dependent variables, we also expand the independent variable in the small parameter series. By means of this method, we obtain more affluent approximate symmetries than those achieved by expanding the dependent variable in the small parameter series only.

The outline of the paper is as follows. In Section 2, the method presented by Fushchich and Shtelen is reviewed and the new method for the approximate symmetry is introduced. Two examples are considered to illustrate this new method in Section 3. In Section 4, new reduced equations and approximate solutions are constructed. Some concluding remarks are discussed in the last section.

2. Main results

In this section, we first review the method originated by Fushchich and Shtelen (Method I), then give the main idea of the new method (Method II).

2.1. Method I

In this method, which is developed by Fushchich and Shtelen and used in Refs.[15]–[17], the dependent variable is expanded in a perturbation series. The separated equations at each order are assumed to be coupled and the approximate symmetry of the original equation is defined as the exact symmetry of these coupled equations.

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Consider the general mth order nonlinear evolution equation

$$E = E(x, t, u, u_1, u_2, \ldots, u_m, u_t; \epsilon) = 0, \qquad (1)$$

where $u_t = \partial u/\partial t, u_k = \partial^k u/\partial x^k, 1 \leq k \leq m$ and ϵ is a small parameter, E is a smooth function of the indicated variables.

Expanding the dependent variable in the small parameter yields

$$u = u_0 + \epsilon u_1 + \ldots, \quad 0 < \epsilon \ll 1, \qquad (2)$$

then inserting expansion (2) into the original Eq.(1) and separating at each order of the perturbed parameter, one has

$$\text{Order } \epsilon^0\text{: } E_0 = 0, \quad \text{Order } \epsilon^1\text{: } E_1 = 0, \qquad (3)$$

hence the exact symmetry of system (3) is the approximate symmetry of the original Eq.(1).

2.2. Method II

In this method, both the independent variable and the dependent variable are simultaneously expanded in the perturb parameter series.

The dependent variable expands as Eq.(2) and the independent variable is expanded to

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} + \epsilon \frac{\partial}{\partial s}, \quad \tilde{t} = t, \ s = \epsilon t, \quad 0 < \epsilon \ll 1. \qquad (4)$$

Inserting expansions (2) and (4) into the original Eq.(1) and separating at each order of the small parameter, then we have the following equations with respect to different order ϵ^0, ϵ^1 and ϵ^2

$$\begin{aligned} &\text{Order } \epsilon^0\text{: } \widetilde{E_0} = 0, \quad \text{Order } \epsilon^1\text{: } \widetilde{E_1} = 0, \\ &\text{Order } \epsilon^2\text{: } \widetilde{E_2} = 0, \end{aligned} \qquad (5)$$

so the approximate symmetry of the original Eq.(1) is defined as the exact symmetry of system (5).

Due to the increase of the number of equations, from two to three, we easily find that calculation of this method increases enormously and the problem can only be handled by a symbolic computer package. Hence, a differential characteristic set algorithm[18,19] is used in the calculations.

3. Applications

To illustrate the new approach for obtaining approximate symmetry and contrast it with the approach developed by Fushchich and Shtelen, we consider two examples.

3.1. Perturbed Burgers equation

We first consider the Burgers equation with a first-order dissipation

$$u_{xx} - uu_x - u_t = \epsilon u_x, \qquad (6)$$

which was discussed in detail in Refs.[20, 21].

3.1.1. Method I

Expanding the dependent variable to the first order yields

$$u = v + \epsilon w, \qquad 0 < \epsilon \ll 1. \qquad (7)$$

Substituting expansion (7) into Eq.(6) and separating at each order of perturbation parameter, one has the following equations with respect to ϵ^0 and ϵ^1:

$$\begin{aligned} &v_t = v_{xx} - vv_x, \\ &w_t = w_{xx} - wv_x - vw_x - v_x. \end{aligned} \qquad (8)$$

Approximate symmetries of Eq.(6) correspond to the exact symmetries of system (8).

Now, we consider a one-parameter Lie symmetry group of local transformations with an infinitesimal operator of the form

$$\begin{aligned} X = &\xi(x, t, v, w)\partial_x + \tau(x, t, v, w)\partial_t \\ &+ \eta(x, t, v, w)\partial_v + \phi(x, t, v, w)\partial_w, \end{aligned} \qquad (9)$$

which leaves Eq.(8) invariant. With the aid of the differential characteristic set,[18,19] performing the standard Lie group algebra,[1,2] one finally has

$$\begin{aligned} &\xi = (-c_1 t + c_2)x + c_4 t + c_5, \\ &\tau = -c_1 t^2 + 2c_2 t + c_6, \\ &\eta = -c_1 x - (-c_1 t + c_2)v + c_4, \\ &\phi = (c_1 t + c_3)(1 + w). \end{aligned} \qquad (10)$$

3.1.2. Method II

The dependent variable and the independent variable expand as Eqs.(7) and (4), respectively. We have the following equations with respect to the ϵ^0, ϵ^1 and ϵ^2

$$\begin{aligned} &v_{\tilde{t}} = v_{xx} - vv_x, \\ &w_s = -(1 + w)w_x, \\ &w_{\tilde{t}} = w_{xx} - wv_x - vw_x - v_x - v_s. \end{aligned} \qquad (11)$$

Assuming that Eq.(11) admits the Lie group of point transformation with the infinitesimal operator of the form

$$X = \xi\partial_x + \tau\partial_{\tilde{t}} + \eta\partial_s + \phi\partial_v + \psi\partial_w, \tag{12}$$

where ξ, τ, η, ϕ, ψ are functions of x, $\tilde{t}$, s, v, w.

With a similar method as Method I, the solutions are

$$\begin{aligned}
&\xi = c_1 s + (c_2 x + c_3)\tilde{t} + c_4 x + c_5,\\
&\tau = c_2\tilde{t}^2 + 2c_4\tilde{t} + c_6,\\
&\eta = (c_2 s - c_7)\tilde{t} + (c_4 - c_8)s + c_9,\\
&\phi = -(c_2\tilde{t} + c_4)v + c_2 x - (c_2 s - c_7)(1 + w) + c_3,\\
&\psi = c_8(1 + w) + c_1.
\end{aligned} \tag{13}$$

Comparing Eq.(13) with Eq.(10), we find six new approximate symmetries with infinitesimal operators of the form

$$\begin{aligned}
&X_1 = x\tilde{t}\partial_x + \tilde{t}^2\partial_{\tilde{t}} + st\partial_s - (\tilde{t}v - x + s(1 + w))\partial_v,\\
&X_2 = x\partial_x + 2\tilde{t}\partial_{\tilde{t}} + s\partial_s - v\partial_v,\\
&X_3 = -\tilde{t}\partial_s + (1 + w)\partial_v,\\
&X_4 = -s\partial_s + (1 + w)\partial_w,\\
&X_5 = s\partial_s, X_6 = s\partial_x + \partial_w.
\end{aligned} \tag{14}$$

3.2. Perturbed wave equation

The second example is the nonlinear perturbed wave equation

$$u_{tt} - u_x u_{xx} + \epsilon u_t = 0, \tag{15}$$

which was analysed in Refs.[14, 22, 23].

3.2.1. Method I

Substituting expansion (7) into Eq.(15) and separating at each order of the perturbation parameter, we have

$$\begin{aligned}
&v_{tt} - v_x v_{xx} = 0,\\
&w_{tt} - v_{xx} w_x - v_x w_{xx} + v_t = 0.
\end{aligned} \tag{16}$$

Approximate symmetries of Eq.(15) correspond to the exact symmetries of system (16).

Hence, we consider a one-parameter Lie symmetry group of local transformations with an infinitesimal operator of the form as Eq.(9), one finally has

$$\begin{aligned}
&\xi = c_1 x + c_2, \tau = c_3 t + c_4,\\
&\eta = (3c_1 - 2c_3)v + c_5 t + c_6,\\
&\phi = (3c_1 - c_3)w - \frac{1}{2}c_5 t^2 + c_7 t + c_8.
\end{aligned} \tag{17}$$

3.2.2. Method II

The dependent variable expand as Eq.(7) and the independent variable is expanded to Eq.(4), then we have the following equations

$$\begin{aligned}
&v_{\tilde{t}\tilde{t}} - v_x v_{xx} = 0,\\
&2w_{\tilde{t}s} + v_{ss} - w_x w_{xx} + w_{\tilde{t}} + v_s = 0,\\
&2v_{\tilde{t}s} + w_{\tilde{t}\tilde{t}} - v_{xx} w_x - v_x w_{xx} + v_{\tilde{t}} = 0.
\end{aligned} \tag{18}$$

Assuming Eq.(18) admits the infinitesimal operator of the form

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_s + \phi\partial_v + \psi\partial_w, \tag{19}$$

where ξ, τ, η, ϕ, ψ are functions of x, t, s, v, w.

With a similar method as Method I, one obtains

$$\begin{aligned}
&\xi = c_1 x + c_2, \tau = c_3\tilde{t} + c_4, \eta = c_5,\\
&\phi = (3c_1 - 2c_3)v + f(s)\\
&\quad + \tilde{t}\mathrm{e}^{-\frac{s}{2}}(c_7\cos\frac{s}{2\sqrt{3}} + c_6\sin\frac{s}{2\sqrt{3}}),\\
&\psi = (3c_1 - c_3)w + g(s)\tilde{t} + h(s)\\
&\quad - \frac{\tilde{t}^2\mathrm{e}^{-\frac{s}{2}}}{2\sqrt{3}}\left(c_6\cos\frac{s}{2\sqrt{3}} - c_7\sin\frac{s}{2\sqrt{3}}\right),
\end{aligned} \tag{20}$$

where $f(s), g(s)$ satisfy $g(s)+f'(s)+2\,g'(s)+f''(s) = 0$ and $h(s)$ is an arbitrary function of s.

Comparing Eq.(20) with Eq.(17), we have three new finite-dimensional approximate infinitesimal operators

$$\begin{aligned}
&X_1 = \tilde{t}\mathrm{e}^{-\frac{s}{2}}\cos\frac{s}{2\sqrt{3}}\partial_v + \frac{\tilde{t}^2\mathrm{e}^{-\frac{s}{2}}}{2\sqrt{3}}\sin\frac{s}{2\sqrt{3}}\partial_w,\\
&X_2 = \tilde{t}\mathrm{e}^{-\frac{s}{2}}\sin\frac{s}{2\sqrt{3}}\partial_v - \frac{\tilde{t}^2\mathrm{e}^{-\frac{s}{2}}}{2\sqrt{3}}\cos\frac{s}{2\sqrt{3}}\partial_w,\\
&X_3 = \partial_s;
\end{aligned} \tag{21}$$

and one infinite-dimensional approximate infinitesimal operator

$$X_4 = f(s)\partial_v + (g(s)\tilde{t} + h(s))\partial_w. \tag{22}$$

4. Approximate solutions

In this section, we utilize the new approximate symmetries obtained in the previous section to gain some new approximate, up to the first order in ϵ, invariant solutions of the corresponding equations.

Case I. Choosing $X = s\partial_x + \partial_w$, from determining equations for similarity variables

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}\tilde{t}}{0} = \frac{\mathrm{d}s}{0} = \frac{\mathrm{d}v}{0} = \frac{\mathrm{d}w}{1}, \tag{23}$$

we can obtain similarity variables $z = \tilde{t}, y = s, T = v, U = -\frac{x}{s} + w$. Substituting these similarity variables into Eq.(11), one finally has

$$\begin{aligned} &T_z = 0, \\ &T + yT_y + yU_z = 0, \\ &U + yU_y + 1 = 0, \end{aligned} \tag{24}$$

which have solutions $v = \frac{c_2}{y} - c_1$, $w = -1 + \frac{1}{y}(c_1\tilde{t} + x + c_2)$, so approximate solutions of Eq.(6) are

$$u = \frac{c_2}{\epsilon t} - \epsilon + \frac{1}{t}(x + c_2), \tag{25}$$

where c_1, c_2 are arbitrary constants.

Case II. $X = -s\partial_s + (1 + w)\partial_w$.

Equation (18) can be reduced to ($z = x, y = \tilde{t}$, $T = v$, $U = s(1 + w)$)

$$\begin{aligned} &U(1 - U_z) = 0, \\ &T_y + TT_z - T_{zz} = 0, \\ &U_y - U_{zz} - (UT)_z = 0. \end{aligned} \tag{26}$$

Especially, we have special approximate solutions of Eq.(6)

$$u = -\epsilon + \frac{x + c_2}{t} - \frac{2}{x + c_1 t + c_2}, \tag{27}$$

where c_1, c_2 are arbitrary constants.

Case III. $X = \partial_x + \tilde{t}\mathrm{e}^{-\frac{s}{2}} \cos\frac{s}{2\sqrt{3}}\partial_v + \frac{\tilde{t}^2}{2\sqrt{3}}\mathrm{e}^{-\frac{s}{2}} \sin\frac{s}{2\sqrt{3}}\partial_w$.

Equation (18) can be reduced to $\left(z = \tilde{t},\ y = s,\ T = \mathrm{e}^{\frac{s}{2}}v - x\tilde{t}\cos\frac{s}{2\sqrt{3}},\ U = \mathrm{e}^{\frac{s}{2}}w - \frac{x\tilde{t}^2}{2\sqrt{3}}\sin\frac{s}{2\sqrt{3}}\right)$

$$\begin{aligned} &T_{zz} = 0, \\ &2T_{zy} + U_{zz} = 0, \\ &8U_{zy} - T + 4T_{yy} = 0. \end{aligned} \tag{28}$$

Its solutions are

$$\begin{aligned} U = &\left(\frac{c_2}{2\sqrt{3}}\cos\frac{y}{2\sqrt{3}} - \frac{c_1}{2\sqrt{3}}\sin\frac{y}{2\sqrt{3}}\right)z^2 \\ &+ f(y)z + h(y), \\ T = &\left(c_1\cos\frac{y}{2\sqrt{3}} + c_2\sin\frac{y}{2\sqrt{3}}\right)z + g(y), \end{aligned} \tag{29}$$

where $f(y)$, $g(y)$ satisfy $8f'(y) - g(y) + 4g''(y) = 0$, $h(y)$ is an arbitrary function and c_1, c_2 are arbitrary constants.

Hence, we can obtain new approximate solutions of Eq.(15)

$$\begin{aligned} u = v + \epsilon w = &\mathrm{e}^{-\frac{\epsilon t}{2}}T + x\mathrm{e}^{-\frac{\epsilon t}{2}}\cos\frac{\epsilon t}{2\sqrt{3}} \\ &+ \epsilon\left(\mathrm{e}^{-\frac{\epsilon\tilde{t}}{2}}U + \frac{x\tilde{t}^2}{2\sqrt{3}}\mathrm{e}^{-\frac{\epsilon\tilde{t}}{2}}\sin\frac{\epsilon\tilde{t}}{2\sqrt{3}}\right), \end{aligned} \tag{30}$$

where U, T is of the form (29).

5. Conclusion

A new method for obtaining the approximate symmetry of the perturbed partial differential equation is introduced. Compared with the method developed by Fushchich and Shtelen, this new method can gain more abundant approximate symmetries. Two examples are used to illustrate the fact that this method is more effective than Method I. Moreover, the new method uses only standard Lie algorithms, so this approximate symmetry approach may readily be extended to determine approximate non-classical symmetries and approximate potential symmetries. It would be interesting to investigate them in our future work.

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