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Periodic, quasiperiodic and chaotic discrete breathers in a parametrical driven two-dimensional discrete diatomic Klein–Gordon lattice*

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We study a two-dimensional (2D) diatomic lattice of anharmonic oscillators with only quartic nearest-neighbor interactions, in which discrete breathers (DBs) can be explicitly constructed by an exact separation of their time and space dependence. DBs can stably exist in the 2D discrete diatomic Klein–Gordon lattice with hard and soft on-site potentials. When a parametric driving term is introduced in the factor multiplying the harmonic part of the on-site potential of the system, we can obtain the stable quasiperiodic discrete breathers (QDBs) and chaotic discrete breathers (CDBs) by changing the amplitude of the driver. But the DBs and QDBs with symmetric and anti-symmetric profiles that are centered at a heavy atom are more stable than at a light atom, because the frequencies of the DBs and QDBs centered at a heavy atom are lower than those centered at a light atom.

Keywords: discrete breather, quasi-periodic discrete breather, chaotic discrete breather, two-dimensional discrete diatomic Klein–Gordon lattice

PACC: 6320R, 6320P, 0547

1. Introduction

Localized oscillations in one-dimensional (1D) Hamiltonian lattices have been known to exist for a long time, and localized excitations in nonlinear lattice systems have attracted extensive attention during the last decade since the discovery of discrete breathers (DBs) or intrinsic localized modes (ILMs) by Sievers and Takeno.[1] DBs and ILMs are stable, time-periodic and space-localized excitations that exist and propagate in a nonlinear perfect lattice system. Their rigorous existence proofs for DBs in networks of weakly coupled anharmonic oscillators were given in Ref.[2]–[10]. Moreover, to obtain exact DBs various high precision numerical schemes were developed. However, it is difficult to tune onto the exact breather solution in any physical and ever numerical experiment and, therefore one should investigate the behavior of the solutions in the vicinity of a given discrete breather (DB) rather than the behavior of the DB itself. So there is a rapidly increasing interest in the investigation of the existence of the quasiperiodic and even chaotic discrete breathers in a nonlinear discrete system.

Quasiperiodic discrete breathers (QDBs) that are quasiperiodic in time and localized in space have been reported in Refs.[10]–[13]. CDBs were first indicated by Burlakov *et al*,[14,15] but the term "chaotic discrete breathers" was introduced by Creteny *et al.*[16] The typical features of CDBs can be described that the destabilization of the initial highest-frequency disturbances leads to a number of localized breather-like structures which gather the energy of the system, which then further merge into a few localized structures or a single one with higher energy. In addition, each localized structure is qualitatively similar to DBs, but moves irregularly. In most situations, CDBs finally break and damp out into a thermalization (equilibrium) state. The most evident difference between a DB and CDB is that the former is exactly periodic and mainly static, while the latter moves in an erratic (chaotic) way and mostly has a finite lifetime, thus, it was named a "chaotic discrete breather".

The investigation of the behavior of the QDBs and CDBs can play a significant role in the dynamical system. Some studies of the QDBs and CDBs

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have been presented in Refs.[10]–[16], but they are restricted to a simple 1D lattice such as the Fermi–Pasta–Ulam (FPU), the Klein–Gordon, and the discrete nonlinear Schrödinger ones. Now the properties of 2D nonlinear models are gradually catching peoples' attention.[17−20] Just as with DBs, the QDBs and CDBs can exist not only in a 1D system but also in a 2D system; therefore, the investigation of the behavior of the QDBs and CDBs in the 2D system has a great significance for the analysis of the 2D nonlinear dynamic system. Kousvke *et al*[21] investigated the situation of CDBs of two types in a 2D Morse monatomic lattice with an on-site harmonic potential. In this paper, we restrict our attention to the DBs, QDBs, and CDBs in a parametrical driven 2D discrete diatomic Klein–Gordon lattice.

2. The model

As Maniadis *et al*,[12] we consider a 2D discrete diatomic Klein–Gordon lattice of coupled nonlinear oscillators with two bands in the linear waves spectrum: a 2D face-centered square lattice consisting of alternating heavy (M) and light (m) atoms that interact with their nearest-neighbors, and without linear dispersion. It can be described by the Hamiltonian

$$H=\sum_{m,n}\left[\frac{1}{2}M\dot{u}^2_{2m-1,2n-1}+\frac{1}{2}m\dot{u}^2_{2m,2n}+V_1\left(u_{2m-1,2n-1}\right)+V_2\left(u_{2m,2n}\right)\right.$$
$$\left.+\frac{1}{4}\lambda\left(u_{2m,2n}-u_{2m-1,2n-1}\right)^4+\frac{1}{4}\lambda\left(u_{2m+1,2n+1}-u_{2m,2n}\right)^4\right], \tag{1}$$

where $u_{2m-1,2n-1}$ and $u_{2m,2n}$ denote the displacements of the $(2m-1,\,2n-1)$-th atom and $(2m,\,2n)$-th atom from their equilibrium positions, respectively, λ is the coupling constant. The $V_1(u_{2m-1,2n-1})$ and $V_2(u_{2m,2n})$ are on-site potentials, and in this paper they can be chosen to be in the forms as follows:

$$V_1\left(u_{2m-1,2n-1}\right)=\frac{1}{2}\alpha u^2_{2m-1,2n-1}+\frac{1}{4}\beta u^4_{2m-1,2n-1}, \tag{2a}$$
$$V_2\left(u_{2m,2n}\right)=\frac{1}{2}\alpha u^2_{2m,2n}+\frac{1}{4}\beta u^4_{2m,2n}, \tag{2b}$$

where α, β and K are constants. $\beta>0$ and $\beta<0$ correspond to the cases of the so-called "hard nonlinearity" and "soft nonlinearity", respectively. The Hamiltonian (1) yields the equation of motion

$$m\ddot{u}_{2m-1,2n-1}=-\alpha u_{2m-1,2n-1}-\beta u^3_{2m-1,2n-1}+\lambda\big[\left(u_{2m,2n}-u_{2m-1,2n-1}\right)^3+\left(u_{2m-2,2n-2}-u_{2m-1,2n-1}\right)^3$$
$$+\left(u_{2m,2n-2}-u_{2m-1,2n-1}\right)^3+\left(u_{2m-2,2n}-u_{2m-1,2n-1}\right)^3\big], \tag{3a}$$
$$M\ddot{u}_{2m,2n}=-\alpha u_{2m,2n}-\beta u^3_{2m,2n}+\lambda\big[\left(u_{2m+1,2n+1}-u_{2m,2n}\right)^3+\left(u_{2m-1,2n-1}-u_{2m,2n}\right)^3$$
$$+\left(u_{2m+1,2n-1}-u_{2m,2n}\right)^3+\left(u_{2m-1,2n+1}-u_{2m,2n}\right)^3\big]. \tag{3b}$$

3. The DBs, QDBs and CDBs

The solutions of Eqs.(3) can be written as a product of a purely spatial part and a function of time only as follows:

$$u_{2m-1,2n-1}=\psi_{2m-1,2n-1}G\left(t\right), \tag{4a}$$
$$u_{2m,2n}=\psi_{2m,2n}G\left(t\right), \tag{4b}$$

where $\psi_{2m-1,2n-1}$ and $\psi_{2m,2n}$ correspond to the spatial profiles of the solutions and $G\left(t\right)$ describes its time evolution. This can be easily seen by substituting Eqs.(4) into Eqs.(3) and writing it as an equality between an expression that depends only on (n,m) and one that is solely time-dependent. Equating both of these expressions to a constant C, one thus finds that $G\left(t\right)$ satisfies the Duffing equation

$$m\ddot{G}\left(t\right)+\alpha G\left(t\right)=-CG\left(t\right)^3, \tag{5a}$$
$$M\ddot{G}\left(t\right)+\alpha G\left(t\right)=-CG\left(t\right)^3, \tag{5b}$$

while the spatial part of the solution obeys the following equations

$$C\psi_{2m-1,2n-1} - \beta\psi^3_{2m-1,2n-1} + \lambda\Big[(\psi_{2m,2n} - \psi_{2m-1,2n-1})^3 + (\psi_{2m-2,2n-2} - \psi_{2m-1,2n-1})^3 + (\psi_{2m,2n-2} - \psi_{2m-1,2n-1})^3 + (\psi_{2m-2,2n} - \psi_{2m-1,2n-1})^3\Big] = 0, \tag{6a}$$

$$C\psi_{2m,2n} - \beta\psi^3_{2m,2n} + \lambda\Big[(\psi_{2m+1,2n+1} - \psi_{2m,2n})^3 + (\psi_{2m-1,2n-1} - \psi_{2m,2n})^3 + (\psi_{2m+1,2n-1} - \psi_{2m,2n})^3 + (\psi_{2m-1,2n+1} - \psi_{2m,2n})^3\Big] = 0. \tag{6b}$$

Above algebraic equations can be solved by using a numerical method, and the numerical result can be shown in Fig.1. Because equation (6a) has the same form as Eq.(6b), they have the same type of solution.

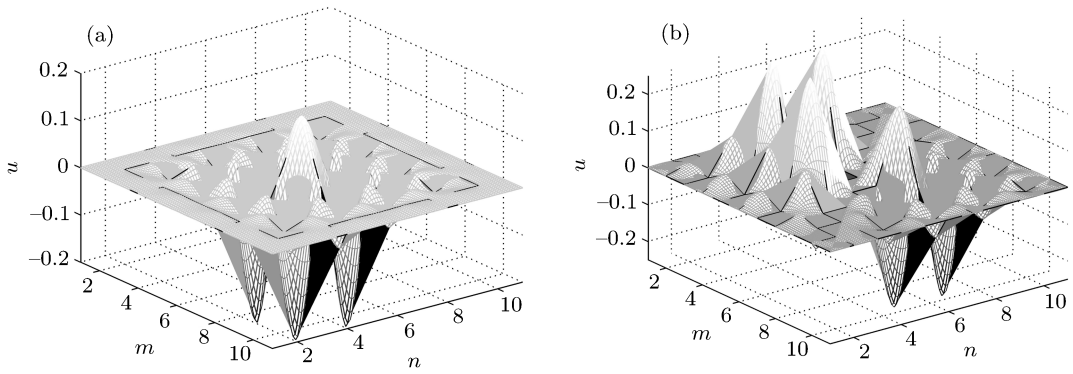


Fig.1. The breather profiles $\psi_{2m-1,2n-1}$ and $\psi_{2m,2n}$ for $C = 1, \beta = 1$ and λ=0.1. (a) Symmetric DB, (b) antisymmetric DB.

Figure 1 shows the symmetric and antisymmetric DBs profiles of the 2D discrete diatomic Klein–Gordon lattice with hard on-site potential. The exact breather solutions are then calculated as the products of these profiles with the periodic solution of Eqs.(5) having the desired frequency ω_b. For $C = 1$, equations (5) have periodic solutions for all initial displacements $G(0)$ and $\dot{G}(0)$, and the frequency of a periodic solutions is larger than the equilibrium frequency of the potential [$\omega_b > \omega_0 = V''(0) = \alpha$, here let $\alpha = 1$]. We solve Eqs.(5) by the numerical method, and the phase plane of this Duffing equation and the evolution of $G(t)$ with time can be shown in Fig.2.

From Fig.2(a) we know that equations (5) have a periodic solution, and from Figs.2(b) and 2(c) we know that the periodic solution can stably exist for longer times. Hence the 2D DBs can stably exist in the 2D Klein–Gordon discrete diatomic lattice with the hard on-site potential. We also investigate the condition of this with the soft on-site potential, and there are none of the stable DBs in it. But we know that the DBs with symmetric profiles centered at a heavy atom are more stable than at a light atom by comparing Fig.2(b) with Fig.2(c). The frequencies of the DBs centered at a heavy atom are lower than those centered at a light atom. We also investigate the conditions of the DBs with anti-symmetric profiles, and obtain the same result.

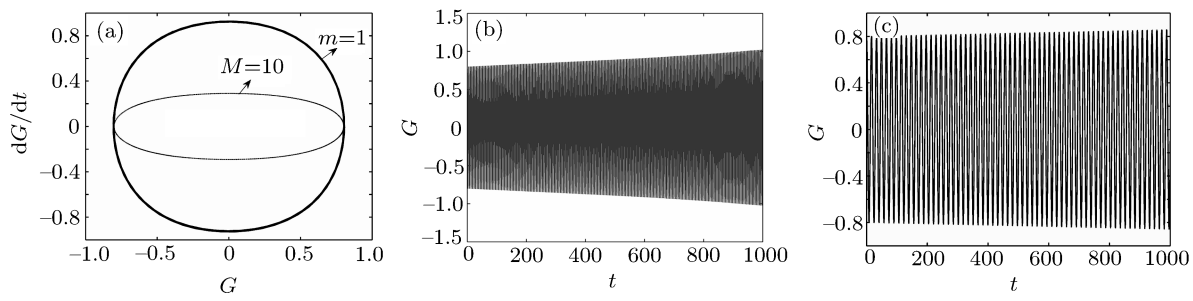


Fig.2. The numerical results of Eqs.(5) for $\alpha = 1$ and $C = 1$, (a) the phase portrait of the periodic solution of Eqs.(5); (b) the solution curve of Eq.(5a) with $m = 1$; (c) the solution curve of Eq.(5b) with $M = 10$.

In order to obtain the evolution from DBs to QDBs, eventually to CDBs, we introduce a parametric driving term in the factor multiplying the harmonic part of the on-site potential of all oscillators, then equations (3) can be rewritten as

$$\begin{aligned} m\ddot{u}_{2m-1,2n-1} = &-\alpha\left[1-\varepsilon\cos\left(\omega_d t\right)\right]u_{2m-1,2n-1} - \beta u_{2m-1,2n-1}^3 + \lambda\big[\left(u_{2m,2n}-u_{2m-1,2n-1}\right)^3 \\ &+\left(u_{2m-2,2n-2}-u_{2m-1,2n-1}\right)^3 + \left(u_{2m,2n-2}-u_{2m-1,2n-1}\right)^3 \\ &+\left(u_{2m-2,2n}-u_{2m-1,2n-1}\right)^3\big], \end{aligned} \tag{7a}$$

$$\begin{aligned} M\ddot{u}_{2m,2n} = &-\alpha\left[1-\varepsilon\cos\left(\omega_d t\right)\right]u_{2m,2n} - \beta u_{2m,2n}^3 + \lambda\big[\left(u_{2m+1,2n+1}-u_{2m,2n}\right)^3 \\ &+\left(u_{2m-1,2n-1}-u_{2m,2n}\right)^3 + \left(u_{2m+1,2n-1}-u_{2m,2n}\right)^3 \\ &+\left(u_{2m-1,2n+1}-u_{2m,2n}\right)^3\big], \end{aligned} \tag{7b}$$

where ε and ω_d represent the amplitude and frequency of the driver. This type of parametric driving can be experimentally implemented, and it does not affect the mathematical property of having solutions of the type (4). Thus separating again the space and time-dependent parts as before, the Duffing equations (5) now become

$$m\ddot{G}\left(t\right) + \alpha\left[1-\varepsilon\cos\left(\omega_d t\right)\right]G\left(t\right) = -CG\left(t\right)^3, \tag{8a}$$

$$M\ddot{G}\left(t\right) + \alpha\left[1-\varepsilon\cos\left(\omega_d t\right)\right]G\left(t\right) = -CG\left(t\right)^3, \tag{8b}$$

while the spatial of the solution can again be constructed independently by solving the identical Eqs.(6). Thus we can study breather solutions of our lattice by selecting the phase planes of $G\left(t\right)$ and $\dot{G}\left(t\right)$ with the different amplitudes of the driver and combining the corresponding solution $G\left(t\right)$ with the spatial profiles $\psi_{2m-1,2n-1}$ and $\psi_{2m,2n}$ given by Eqs.(6). Figure 3 shows the numerical results of Eqs.(8).

From Figs.3(a) and 3(b) we can conclude that the QDBs can stably exist for very long time in this system with the patters centered at a light atom, while the localized modes with the patters centered at a heavy atom are still the stable DBs, and it further demonstrates that the localized modes with the patters centered at a heavy atom are more stable than at a light atom.

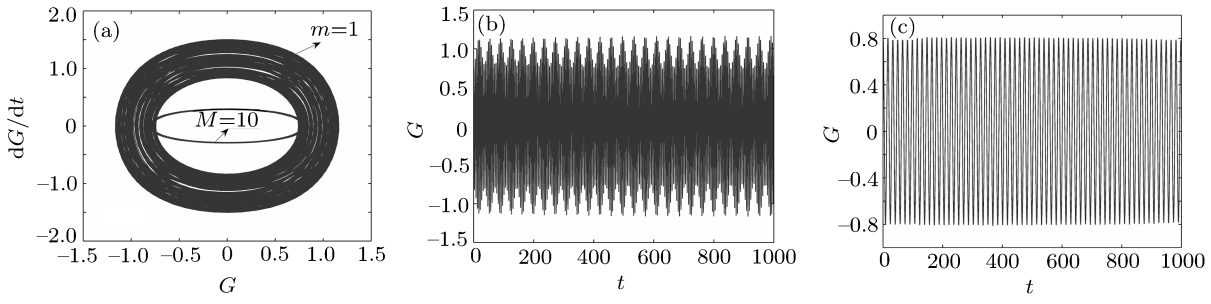


Fig.3. The numerical results of Eqs.(8) for $\alpha = 1$, $C = 1$ and $\varepsilon = 0.2$, (a) the phase portrait of the periodic and quasi-periodic solutions of Eqs.(8); (b) the solution curve of Eq.(8a) with $m = 1$; (c) the solution curve of Eq.(8b) with $M = 10$.

When $\varepsilon = 1.3$ the numerical results of Eqs.(8) are shown in Fig.4.

From Fig.4 we can obtain that the QDBs in this system with the patters centered at a light atom change to CDBs, when ε increases to ε=1.3, while the DBs with the patters centered at a heavy atom change to the stable QDBs, and it further demonstrates that the localized modes with the patters centered at a heavy atom are more stable than at a light atom.

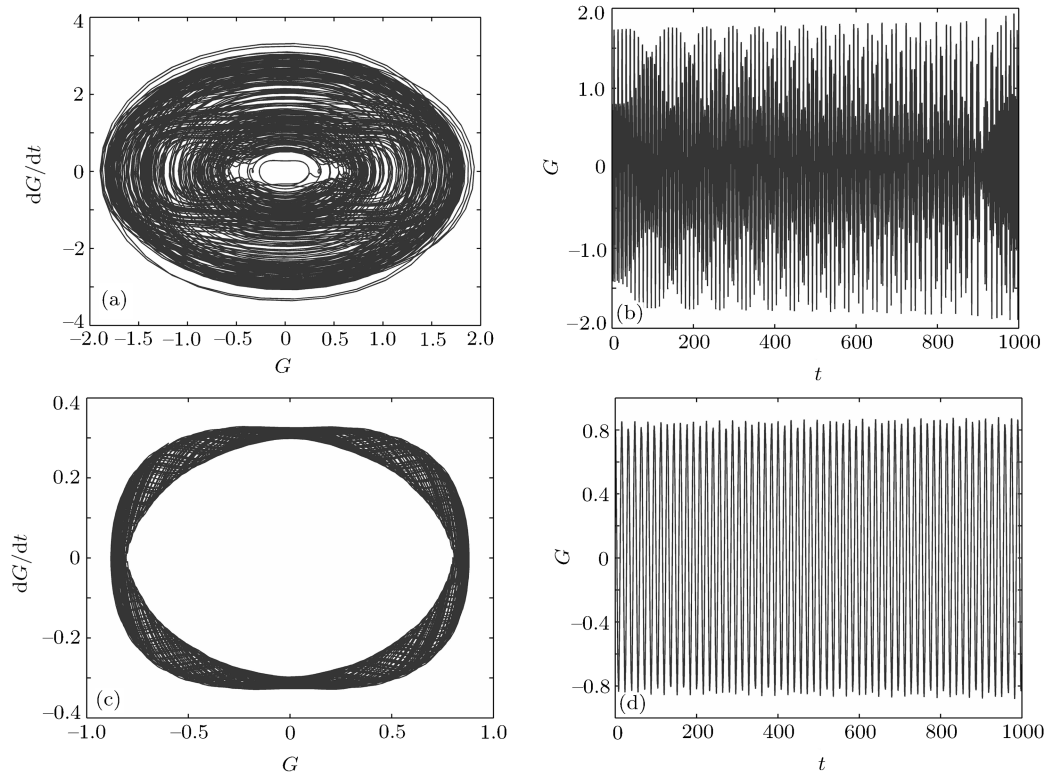


Fig.4. The numerical results of Eqs.(8) for $\alpha = 1$, $C = 1$ and $\varepsilon = 1.3$, (a) The phase portrait of the chaotic solution of Eq.(8a); (b) the solution curve of Eq.(8a) with $m = 1$; (c) the phase portrait of the quasi-periodic solution of Eq.(8b); (d) the solution curve of Eq.(8b) with $M = 10$.

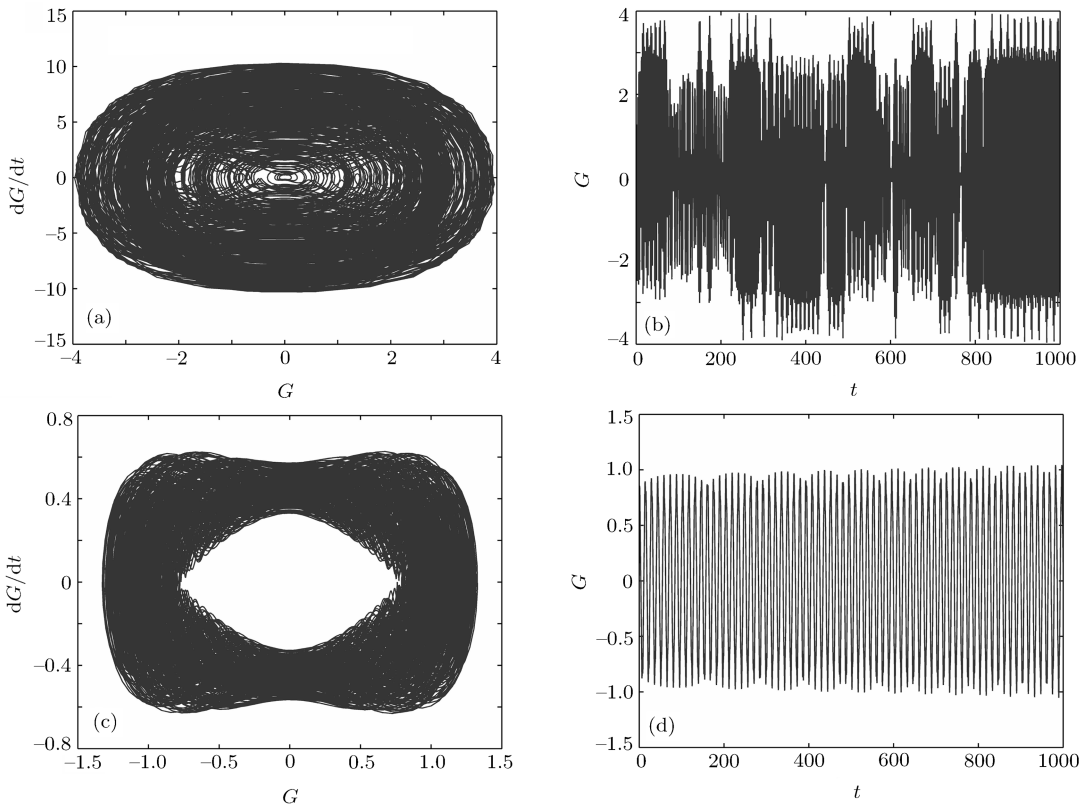


Fig.5. The numerical results of Eqs.(8) for $\alpha = 1$, $C = 1$ and $\varepsilon = 2.2$. (a) The phase portrait of the completely chaotic solution of Eq.(8a); (b) the solution curve of Eq.(8a) with $m = 1$; (c) the phase portrait of the quasi-periodic solution of Eq.(8b); (d) the solution curve of Eq.(8b) with $M = 10$.

When the ε further increases to $\varepsilon = 2.2$, the CDBs in this system with the patters centered at a light atom change to completely CDBs, while the localized modes centered at a heavy atom are still stable QDBs.

4. Conclusions

In this paper, we investigate a 2D discrete diatomic Klein–Gordon lattice of coupled nonlinear oscillators with two bands in the linear waves spectrum: a 2D face-centered square lattice consisting of alternating heavy (M) and light (m) atoms that interact with their nearest-neighbors, and without linear dispersion. We obtain the following conclusions. First, the 2D DBs can stably exist in the 2D Klein–Gordon discrete diatomic lattice with the hard on-site potential. We can also obtain the QDBs and CDBs by introducing a parametric driving term in the factor multiplying the harmonic part of the on-site potential of all oscillators. Second, the stabilities of the localized modes become weak, when ε increases. Nevertheless, the localized modes with symmetric profiles that are centered at a heavy atom are more stable than at a light atom, because the frequencies of the localized modes centered at a heavy atom are lower than those centered at a light atom. That is the localized modes with lower frequency are more stable than those with higher frequency. Besides, we also investigate the conditions of the anti-symmetric profiles, and obtain the same result. This is the difference from the conditions in a 1D system, because the anti-symmetric modes in a 2D system have symmetries, too. Hence, the anti-symmetric modes in a 1D system are unstable, while those in a 2D system are stable. Finally, the localized modes in the system with soft on-site potential are unstable, because the interaction of this system is attractive ($\lambda = 1 > 0$). When the interaction of the system becomes repulsive, the results can be the same as the conditions of the models with a hard on-site potential and the attractive interaction.

Certainly, this system is a very particular case with no linear dispersion, so there are no resonances between QDBs and normal modes, and QDBs can exist for very long times, because the resonances can lead to a radiation of energy outside their core when certain superpositions of incommensurate frequencies, corresponding to these QDBs, get inside the phonon band. Naturally, this process causes a decay of the QDBs as localized dynamical objects. However, in many cases, this decay can be weak enough and, therefore, QDBs can exist for a very long time, and this method introducing the driving parameter has the realistic meaning. If some excitations exist during a very long time, they must have a certain physical meaning and should be worth thoroughly investigating. The QDBs represent an example of such dynamical objects and, actually, they are much more relevant entities than strictly time-periodic breathers.

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