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The topological structure of the integral quantum Hall effect in magnetic semiconductor-superconductor hybrids*

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An unconventional integer quantum Hall regime was found in magnetic semiconductor-superconductor hybrids. By making use of the decomposition of the gauge potential on a $U(1)$ principal fibre bundle over k-space, we study the topological structure of the integral Hall conductance. It is labeled by the Hopf index β and the Brouwer degree η. The Hall conductance topological current and its evolution is discussed.

Keywords: integral quantum Hall effect, gauge potential, Hopf index, vortex
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1. Introduction

The discovery of the quantization of the Hall conductance by Von Klitzing, Dorda and Pepper[1] is one of the most important results in condensed matter physics. By now, an unconventional integer quantum Hall regime was found in magnetic semiconductor-superconductor hybrids.[2] A superconductor (SC) film deposited on top of a diluted magnetic semiconductor quantum well under the influence of an external magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$ for all values of B_0 between the two possible asymptotic limits was studied. For very low applied fields, the SC film is populated with few isolated vortices. Each vortex produces a highly inhomogeneous magnetic field which localizes spin polarized states in the diluted magnetic semiconductor (DMS). The overlap between the magnetic fields of independent vortices is small. If B_0 is a high applied external magnetic field, the magnetic fields of different vortices begin to overlap significantly. In this limit, the total magnetic field in the DMS layer is almost homogeneous. Its average is $B_0\boldsymbol{e}_z$. But it has a small additional periodic modulation (the same effect can be achieved at a fixed B_0 by increasing the distance z between the SC and DMS layers). For such the quasi-homogeneous magnetic fields, the Zeeman interaction can no longer induce trapping instead it reverts to its traditional role of lifting the spin degeneracy. The small periodic modulation insures the fact that the system still corresponds to a $\phi/\phi_0 = 1/2$ Hofstadter butterfly. But now it is in the other asymptotic limit, namely that of a weak periodic modulation. In this case each Landau level is expected to split into q subbands with a bandwidth controlled by the amplitude of the weak modulation. The case $\phi/\phi_0 = 1/2$ has $q = 1$ and there are no additional gaps in the band structure. As a result, one expects to see the usual integer quantum Hall effect (IQHE) in this regime. In this article, we introduce the corresponding Hamiltonian for the free carriers in the DMS quantum well, by using the so-called ϕ-mapping method. We study the topological structure of quantum Hall effect in the magnetic semiconductor-superconductor hybrids. The integral Hall conductance density in the magnetic Brillouin zone is found to take the same form as the decomposition formula of the $U(1)$ gauge potential which is composed of the Bloch wave functions. Using the so-called ϕ-mapping method, it is verified that the Hall conductance density takes the form of a generalized function $\delta(\phi)$ with $\phi = (\phi^1, \phi^2)$. The result comes from the theory itself without any assumptions. Only the zeroes of the magnetic Bloch wave function contribute to the Hall conductance. The Hall conductance topological current and its evolution are discussed. The Hall topological charges are found generating or annihilating at the limit points of wave

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function and splitting and merging at the bifurcation points of the wave function. Different directions of the bifurcation are given.

This paper is arranged as follows. In Section 2 we study the topological structure of the quantum Hall effect in the magnetic semiconductor-superconductor hybrids. In Section 3 we give the Hall conductance topological current and discuss its evolution. In Section 4 a short summary of this paper is given.

2. The topological structure of quantum Hall effect in magnetic semiconductor-superconductor hybrids

The corresponding Hamiltonian for the free carriers in the DMS quantum well, in the presence of an SC vortex lattice, is given by

$$H = \frac{1}{2m}[\boldsymbol{p} + e\boldsymbol{A}_L(\boldsymbol{r};z)]^2 - \frac{1}{2}g_{\rm eff}\mu_B\boldsymbol{\sigma}\cdot\boldsymbol{B}_L(\boldsymbol{r};z). \quad (1)$$

Here $-e$ is the charge of the charge carriers, assumed to be electrons. Holes can be treated similarly. We use $\boldsymbol{r} = (x, y)$ to describe the 2D position of the charge carrier inside the narrow (2D) DMS quantum well (QW). Before constructing the solutions for the full Hamiltonian of Eq.(1), we first briefly review the solutions in the presence of only a homogeneous field B_0, in order to fix the notation. In this case the Zeeman term lifts the spin degeneracy, but it is the orbital coupling that essentially determines the energy spectrum of the carriers, which consists of spin-polarized Landau levels (LLs):[2]

$$E_{N,\sigma} = \hbar\omega_c\left(N + \frac{1}{2}\right) - \frac{1}{2}g_{\rm eff}\mu_B B_0\sigma. \quad (2)$$

In the Landau gauge the corresponding eigenstates are

$$\phi_{N,k_y,\sigma}(\boldsymbol{r}) = \frac{\exp {\rm i}k_y y \exp\left(-\frac{1}{2}\right)\left(\frac{x}{l} + lk_y\right)^2}{\sqrt{b}\sqrt{l\sqrt{\pi}2^N N!}} \times {\rm H}_N\left(\frac{x}{l} + lk_y\right)\chi_\sigma, \quad (3)$$

where $l = \sqrt{\dfrac{\hbar}{eB_0}}$ is the magnetic length and $\omega_c = \dfrac{eB_0}{m}$ is the cyclotron frequency. $N \geq 0$ is the index of the LL, ${\rm H}_N(\xi)$ are Hermite polynomials, and k_y is a momentum. χ_σ are spin eigenstates, $\sigma_z\chi_\sigma = \sigma\chi_\sigma, \sigma = \pm 1$. Each LL is highly degenerate and can accommodate up to one electron per $2\pi l^2$ sample area. The filling factor $\nu = n2\pi l^2$, where n is the 2D electron density in the DMS QW, counts how many LLs are fully or partially filled.

In order to construct solutions that include both the periodic and homogeneous part of the magnetic field, we need to first consider the symmetries of Hamiltonian (1). Because of the orbital coupling to the nonperiodic part of the vector potential $\boldsymbol{A}_0$, the ordinary lattice translation operators

$$T(\boldsymbol{R}) = \exp\left(\frac{\rm i}{\hbar}\boldsymbol{R}\cdot\boldsymbol{P}\right)$$

do not commute with the Hamiltonian. Instead, one needs to define the so-called magnetic translation operators[3]

$$T_M(\boldsymbol{R}) = \exp\left(-\frac{{\rm i}e}{\hbar}B_0 R_x y\right)\exp\left(\frac{\rm i}{\hbar}\boldsymbol{R}\cdot\boldsymbol{p}\right). \quad (4)$$

With this choice, the wave functions must also be eigenstates of the magnetic translation operators

$$T_M(\boldsymbol{R})\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \exp({\rm i}\boldsymbol{k}\cdot\boldsymbol{R})\psi_{\boldsymbol{k}}(\boldsymbol{r}). \quad (5)$$

We thus need to expand the eigenstates in a complete basis set of wave function which satisfies Eq.(5). Such a basis can be constructed from the LL eigenstates, since the large degeneracy of each LL allows one to construct linear combinations satisfying Eq.(5):[4]

$$\varphi_{N,\boldsymbol{k},\sigma}(\boldsymbol{r}) = \frac{1}{\sqrt{N_T}}\sum_{n=-\infty}^{\infty}\exp({\rm i}k_x na)\phi_{N,k_y+(\frac{2\pi}{b})n,\sigma}(\boldsymbol{r}), \quad (6)$$

where N_T is the number of magnetic unit cells and $\boldsymbol{k}$ is a wave vector in the magnetic Brillouin zone.

As a result, we search for eigenstates of Hamiltonian (1) of the general form

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \sum_{N,\sigma} d_{N\sigma}(\boldsymbol{k})\varphi_{N,\boldsymbol{k},\sigma}(\boldsymbol{r}). \quad (7)$$

Here, $d_{N\sigma}(\boldsymbol{k})$ are complex coefficients characterizing the contribution of states from various LLs to the true eigenstates.[2]

In our case, it is necessary to use a more general method to calculate the Hall conductance. We briefly review it here. This approach is inspired by the work of Kohmoto[5] and Usov,[6] who showed that it is possible to use the topology of the band structure to calculate, in a relatively simple way, the contribution to the Hall conductance of each filled subband.

When the Fermi level is inside a gap, the total Hall conductivity equals the sum of the contributions

from all the fully occupied bands $\sigma_{xy} = \sum_a \sigma^a_{xy}\theta(E_F - E^a_{\boldsymbol{k}})$. The contribution of each occupied band is[4]

$$\sigma^a_{xy} = \mathrm{i}\frac{e^2}{2\pi h}\int \mathrm{d}\boldsymbol{k}\int \mathrm{d}\boldsymbol{r} \times\left[\frac{\partial u^{a*}_{\boldsymbol{k}}}{\partial k_x}\frac{\partial u^a_{\boldsymbol{k}}}{\partial k_y} - \frac{\partial u^{a*}_{\boldsymbol{k}}}{\partial k_y}\frac{\partial u^a_{\boldsymbol{k}}}{\partial k_x}\right], \tag{8}$$

where the integrals are carried over the magnetic Brillouin zone and the magnetic unit cell, respectively. Here, $u^a_{\boldsymbol{k}}(\boldsymbol{r}) = \psi^a_{\boldsymbol{k}}(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$ is the Bloch part of the band eigenstate. Using Eqs.(6) and (7) we can perform the real space integrals explicitly to obtain

$$\sigma^a_{xy} = \frac{e^2}{h} + \sum_{N\sigma}\frac{e^2}{2\pi\mathrm{i}h}\iint_{T^2_{\mathrm{MBZ}}}\mathrm{d}k^1\mathrm{d}k^2 \times[\nabla_k\times\langle d^a_{N\sigma}\mid\nabla_k\mid d^a_{N\sigma}\rangle]_z. \tag{9}$$

The second term can be rewritten as

$$\Delta\sigma^a_{xy} = \sum_{N\sigma}\frac{e^2}{2\pi\mathrm{i}h}\iint_{T^2_{\mathrm{MBZ}}}\mathrm{d}k^1\mathrm{d}k^2 \times[\nabla_k\times\langle d^a_{N\sigma}\mid\nabla_k\mid d^a_{N\sigma}\rangle]_z. \tag{10}$$

The integration is over the magnetic Brillouin zone: $0 \le k^1 \le \dfrac{2\pi}{qa}, 0 \le k^2 \le \dfrac{2\pi}{b}$. An important observation here is that the magnetic Brillouin zone is topologically a torus T^2 rather than being rectangular in k-space. Since a torus does not have a boundary, the application of Stokes' theorem to Eq.(10) would give $\Delta\sigma^a_{xy} = 0$ if all $\langle d^a_{N\sigma}\mid\nabla_k\mid d^a_{N\sigma}\rangle$ are uniquely defined on the entire torus T^2. The non-zero value of $\Delta\sigma^a_{xy}$ is a consequence of the non-trivial topology of $[\langle d^a_{N\sigma}\mid\nabla_k\mid d^a_{N\sigma}\rangle]_z$. From the theory of fiber bundles we know that the non-trivial topological property of the bundle may be manifested on a section on it. Therefore, there must exist some zeroes of $d^a_{N\sigma}$. For convenience, to discuss the Hall conductance in detail we choose a loop in the magnetic Brillouin zone on which the probability density ρ is a constant. It divides the magnetic Brillouin into two parts Σ_1 and Σ_2. Now we can use Stokes' theorem to Eq.(10)[7]

$$\Delta\sigma^a_{xy} = \frac{e^2}{2\pi\mathrm{i}h}\Big(\oint_{\partial\Sigma_1}\mathrm{d}\boldsymbol{k}\cdot\sum_{N\sigma}\langle d^a_{N\sigma}\mid\nabla_k\mid d^a_{N\sigma}\rangle + \oint_{\partial\Sigma_2}\mathrm{d}\boldsymbol{k}\cdot\sum_{N\sigma}\langle d^a_{N\sigma}\mid\nabla_k\mid d^a_{N\sigma}\rangle\Big). \tag{11}$$

If we define

$$A_{N\sigma} = \frac{1}{2\mathrm{i}}\frac{1}{\rho_{N\sigma}}(\mathrm{d}d^{a*}_{N\sigma}d^a_{N\sigma} - d^{a*}_{N\sigma}\mathrm{d}d^a_{N\sigma}), \tag{12}$$

$$\mathrm{d} = \frac{\partial}{\partial k^i}\mathrm{d}k^i, \tag{13}$$

where

$$\rho_{N\sigma} = d^{a*}_{N\sigma}d^a_{N\sigma}.$$

Then equation (11) can be further expressed as

$$\Delta\sigma^a_{xy} = \frac{e^2}{2\pi\mathrm{i}h}\sum_{N\sigma}\Big(\int\mathrm{d}^2r\rho_{N\sigma}\oint_{\partial\Sigma_1}\mathrm{d}k^a A^a_{N\sigma} + \int\mathrm{d}^2r\rho_{N\sigma}\oint_{\partial\Sigma_2}\mathrm{d}k^a A^a_{N\sigma}\Big). \tag{14}$$

Using Stokes' theorem again, we get

$$\Delta\sigma^a_{xy} = \frac{e^2}{2\pi\mathrm{i}h}\sum_{N\sigma}\int\mathrm{d}^2r\rho_{N\sigma}\iint_{T^2}\mathrm{d}A_{N\sigma}. \tag{15}$$

Given a transformation

$$d'_{N\sigma}(k^1,k^2) = d_{N\sigma}(k^1,k^2)\exp[\mathrm{i}f(k^1,k^2)], \tag{16}$$

where $f(k^1,\ k^2)$ is an arbitrary smooth real function of k^1 and k^2 and is independent of x and y, $d'_{N\sigma}(k^1,k^2)$ also satisfies Eq.(7). So we recognize that $d_{N\sigma}(k^1,k^2)$ is a section of the $U(1)$ bundle over k-space. Since equation (15) is a change of the overall phase of the wave function, any physical quantity remains invariant under this transformation. From Eq.(15) the corresponding transformation of $A_{N\sigma}$ is given by

$$A'_{N\sigma} = A_{N\sigma} - \mathrm{i}df(k^1,k^2). \tag{17}$$

Therefore, $A_{N\sigma}$ is indeed a connection 1-form (or gauge potential) of a principal $U(1)$ fiber bundle over T^2. It is easy to see from Eqs.(14) and (16) that $\Delta\sigma^a_{xy}$ is invariant under the transformation (15). Therefore, the integral Hall conductance is quantized by the first Chern class of the $U(1)$ principal fiber bundle over k-space. For the Chern number being a topological invariant, the quantization of integral Hall conductance is a topological quantization. Because the wave function can be denoted by

$$d_{N\sigma}(k^1,k^2) = \phi^1 + \mathrm{i}\phi^2, \tag{18}$$

we regard $d_{N\sigma}(k^1,k^2)$ as the complex representation of the vector field

$$\boldsymbol{\phi} = (\phi^1,\phi^2). \tag{19}$$

Then we define the unit vector of the wave function,

$$n^a = \frac{\phi^a}{\|\phi\|},\quad \|\phi\| = \sqrt{\phi^a\phi^a}, \tag{20}$$

$$n^a n^a = 1, a = 1,2.$$

In fact $\boldsymbol{n}$ is identified as a section of the sphere bundle over T^2, i.e. $\boldsymbol{n}$ is a section of the SO(2) principal fiber bundle. It is obvious that the zeros of $\boldsymbol{\phi}$ are just the singular points of $\boldsymbol{n}$. Now $A_{N\sigma}$ can be rewritten as

$$A_{N\sigma} = -\mathrm{i}\epsilon^{ab} dn^a n^b. \tag{21}$$

This formula of $A_{N\sigma}$ takes the same form as the decomposition formula of the spin connection 1-form of the $U(1)$ fiber bundle in Duan, Yang and Jiang's paper.[8] This formula is important for it reveals the inner structure of the $U(1)$ gauge potential and is also a useful tool in discussing the topological structure of the $U(1)$ Chern class.

Substituting Eq.(20) into Eq.(14), the Hall conductance is changed into

$$\begin{aligned} \Delta\sigma^a_{xy} &= \frac{e^2}{2\pi \mathrm{i} h} \sum_{N\sigma} \int \mathrm{d}^2 r \rho_{N\sigma} \iint_{T^2} \mathrm{i}\epsilon^{ab}\epsilon_{ij} \\ &\times \frac{\partial n^a}{\partial k_i} \frac{\partial n^b}{\partial k_j} \mathrm{d}k^1 \mathrm{d}k^2 \\ &= \frac{e^2}{2\pi h} \sum_{N\sigma} \int \mathrm{d}^2 r \rho_{N\sigma} \int_{T^2} \epsilon^{ab}\epsilon_{ij} \frac{\partial}{\partial \phi^c} \frac{\partial}{\partial \phi^b} \\ &\times \ln \|\phi\| \frac{\partial \phi^c}{\partial k_i} \frac{\partial \phi^b}{\partial k_j} \mathrm{d}k^1 \mathrm{d}k^2. \end{aligned} \tag{22}$$

By defining the Jacobian determinant $D\left(\frac{\phi}{u}\right)$ as

$$\epsilon^{ij} D\left(\frac{\phi}{k}\right) = \epsilon^{ab} \frac{\partial \phi^i}{\partial k_a} \frac{\partial \phi^j}{\partial k_b} \tag{23}$$

and making use of the Laplacian relation in ϕ-space

$$\frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi^i} \ln \|\phi\| = 2\pi \delta^2(\phi),$$

we rewrite Eq.(21) in a compact form

$$\Delta\sigma^a_{xy} = \frac{e^2}{h} \sum_{N\sigma} \int \mathrm{d}^2 r \rho_{N\sigma} \int_{T^2} D\left(\frac{\phi}{k}\right) \delta^2(\phi) \mathrm{d}k^1 \mathrm{d}k^2, \tag{24}$$

which shows that only those points, at which $\boldsymbol{\phi} = 0$, contribute to $\Delta\sigma^a_{xy}$.

Suppose that the vector field $\phi^i(\boldsymbol{k})$ possesses N zeroes on T^2 and let the i-th zero be $\boldsymbol{k} = z_i$

$$\phi^i(z_i) = 0.$$

According to the derivation of Duan and Liu[9] and the implicit function theorem, the δ-function $\delta^2(\phi)$ can be expanded as[10]

$$\delta^2(\phi) D\left(\frac{\phi}{k}\right) = \sum_{l=1}^{N} \beta_l \eta_l \delta^2(\boldsymbol{k} - z_l).$$

The positive integer β_l is called the Hopf index of the map $\boldsymbol{k} \longrightarrow \boldsymbol{\phi}$,[11] which means that when the point $\boldsymbol{k}$ covers the neighborhood of the zero $\boldsymbol{k} = z_l$ once, the function ϕ^i covers the corresponding region in ϕ-space β_l times

$$\eta_l = \left. \frac{D\left(\frac{\phi}{k}\right)}{\left|D\left(\frac{\phi}{k}\right)\right|} \right|_{\boldsymbol{k}=z_l} = \pm 1$$

is called the Brouwer degree of the map $\boldsymbol{k} \longrightarrow \boldsymbol{\phi}$. Substituting this expansion of $\delta^2(\phi)$ into Eq.(23), the Hall conductance is quantized at the topological level as

$$\Delta\sigma^a_{xy} = \frac{e^2}{h} \sum_{l=1}^{N} \beta_l \eta_l. \tag{25}$$

Therefore, the topological structure of the Hall conductance density is labeled by the Hopf index β_i and the Brouwer degree η_i. On the other hand,

$$W_l = \beta_l \eta_l$$

is just another definition of the winding number of the Gauss map $\boldsymbol{n}: T^2 \longrightarrow S^1$.[8] In topology, it means that, when the point $\boldsymbol{k}$ covers the neighborhood of the zero $\boldsymbol{k} = z_l$ once, the unit vector $\boldsymbol{n}$ covers the corresponding region in S^1 W_l times. Then the Hall conductance is

$$\sigma^a_{xy} = \frac{e^2}{h} + \frac{e^2}{h} \sum_{l=1}^{N} W_l. \tag{26}$$

The above discussions are very valuable to establish the theory of the Hall conductance topological current and determine its evolution.

From Eq.(23) the topological charge can be expressed as

$$G = \Delta\sigma^a_{xy} = \frac{e^2}{h} \deg\boldsymbol{\phi}, \tag{27}$$

where $\deg\boldsymbol{\phi}$ is the degree of the mapping ϕ.

Denoting the sum of β_l with $\eta_l = 1$ and $\eta_l = -1$ as W^+ and W^-, respectively, the Hall conductance can be rewritten as

$$\Delta\sigma^a_{xy} = \frac{e^2}{h}(W^+ - W^-), \tag{28}$$

which reveals the contribution of the current with different current-carrying properties. Since the zero of the wave function at $\boldsymbol{k} = z_l$ gives a vortex-like structure, expression (14) gives the vorticity in the magnetic Brillouin zone. So β^+ and β^- denote the vorticity of the vortex and antivortex. Therefore, the vortex and antivortex classify different types of current with positive or negative charges. The Hall conductance

$\Delta\sigma^a_{xy}$ is obtained as the sum of the vorticity in the magnetic Brillouin zone.[5] Also it can be shown that the Chern number of a subband does not change unless the subband merges with a neighboring subband. In this case, the Chern number of the newly formed band equals the sum of the Chern numbers of the two subbands. If a band split into one or more subbands, its Chern number is redistributed over the two or more subbands.

3. The Hall conductance topological current and its evolution

The general Hall conductance topological current and its evolution are first given by Duan and Li.[7] By using this method, we can discuss the Hall conductance topological current and discuss its evolution.

Let us consider the (2+1)-dimensional space-time $T^2 \times R$ with coordinate $k_0 = t \in R$ denoted as a time variable. A generally covariant Hall topological current is defined as

$$j^i = \frac{e^2}{h}\frac{1}{2\pi}\epsilon_{a_1a_2}\frac{\epsilon^{ijk}}{\sqrt{g}}\frac{\partial}{\partial k^j}n^{a_1}\frac{\partial}{\partial k^k}n^{a_2}, \qquad (29)$$
$$i,j,k = 0,1,2,$$

where j^0 is just the Hall conductance density. Define Jacobians determinants $D^i\left(\frac{\phi}{k}\right)$ as

$$\epsilon^{ab}D^i\left(\frac{\phi}{k}\right) = \epsilon^{ijk}\frac{\partial\phi^a}{\partial k_j}\frac{\partial\phi^b}{\partial k_k}, \qquad (30)$$

in which $D^0\left(\frac{\phi}{x}\right) = D\left(\frac{\phi}{x}\right)$ is the usual two-dimensional Jacobian. Making use of the Laplacian relation, we obtain the δ-function-like current

$$j^i = \frac{e^2}{h}\delta(\phi)D^i\left(\frac{\phi}{k}\right). \qquad (31)$$

Supposing that $\phi_a(k)$ possesses N isolated zeroes on M and let the l-th zero be $\boldsymbol{k} = z_l$ we have

$$\phi^a(z_l,t) = 0 \quad (l = 1,2,\ldots,N; a = 1,2), \qquad (32)$$

whose solutions can be expressed as[12]

$$z_l = z_l(t), \qquad (33)$$

which is the trajectory of the l-th zero. From Eq.(31) we can prove that the general velocity of the l-th zero,[13]

$$V^j_l = \frac{\mathrm{d}z^j_l}{\mathrm{d}t} = \left.\frac{D^j\left(\frac{\phi}{k}\right)}{D\left(\frac{\phi}{k}\right)}\right|_{x=z_l}, \quad V^0_l = 1. \qquad (34)$$

Then the Hall topological current j^i can be written in the form of the current density of the system of N classical point particles with topological charge $G_l = \frac{e^2}{h}\eta_l\beta_l$ moving in the (2+1)-dimensional space-time.

$$j^i = \sum_{i=1}^{N} G_l\delta(k - z_l(t))\frac{\mathrm{d}z^i_l}{\mathrm{d}t},$$
$$j^0 = \sum_{i=1}^{N} G_l\delta(k - z_l(t)). \qquad (35)$$

The total charge of the system, i.e. the Hall conductance, is

$$G = \Delta\sigma^a_{xy} = \int_M j^0\sqrt{g}\mathrm{d}^2k$$
$$= \frac{e^2}{h}\sum_{i=1}^{N}\beta_l\eta_l = \sum_{i=1}^{N}G_l. \qquad (36)$$

From Eq.(30) we get a concise expression for the topological current,

$$j^i = j^0\frac{D^i\left(\frac{\phi}{k}\right)}{D\left(\frac{\phi}{k}\right)} = j^0V^i, \qquad (37)$$

which takes the same form as the current density in hydrodynamics. Expressions (34)–(36) give the topological structure of the Hall conductance topological current, which is characterized by the Brouwer degrees and Hopf indices. In our theory the point-like particles with topological charges $G_l = \frac{e^2}{h}\beta_l\eta_l$ are called Hall topological particles, the charges of which are topologically quantized and these particles are just located at the zeros of $\boldsymbol{\phi}(\boldsymbol{k})$, i.e. the singularities of the unit vector $\boldsymbol{n}(\boldsymbol{k})$.

The above discussion is based on the condition that the Jacobian

$$\left.D\left(\frac{\phi}{k}\right)\right|_{z_l} \neq 0. \qquad (38)$$

When $\left.D\left(\frac{\phi}{k}\right)\right|_{z_l} = 0$ it is shown that there is the crucial branch points, namely, limit points and bifurcation points, which will be discussed in the same way.[14] We just give the results briefly.

Firstly, we study the case when the zeros of $\boldsymbol{\phi}$ include some limit points. The limit points are determined by

$$\begin{aligned}&\phi^1(k^1,k^2,t)=0,\\&\phi^2(k^1,k^2,t)=0,\\&\phi^3(k^1,k^2,t)=D\left(\frac{\phi}{k}\right)=0,\end{aligned}$$

and

$$D^A\left(\frac{\phi}{k}\right)\Big|_{(z_l,t*)}\neq 0,\quad A=1,2. \tag{39}$$

We have a unique solution of Eq.(38) in the neighborhood of the limit point (t^*,z_l),

$$t=t(k^1),\quad k^2=k^2(k^1) \tag{40}$$

with $t^*=t(z_l^1)$.

$$t-t^*=\frac{1}{2}\frac{\mathrm{d}^2t}{(\mathrm{d}k^1)^2}\Big|_{(t^*,z_l)}(k^1-z_l^1)^2. \tag{41}$$

If $\frac{\mathrm{d}^2t}{(\mathrm{d}k^1)^2}\big|_{(t^*,z_l)}>0$, we have the branch solutions for $t>t^*$, otherwise, we have the branch solutions for $t<t^*$. The former is related to the origin of the Hall topological charges and the latter is related to the annihilation of the Hall topological charges.

Since the topological current of Hall conductance density is identically conserved, the topological quantum numbers of these two generated or annihilated Hall topological charges must be opposite at the limit point, i.e.

$$\beta_1\eta_1+\beta_2\eta_2=0 \tag{42}$$

which shows that the limit points do not contribute to the total Hall conductance.

Now, let us turn to the other case, in which the restrictions are

$$D\left(\frac{\phi}{k}\right)\Big|_{(z_l,t^*)}=0,\quad D^1\left(\frac{\phi}{k}\right)\Big|_{(z_l,t^*)}=0. \tag{43}$$

These two restrictive conditions will lead to the important fact that the function relationship between t and k^1 is not unique in the neighborhood of (z_l,t^*). In our Hall topological current theory this fact is easily seen from Eq.(33),

$$V_l^1=\frac{\mathrm{d}z_l^1}{\mathrm{d}t}=\frac{D^1\left(\frac{\phi}{k}\right)}{D\left(\frac{\phi}{k}\right)}\Big|_{(z_l,t^*)} \tag{44}$$

which under Eq.(42) directly shows that the direction of the integral curve of Eqs.(43) is indefinite at (z_l,t^*). Therefore, the very point (z_l,t^*) is called a bifurcation point of the Hall topological current. With the aim of finding the different directions of all branch curves at the bifurcation point, we suppose that

$$\frac{\partial\phi^1}{\partial k^2}\Big|_{(z_l,t^*)}\neq 0. \tag{45}$$

From $\phi^1(k^1,k^2,t)=0$, the implicit function theorem says that there exists one and only one function relationship $k^2=k^2(k^1,t)$. Making use of the same method in Ref.[14], we get

$$A\left(\frac{\mathrm{d}k^1}{\mathrm{d}t}\right)^2+2B\frac{\mathrm{d}k^1}{\mathrm{d}t}+C=0 \tag{46}$$

and

$$C\left(\frac{\mathrm{d}t}{\mathrm{d}k^1}\right)^2+2B\frac{\mathrm{d}t}{\mathrm{d}k^1}+A=0. \tag{47}$$

The solutions of Eqs.(45) or (46) give the different directions of the branch at the bifurcation point. The remainder component $\frac{\mathrm{d}k^2}{\mathrm{d}t}=f_1^2\frac{\mathrm{d}k^1}{\mathrm{d}t}+f_t^2$,[15] where the partial derivatives are

$$f_1^2=\frac{\partial k^2}{\partial k^1},\quad f_t^2=\frac{\partial k^2}{\partial t}. \tag{48}$$

Now we have all the different directions of the branch curves. This means the behavior of the Hall topological current at the bifurcation points is determined. The detailed discussion of branch can be found in Ref.[15].

The above solutions show the evolution of the Hall conductance topological current. Besides the generation and the annihilation of the Hall topological charges, it also includes the splitting and merging of Hall topological current. When an original Hall topological current moves through the bifurcation point in k-space, it may split into two Hall topological currents moving along different branch curves or two original Hall topological currents merge into one Hall topological current at the bifurcation points. The identical argument of the Hall topological current says the sum of the topological quantum numbers of these two split Hall topological currents must be equal to that of the original current at the bifurcation point, i.e.

$$\beta_{l_1}\eta_{l_1}+\beta_{l_2}\eta_{l_2}=\beta_l\eta_l \tag{49}$$

for fixed l. Furthermore, from the above, we see that the generation, annihilation and bifurcation of the space-time Hall topological current are not gradual changes, but start at a critical value of the arguments, i.e. a sudden change.

4. Conclusion

In this paper, with the gauge potential decomposition and the so-called ϕ-mapping method, we have obtained the inner topological structure of $\Delta\sigma^a_{xy}$ in magnetic semiconductor-superconductor hybrids. The Hall conductance density is found to take the form of the generalized function $\delta(\phi)$, $\phi = (\phi^1, \phi^2)$ with ϕ^1, ϕ^2 the real part and the imaginary part of the magnetic wave function. The Hopf indices β_l and Brouwer degree η_l of the magnetic wave function shed light on the inner topological structure of the Hall conductance, which also bring light to the effect of vortices with different current-carrying property. When the Jacobian $D\left(\frac{\phi}{k}\right) = 0$, it is shown that it is the crucial branch process case. Based on the implicit function theorem and the Taylor expansion, the evolution of the Hall conductance $\Delta\sigma^a_{xy}$ is determined in the neighborhoods of the branch points of the ϕ-mapping. The branch solutions at the limit points and at the bifurcation points are studied. Because the Hall topological current is identically conserved, the total Hall conductance will be a constant during the evolution. In the limiting point case, this means that the topological quantum numbers of the two generated or annihilated Hall conductance density currents must be opposite at the limiting point. This can be looked upon as the topological origin of the Hall conductance. In the bifurcation case, the total charges of different branch Hall topological currents are invariant, which is the topological cause of the conservation law.

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