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Electric fields inside and outside an anisotropic dielectric sphere*

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Analytical expressions of electric fields inside and outside an anisotropic dielectric sphere are presented by transforming an anisotropic medium into an isotropic one based on the multi-scale transformation of electromagnetic theory. The theoretical expressions are consistent with those in the literature. The inside electric field, the outside electric field and the angle between their directions are derived in detail. Numerical simulations show that the direction of the outside field influences the magnitude of the inside field, while the dielectric constant tensor greatly affects its direction.

Keywords: anisotropic medium, scale transformation, electric field
PACC: 5235H, 4110H, 9580D

1. Introduction

The electromagnetic (EM) scattering features and their applications of spherical targets have been widely studied. The electric fields inside and outside a single isotropic dielectric sphere have been investigated.[1−4] The scattering features of an isotropic dielectric sphere and a conducting sphere, which are irradiated by an EM wave propagating in the z-direction and polarizing in the x-direction, have been studied.[5,6] The scattering properties of spherical targets that are irradiated by a wave from an arbitrary direction was further studied in Ref.[6]. In Refs.[7]–[10], the EM scattering features for an isotropic dielectric ellipsoidal target and the power to seize radiation for a coated sphere in the Gaussian beam were separately investigated by the well-known Mie theory. In a word, the subjects of electromagnetic scattering and their applications of an isotropic dielectric sphere have been discussed in detail, but they have not been fully understood in theory. The main reason for this is the lack of analytical expression of electric field inside an anisotropic dielectric sphere irradiated by a wave from an arbitrary direction. In reality, many particles are much smaller than the wave length in size, such as raindrops, sand-dust storm particulates, fog droplets, etc. Therefore the problem can be approximately considered as an electrostatic one.[3] If the inside electric field is known, the absorption cross section and the scattering cross section of the sphere will be obtained accurately.[11] In the present paper, the expression of the electric field inside an anisotropic dielectric spherical target is presented in detail based on the scale transformation theory of the electromagnetic field. The angle between the field inside the sphere and that outside the sphere is calculated. The influence induced by the dielectric constant tensor on the direction of the inner field is simulated. The method used in the present paper has the features of briefness in computation and distinctness in physical significance.

2. Electric fields inside and outside an anisotropic dielectric sphere

2.1. Foundation of the differential equation

Assume an anisotropic dielectric sphere to have radius R_0 and its centre to be located at the origin of the primary coordinate system Σ. The dielectric constant tensor of this sphere is given as

$$\varepsilon = \varepsilon_0 \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}, \tag{1}$$

where ε_1, ε_2 and ε_3 are the constants. The electric

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potential u inside the anisotropic dielectric sphere can be easily obtained from the following differential equation:

$$\varepsilon_1 \frac{\partial^2 u}{\partial x^2} + \varepsilon_2 \frac{\partial^2 u}{\partial y^2} + \varepsilon_3 \frac{\partial^2 u}{\partial z^2} = 0. \tag{2}$$

Now, a scale coordinate system Σ' is introduced as a new coordinate system. The coordinates of this system are indicated with x', y' and z'. The relation of coordinates between the two systems is written as

$$x' = \frac{x}{\sqrt{\varepsilon_1}}, \quad y' = \frac{y}{\sqrt{\varepsilon_2}} \text{ and } z' = \frac{z}{\sqrt{\varepsilon_3}}.$$

The differential equation of the potential in the scale coordinate system is derived by substituting the above expressions into Eq.(2) and using the condition $u = u'$[12,13] at any spatial point, and it is expressed as

$$\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2} = 0. \tag{3}$$

The condition $u = u'$ is understandable, for the potential is defined as the work done by the electric field to move a unit charge from one point to the reference point, namely W/q, so both the numerator and the denominator are scale invariants. Equation (3) shows that an anisotropic dielectric sphere in the primary coordinate system is transformed into an isotropic dielectric sphere in the scale coordinate system. This manipulation can greatly simplify the electromagnetic scattering problems.

2.2. Expressions of electric fields outside and inside an anisotropic dielectric sphere

The solution of Eq.(3) can be obtained by using the method of separation of variables as follows:

$$u'(R', \theta', \varphi') = \sum_{m,n} a_{m,n} R'^n P_n^m (\cos\theta') \cos m\varphi' + \sum_{m,n} c_{m,n} R'^n P_n^m (\cos\theta') \sin m\varphi'. \tag{4}$$

Expression (4) is a general solution in the scale coordinate system. The parameters in the two coordinate systems are related through the following relationships:[12,13]

$$R' = Rq,$$

$$q = \left(\frac{\sin^2\theta\cos^2\varphi}{\varepsilon_1} + \frac{\sin^2\theta\sin^2\varphi}{\varepsilon_2} + \frac{\cos^2\theta}{\varepsilon_3} \right)^{1/2},$$

$$\sin\theta' = \frac{\sin\theta g}{q}, \quad g = \left(\frac{\cos^2\varphi}{\varepsilon_1} + \frac{\sin^2\varphi}{\varepsilon_2} \right)^{1/2},$$

$$\cos\theta' = \frac{\cos\theta/\sqrt{\varepsilon_3}}{q}, \quad \sin\varphi' = \frac{\sin\varphi/\sqrt{\varepsilon_2}}{g},$$

$$\cos\varphi' = \frac{\cos\varphi/\sqrt{\varepsilon_1}}{g},$$

and

$$u(R, \theta, \varphi) = u'(R', \theta', \varphi').$$

In expression (4), the term that represents a finite potential in the sphere centre is considered. It is concluded that the expression of potential in the primary coordinate system can be obtained by substituting the relations above into expression (4). By utilizing the relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ and the relation between the vectors in right angle system and spherical system, we can also obtain the expression of the dielectric constant tensor in spherical coordinate system as

$$\varepsilon = \varepsilon_0 \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}, \tag{5}$$

where

$$\begin{aligned}
\varepsilon_{11} &= \varepsilon_3 \cos^2\theta + \varepsilon_1 \sin^2\theta \cos^2\varphi + \varepsilon_2 \sin^2\theta \sin^2\varphi, \\
\varepsilon_{12} &= -\varepsilon_3 \cos\theta \sin\theta + \varepsilon_1 \cos\theta \sin\theta \cos^2\varphi \\
&\quad + \varepsilon_2 \cos\theta \sin\theta \sin^2\varphi, \\
\varepsilon_{13} &= (\varepsilon_2 - \varepsilon_1) \cos\varphi \sin\theta \sin\varphi, \\
\varepsilon_{22} &= \varepsilon_3 \sin^2\theta + \varepsilon_1 \cos^2\theta \cos^2\varphi + \varepsilon_2 \cos^2\theta \sin^2\varphi, \\
\varepsilon_{23} &= (\varepsilon_2 - \varepsilon_1) \cos\varphi \cos\theta \sin\varphi, \\
\varepsilon_{33} &= \varepsilon_1 \sin^2\varphi + \varepsilon_2 \cos^2\varphi, \\
\varepsilon_{12} &= \varepsilon_{21}, \quad \varepsilon_{13} = \varepsilon_{31}, \quad \text{and} \quad \varepsilon_{32} = \varepsilon_{23}.
\end{aligned}$$

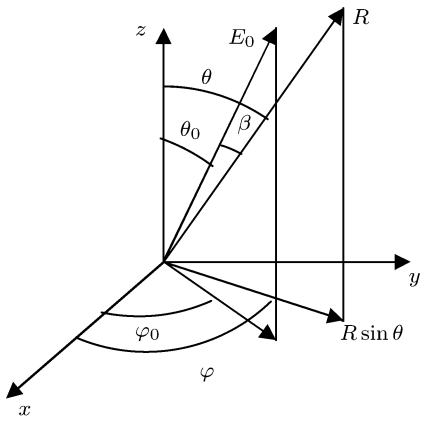


Fig.1. Relation between the observing point and the outside electric field $\boldsymbol{E}_0$.

We suppose that E_0 is the magnitude of incident electric field, and that θ_0 and ϕ_0 are its directional parameters in the primary coordinate system. As shown in Fig.1, since the outside electric potential is not symmetrical, we can write the potential as[2]

$$u_1(R,\theta,\varphi)=\sum_{m,n}\left(e_{m,n}R^n+\frac{f_{m,n}}{R^{n+1}}\right)P_n^m(\cos\theta)\cos m\varphi+\sum_{m,n}\left(g_{m,n}R^n+\frac{h_{m,n}}{R^{n+1}}\right)P_n^m(\cos\theta)\sin m\varphi. \quad (6)$$

When $R\to\infty$, the electric potential

$$u_1\to -RE_0\cos\beta=-RE_0\left[\sin\theta\sin\theta_0\cos(\varphi-\varphi_0)+\cos\theta\cos\theta_0\right].$$

Comparin the coefficients of the above expression with those in expression (6) yields

$$\begin{aligned}u_1(R,\theta,\varphi)=&AR\cos\theta+BR\sin\theta\cos\varphi+DR\sin\theta\sin\varphi\\&+\sum_{m,n}\frac{f_{m,n}}{R^{n+1}}P_n^m(\cos\theta)\cos m\varphi+\sum_{m,n}\frac{h_{m,n}}{R^{n+1}}P_n^m(\cos\theta)\sin m\varphi,\end{aligned}$$

where $A=e_{01}=-E_0\cos\theta_0$, $B=e_{11}=-E_0\sin\theta_0\cos\varphi_0$, $D=g_{11}=-E_0\sin\theta_0\sin\varphi_0$,

$$e_{m,n}=0,\quad m\neq 0,1\quad n\neq 1\quad\text{and}\quad g_{m,n}=0,\quad m\neq 1,\quad n\neq 1.$$

Expressions (4) and (6) are the electric potentials inside and outside the anisotropic dielectric sphere respectively. On the surface of the sphere, the electric potential inside the sphere is equal to that outside the sphere and the electric displacement $\boldsymbol{D}_0$ is continuous in the normal direction, namely

$$u|_{R=R_0}=u_1|_{R=R_0},\quad\text{and}\quad \varepsilon_0\left(\varepsilon_{11}\frac{\partial u}{\partial R}+\varepsilon_{12}\frac{1}{R}\frac{\partial u}{\partial\theta}+\varepsilon_{13}\frac{1}{R\sin\theta}\frac{\partial u}{\partial\varphi}\right)\Big|_{R=R_0}=\varepsilon_0\frac{\partial u_1}{\partial R}\Big|_{R=R_0}. \quad (7)$$

Inserting expressions (4), (5) and (6) into the above conditions yields

$$u(R,\theta,\varphi)=\frac{3A}{2+\varepsilon_3}R\cos\theta+\frac{3B}{2+\varepsilon_1}R\cos\varphi\sin\theta+\frac{3D}{2+\varepsilon_2}R\sin\varphi\sin\theta, \quad (8)$$

and

$$\begin{aligned}u_1(R,\theta,\varphi)=&AR\cos\theta+BR\sin\theta\cos\varphi+DR\sin\theta\sin\varphi\\&+\frac{AR_0^3}{R^2}\frac{1-\varepsilon_3}{2+\varepsilon_3}\cos\theta+\frac{BR_0^3}{R^2}\frac{1-\varepsilon_1}{2+\varepsilon_1}\sin\theta\cos\varphi+\frac{DR_0^3}{R^2}\frac{1-\varepsilon_2}{2+\varepsilon_2}\sin\theta\sin\varphi.\end{aligned} \quad (9)$$

Expressions (8) and (9) are the analytical formulae for electric potentials inside and outside the anisotropic dielectric spherical target respectively.

2.3. Discussion

From expression (6) it follows that when $\theta_0=\phi_0=0$, the incident electric field $\boldsymbol{E}_0$ is in the z-direction, $B=D=0$, and $A=-E_0$. If we suppose that $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon/\varepsilon_0$, now the problem of the electric field in an anisotropic medium under study has become one in an isotropic medium. Expressions (8) and (9) are transformed, respectively, into the following expressions:

$$u(R,\theta,\varphi)=\frac{-3E_0\varepsilon_0}{2\varepsilon_0+\varepsilon}R\cos\theta, \quad (10)$$

and

$$u_1(R,\theta,\varphi)=-E_0R\cos\theta+\frac{-E_0R_0^3}{R^2}\frac{\varepsilon_0-\varepsilon}{2\varepsilon_0+\varepsilon}\cos\theta. \quad (11)$$

From Refs.[3], [4], and [14] we obtain the solutions in the case of a dielectric sphere in the electric field E_0. These solutions are given as

$$V(R,\theta,\varphi)=\frac{-3E_0\varepsilon_0}{2\varepsilon_0+\varepsilon}R\cos\theta,$$

and

$$V_1(R,\theta,\varphi)=-E_0R\cos\theta+\frac{-E_0R_0^3}{R^2}\frac{\varepsilon_0-\varepsilon}{2\varepsilon_0+\varepsilon}\cos\theta.$$

It can be seen that the results are consistent with those in the literature, thereby indirectly testifying the correctness of the results.

Let α and β be the angles between E_0 and the x-axis and between E_0 and the y-axis, respectively, then it will be easily proved that $\cos\alpha = \sin\theta\cos\varphi$ and $\cos\beta = \sin\theta\sin\varphi$. So expression (8) can be written as

$$u(R,\theta,\varphi) = \frac{3A}{2+\varepsilon_3}R\cos\theta + \frac{3B}{2+\varepsilon_1}R\cos\alpha + \frac{3D}{2+\varepsilon_2}R\cos\beta.$$

The above result indicates that the potential inside the anisotropic dielectric sphere is composed of three parts, and each part is determined by both the dielectric constant and the incident electric field in the same direction and is not related to the parameters in the other directions. In the scattering of small particles, for example, Raleigh scattering, the electric field inside the target is of great importance. So we discuss the distribution of the inside electric field in the following. The electric field and the electric displacement are obtained by making a gradient from expression (8) and utilizing the transformation between a vector in the spherical coordinate system and that in the right angle coordinate system as follows:

$$\boldsymbol{E} = -\frac{3D}{2+\varepsilon_2}\hat{y} - \frac{3B}{2+\varepsilon_1}\hat{x} - \frac{3A}{2+\varepsilon_3}\hat{z}. \tag{12}$$

Expression (12) demonstrates that the electric field inside an anisotropic dielectric sphere is a uniform field that makes an angle (δ) with respect to the incident field. This cosine function for the angle is easily derived by taking the scalar product of these two vectors as follows:

$$\cos\delta = \frac{\dfrac{3\sin^2\theta_0\sin^2\varphi_0}{2+\varepsilon_2} + \dfrac{3\sin^2\theta_0\cos^2\varphi_0}{2+\varepsilon_1} + \dfrac{3\cos^2\theta_0}{2+\varepsilon_3}}{\left[\dfrac{9\sin^2\theta_0\sin^2\varphi_0}{(2+\varepsilon_2)^2} + \dfrac{9\sin^2\theta_0\cos^2\varphi_0}{(2+\varepsilon_1)^2} + \dfrac{9\cos^2\theta_0}{(2+\varepsilon_3)^2}\right]^{1/2}}. \tag{13}$$

The electric displacement is obtained as

$$\boldsymbol{D} = -\frac{3D\varepsilon_2}{2+\varepsilon_2}\hat{y} - \frac{3B\varepsilon_1}{2+\varepsilon_1}\hat{x} - \frac{3A\varepsilon_3}{2+\varepsilon_3}\hat{z}. \tag{14}$$

Let τ denote the angle between $\boldsymbol{D}$ and $\boldsymbol{E}_0$, then the cosine function will also be derived as

$$\cos\tau = \frac{\dfrac{9\varepsilon_2\sin^2\theta_0\sin^2\varphi_0}{(2+\varepsilon_2)^2} + \dfrac{9\varepsilon_1\sin^2\theta_0\cos^2\varphi_0}{(2+\varepsilon_1)^2} + \dfrac{9\varepsilon_3\cos^2\theta_0}{(2+\varepsilon_3)^2}}{ED}. \tag{15}$$

Furthermore, we can conclude from expression (10) that the electric field in one direction is determined only by the incident field and the dielectric constant in this direction in the anisotropic medium, which is consistent with the superposition principle of force. Figure 2 demonstrates that the magnitude of electric field inside the sphere varies with dielectric constant tensor. It can be seen that when the dielectric constant tensor increases, the magnitude of the field will decrease. From Fig.3 it follows that the direction of incident field periodically affects the magnitude of the field inside the sphere when the dielectric constant tensor is given. The variations of the angle between the incident field direction and the inside field direction with incident direction and dielectric constant tensor are shown in Figs.4 and 5 respectively. It is concluded that the influence induced by the dielectric constant tensor on the inside field direction is bigger than that induced by the incident field direction. This angle is generally $\pi/6$. This numerical result means that we can consider the two directions of fields to be neglected in some approximate calculations. The incident wave direction induces the magnitude of the inside field of an anisotropic dielectric sphere periodically.

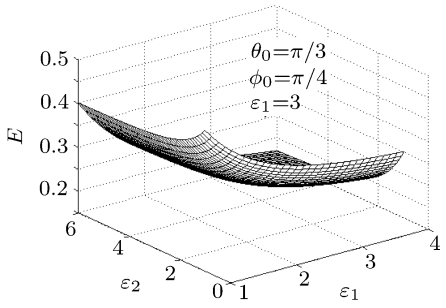


Fig.2. Electric field inside sphere versus the dielectric constant tensor.

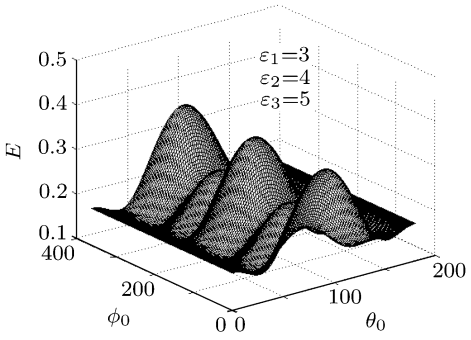


Fig.3. Electric field inside sphere versus incident direction.

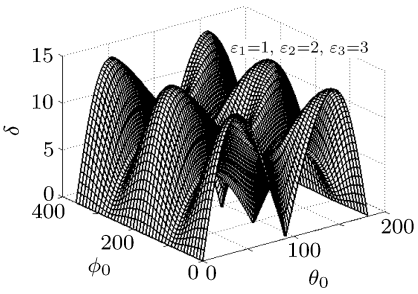


Fig.4. Direction of electric field inside sphere versus incident direction.

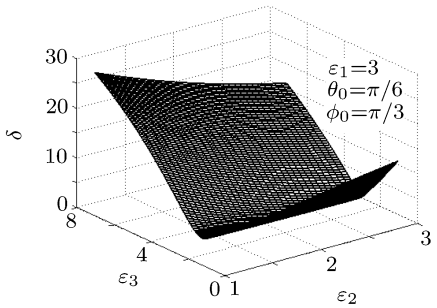


Fig.5. Direction of electric field inside sphere versus dielectric constant tensor.

3. Conclusion

In this paper, the electric fields inside and outside an anisotropic dielectric sphere are investigated. We use the scale transformation theory of the electromagnetic field to reconstruct the Laplace equation and then obtain two analytical expressions of the electric potentials inside and outside the anisotropic dielectric sphere in detail. The obtained results are consistent with those in the literature when the dielectric constant tensor becomes that in an isotropic medium. The angle between the total fields inside and outside the anisotropic dielectric sphere is derived. The effects induced by the incident direction and the dielectric constant tensor on the direction and the magnitude of the inside electric field are simulated. The results obtained show that the incident field direction affects the magnitude of the electric field inside the sphere greatly and the dielectric constant tensor also exerts a great influence on the direction of the inside field. Particles such as sand-dust storm particulates are generally anisotropic, so the results obtained provide a good theoretical foundation for studying the scattering features of small particles. How to use the scale transformation theory to study the electromagnetic fields inside and outside an anisotropic dielectric target irradiated by the time-varying electromagnetic wave is our next research subject.

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