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Solutions to the equations describing materials with competing quadratic and cubic nonlinearities*

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The Lie group theoretical method is used to study the equations describing materials with competing quadratic and cubic nonlinearities. The equations share some of the nice properties of soliton equations. From the elliptic functions expansion method, we obtain large families of analytical solutions, in special cases, we have the periodic, kink and solitary solutions of the equations. Furthermore, we investigate the stability of these solutions under the perturbation of amplitude noises by numerical simulation.

Keywords: competing nonlinearities, the elliptic functions expansion, soliton, numerical simulation
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1. Introduction

Materials with a significant $\chi^{(3)}$ nonlinearity show us the phenomenon of the solitonlike beam or pulse propagation. These materials can be modeled by the nonlinear Schrödinger equation and its variants,[1] which are known to be integrable and have exact soliton and periodic solutions. However, the nonlinearity effect has a low efficiency, and the phase matching is hard to achieve for the parametrically coupled processes.

Recently it has been demonstrated experimentally that nonlinearity-induced phase shifts can be achieved in $\chi^{(2)}$ materials (i.e., optical materials without the inversion symmetry) as the result of cascaded effect, and the solitary waves can be produced through the $\chi^{(2)}$:$\chi^{(2)}$ cascading.[2,3] This effect occurs in parametrically coupled fields with quadratic nonlinearities in which the nonlinearities in the fundamental and first harmonic field interact in a way that mimics the $\chi^{(3)}$ nonlinear effect. Solitary waves in quadratic nonlinear materials have attracted growing attention because of the possibility to employ large second-order nonlinearities for the needs of all-optical switching.[4] Such solitary waves have been observed experimentally.[2,3] Theoretical and experimental results also indicate that multidimensional bright solitons can exist in noncentrosymmetric materials with quadratic nonlinearities.[5] However, in $\chi^{(2)}$ materials there always exists next-order, $\chi^{(3)}$ nonlinearity, which, under certain conditions, might become important and strongly compete with $\chi^{(2)}$ nonlinearity.[6]

For the system with competing nonlinearities, namely, quadratic and cubic nonlinearities, the focusing Kerr media supported by cascaded nonlinearities has been discussed.[7,8]

First we consider the beam propagation in noncentrosymmetric lossless media with defocusing cubic nonlinearity, described by the dimensionless equations,[9]

$$\begin{aligned} &\mathrm{i}\frac{\partial \omega}{\partial t}+\frac{\partial^2 \omega}{\partial x^2}+\omega^*\,\nu-(|\omega|^2+\rho\,|\nu|^2)\,\omega=0,\\ &2\mathrm{i}\frac{\partial \nu}{\partial t}+\frac{\partial^2 \nu}{\partial x^2}-\beta\,\nu+\frac{1}{2}\,\omega^2-(\eta\,|\nu|^2+\rho\,|\omega|^2)\,\nu=0, \end{aligned} \quad (1)$$

where ω and ν are the complex envelops of the fundamental-harmonic (FH) and second-harmonic (SH) waves, respectively. Two variables, namely, t and x, are the normalized distance along the wave guide and transverse coordinate, respectively. System (1) is rather complicated for analyzing all possible localized solutions. A qualitative analytical explanation to the stable vortex solitons of the similar equations is

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proposed.[10] So far as we know, many numerical calculation methods[7,8] have been used to study system (1), but only one exact analytical solution of system (1) has been shown in Ref.[6] in the case of $\beta = 1$, ν and ω are real, and the constraints $\omega = \sqrt{2}\nu$ are fulfilled.

In this paper, we try to use Lie group symmetry reduction and improved elliptic functions expansion methods to get analytical solutions of system (1). Some of the new analytical solutions of system (1) are obtained in the cases of some special relations between parameters. In special cases, we obtained the periodic, kink and solitary wave solutions, and found that these solutions can propagate stably for a given distance under the perturbation of the amplitude noise whose maximal value is 0.08. For long distance and extremely high bit rate communications, optical soliton communication is expected to be one of the most promising candidates, the results obtained in this article are also expected to play a role in analyzing the optical soliton communications in the future.

2. Equation reduction

We find out the Lie point symmetry group of system (1) and use the symmetry reduction method, then we can reduce system (1) to ordinary differential equations and obtain analytical solutions. Put

$$\omega(x,t) = R_1(x,t)\,\mathrm{e}^{\mathrm{i}\phi_1(x,t)}, \tag{2}$$

$$\nu(x,t) = R_2(x,t)\,\mathrm{e}^{\mathrm{i}\phi_2(x,t)}, \tag{3}$$

$$0 \le R_{1,2} < \infty, \qquad 0 \le \phi_{1,2} < 2\pi, \tag{4}$$

and apply a standard algorithm to find their symmetry group.[11–13] In the generic case, when all constants in system (1) are arbitrary, the Lie symmetry group of the system is just three-dimensional, consisting of translations in x and t and correlated shifts of the phase shift $\phi_1 \to \phi_1 + \lambda$ and $\phi_2 \to \phi_2 + 2\lambda$. The Lie algebra of the symmetry group has a basis consisting of the operators

$$P_1 = \partial_x, \quad P_0 = \partial_t, \quad W = \partial_{\phi_1} + 2\partial_{\phi_2}. \tag{5}$$

Under the subgroup, solutions invariant could be generated by

$$X = c\,k\,P_1 + k\,P_0 + \kappa_1\,k(c_1 - c)\,W, \tag{6}$$

where c, k, κ_1, c_1 are real constants.

The invariant solutions then have the forms

$$\begin{aligned}
&\omega(\xi) = R_1(\xi)\,\mathrm{e}^{\mathrm{i}\phi_1(\xi)}\exp[-\mathrm{i}\,\kappa_1(x - c_1\,t)],\\
&\nu(\xi) = R_2(\xi)\,\mathrm{e}^{\mathrm{i}\phi_2(\xi)}\exp[-2\mathrm{i}\,\kappa_1\,(x - c_1\,t)],\\
&\xi = k\,(x - c\,t).
\end{aligned} \tag{7}$$

The four real functions R_1, R_2, ϕ_1, ϕ_2 satisfy the following system of coupled ordinary differential equations,

$$\begin{aligned}
&-k^2(2R_1'\,\phi_1' + R_1\,\phi_1'') + k\,(2\kappa_1 + c)\,R_1'\\
&-R_1\,R_2\,\sin\phi = 0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
&-k^2(2R_2'\,\phi_2' + R_2\,\phi_2'') + k\,(4\kappa_1 + 2c)R_2'\\
&+\frac{1}{2}\,R_1^2\,\sin\phi = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&-k^2\,(R_1'' - R_1\,\phi_1'^2) - k\,(c + 2\kappa_1)\,R_1\,\phi_1'\\
&+(\kappa_1^2 + c_1\,\kappa_1)\,R_1 + R_1^3\\
&+\rho\,R_1\,R_2^2 - R_1\,R_2\,\cos\phi = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&-k^2(R_2'' - R_2\,\phi_2'^2) - k\,(2c + 4\,\kappa_1)\,R_2\,\phi_2'\\
&+(4\kappa_1^2 + 4c_1\,\kappa_1 + \beta)\,R_2\\
&-\frac{1}{2}\,R_1^2\,\cos\phi + \eta\,R_2^3 + \rho\,R_1^2\,R_2 = 0,
\end{aligned} \tag{11}$$

$$\phi \equiv \phi_2 - 2\phi_1.$$

The equations (8)–(11) are a set of four coupled real ordinary differential equations which are difficult to decouple and solve. By given a supplementary restriction on the phase, the equations will be greatly simplified, namely,

$$\phi = m\pi, \tag{12}$$

where m is an integer, then we know

$$\sin\phi = 0, \quad \cos\phi = \epsilon, \quad \epsilon^2 = 1. \tag{13}$$

We call such solutions "phase-locked solutions." Let us first assume

$$k \neq 0, \tag{14}$$

and substitute (13) into Eqs.(8)–(11), then integrate Eqs.(8) and (9), we have

$$\phi_1' = A + \frac{M_1}{R_1^2}, \quad \phi_2' = 2A + \frac{M_2}{R_2^2}, \tag{15}$$

where

$$M_1 = -\frac{N_1}{k^2}, \quad M_2 = -\frac{N_2}{k^2}, \tag{16}$$

and N_1 and N_2 are integration constants.

We can rewrite Eqs.(10) and (11) as

$$\begin{aligned}
&R_1'' - R_1\,\phi_1'^2 + 2A\,R_1\,\phi_1' - D\,R_1\\
&+\frac{R_1\,R_2\,\epsilon}{k^2} - \frac{R_1^3 + \rho\,R_1\,R_2^2}{k^2} = 0,
\end{aligned} \tag{17}$$

$$R_2'' - R_2\,\phi_2'^2 + 4A\,R_2\,\phi_2' - B\,R_2 + \frac{R_1^2\,\epsilon}{2k^2} - \frac{\eta\,R_2^3 + \rho\,R_1^2\,R_2}{k^2} = 0, \tag{18}$$

where

$$A = \frac{2k\,\kappa_1 + c\,k}{2k^2},$$
$$B = \frac{4\kappa_1^2 + 4c_1\,\kappa_1 + \beta}{k^2},$$
$$D = \frac{c_1\,\kappa_1 + \kappa_1^2}{k^2}.$$

2.1. Case $M_1\,M_2 \neq 0$

The phase locking condition (13) imposes a relation between R_1 and R_2, namely, $2A + M_2/R_2^2 = 2A + 2M_1/R_1^2$, we can see

$$\frac{M_2}{R_2^2} = \frac{2M_1}{R_1^2}, \tag{19}$$

so that equations (17), (18) and (19) are overdetermined. Expressing R_1, ϕ_1, ϕ_2 in terms of R_2, substituting it into Eqs.(17) and (18), and putting

$$A_0 = A^2 - D, \quad B_0 = 4A^2 - B, \quad D_0 = \frac{\epsilon}{k^2}, \tag{20}$$

then we obtain two second-order ordinary differential equations for R_2

$$R_2'' - \frac{(2M_1 + \rho\,M_2)\,R_2^3}{M_2\,k^2} + D_0\,R_2^2 + A_0\,R_2 - \frac{M_2^2}{4R_2^3} = 0,$$
$$R_2'' - \frac{(2\rho\,M_1 + \eta\,M_2)\,R_2^3}{M_2\,k^2} + \frac{M_1\,D_0\,R_2}{M_2} + B_0\,R_2 - \frac{M_2^2}{4R_2^3} = 0. \tag{21}$$

These turn out to be compatible only for R_2 constant. This case will be considered separately in subsubsection 3.6.1 below.

2.2. Case $M_1\,M_2 = 0$

For this case, we have

$$\phi_1' = A, \qquad \phi_2' = 2A. \tag{22}$$

The equations (17) and (18) can be rewritten as

$$k^2 R_1'' + A_0\,k^2 R_1 + D_0\,k^2 R_1\,R_2 - R_1^3 - \rho\,R_1\,R_2^2 = 0, \tag{23}$$

$$k^2 R_2'' + B_0\,k^2 R_2 + \frac{1}{2}\,D_0\,k^2 R_1^2 - \eta\,R_2^3 - \rho\,R_1^2\,R_2 = 0. \tag{24}$$

The equations (23) and (24) are nonlinear ordinary differential equations. In order to obtain the analytical solutions, we use the method of the improved Jacobi elliptic functions expansion.

3. Exact solutions

The main idea of the improved Jacobi elliptic functions expansion method[14−16] is to take the full advantage of the elliptic function which the Jacobi elliptic functions satisfy and use its solutions to replace the Jacobi elliptic functions.

Let us look for solutions $R_1(\xi)$ and $R_2(\xi)$ satisfying Eq. (24), and also the elliptic function equation

$$R_2'^2 = \gamma_4\,R_2^4 + \gamma_3\,R_2^3 + \gamma_2\,R_2^2 + \gamma_1\,R_2 + \gamma_0, \tag{25}$$

the compatibility of Eqs.(24) and (25) imposes eight relations between the constants in Eqs.(24) and (25), which allow the following solutions.

3.1. $\gamma_4 = 3/2k^2$, $A_0 = B_0$

The constants γ_ν in Eq.(25) are completely specified,

$$\gamma_3 = -\frac{2D_0}{3}, \quad \gamma_2 = -B_0, \quad \gamma_1 = 0, \tag{26}$$

and γ_0 is arbitrary. The other amplitude $R_1(\xi)$ is given directly by the expression $R_1 = \sqrt{2}\,R_2$. In the entire range of values of R_2, a condition will be analyzed below. Thus we have a two-parameter family of solutions depending on two parameters, being γ_0 and a constant arising in the integration of equation (25). The coefficients γ_4, γ_3, γ_2, γ_1 and γ_0 are fixed in terms of k, B_0 and D_0. The constraints on the constants in the original system (1) ρ, η and β which need to satisfy $\rho = \eta = 1$ and $\beta = -12k^2$, and those introduced in the reduction procedure (7) namely, c, k, c_1 and κ_1, need to satisfy $A_0 = B_0 = -4$, namely, $16k^2 - 4\kappa_1(c - c_1) + c^2 = 0$. In fact, the solutions of Eq.(25) can be expressed as a finite series of Jacobi functions

$$R_2 = \sum_{i=-n}^{n} a_i\,\mathrm{sn}^i(\xi, p) \tag{27a}$$

or

$$R_2 = \sum_{i=-n}^{n} a_i\,\mathrm{cn}^i(\xi, p). \tag{27b}$$

By balancing the highest order of the linear term with the nonlinear term in Eq.(25), we can determine $n = 1, -1$.

Substituting Eqs.(27) into Eq.(25) and setting the coefficients of $\mathrm{sn}^i(\xi,p)$ to zero, we can deduce the following set of algebraic polynomials with the respect unknowns a_0, a_1 and p, namely,

$$-\frac{3}{2}a_1^4+k^2a_1^2p^2=0, \tag{28}$$

$$-6a_1^3\,a_0+\frac{2}{3}D_0k^2\,a_1^3=0, \tag{29}$$

$$B_0k^2a_1^2-a_1^2k^2p^2-k^2a_1^2-9a_1^2a_0^2+2D_0k^2a_1^2a_0=0, \tag{30}$$

$$2D_0k^2a_1a_0^2+2B_0k^2a_1a_0-6a_1a_0^3=0, \tag{31}$$

$$-\gamma_0k^2-\frac{3}{2}a_0^4+\frac{2}{3}D_0k^2a_0^3+B_0k^2a_0^2+k^2a_1^2=0. \tag{32}$$

From the above equations, we have

$$\begin{aligned}
&a_0=\frac{D_0k^2}{9},\quad a_1=\pm\frac{\sqrt{(2-54k^2)}}{9},\\
&p=\pm\sqrt{\frac{1}{27k^2}-1},\\
&D_0=-2187\gamma_0+54\pm243\sqrt{81\gamma_0^2-4\gamma_0},\\
&B_0=-\frac{2}{27k^2},
\end{aligned} \tag{33}$$

where γ_0 is arbitrary and k needs to satisfy $1/54<k^2<1/27$, we obtain the exact periodic solutions

$$R_1=\sqrt{2}R_2=\frac{\sqrt{2}D_0k^2}{9}+\frac{2\sqrt{1-27k^2}}{9}\mathrm{sn}(\xi-\xi_0,p), \tag{34}$$

$$R_1=\sqrt{2}R_2=\frac{\sqrt{2}D_0k^2}{9}+\frac{\sqrt{6k^2}}{3}\mathrm{sn}^{-1}(\xi-\xi_0,p). \tag{35}$$

If $k^2=1/54$, namely, $p=1$ and $\beta=-2/9$, then we obtain the kink wave solutions,

$$R_1=\sqrt{2}R_2=\frac{\sqrt{2}}{9}\pm\frac{\sqrt{2}}{9}\tanh(\xi-\xi_0), \tag{36}$$

$$R_1=\sqrt{2}R_2=\frac{\sqrt{2}}{9}\pm\frac{\sqrt{2}}{9}\coth(\xi-\xi_0). \tag{37}$$

3.2. $\gamma_4=5/2k^2$, $A_0=B_0$

In this case, we obtain,

$$\gamma_3=-\frac{2D_0}{3},\quad \gamma_2=-B_0,\quad \gamma_1=0,\quad \gamma_0=0, \tag{38}$$

and also have the relation $R_1=\sqrt{2}\,R_2$. Noticing that equation (25) will have solutions depending on just one parameter, an integration constant, since coefficients γ_ν are fixed in terms of k, B_0 and D_0. The constraints on the constants in the original system (1) ρ, η and β need to satisfy $\rho=3$, $\eta=-1$ and $\beta=-2/15$, and c, c_1, and κ_1 which were introduced in the reduction procedure (7) need to satisfy $A_0=B_0=-2/45k^2$, namely, $180\kappa_1(c-c_1)+45c^2+8=0$.

Similarly, in this case, $p=1$ is the requirement. Then we obtain the kink wave solutions

$$R_1=\sqrt{2}R_2=\pm\frac{\sqrt{2}}{15}\pm\frac{\sqrt{2}}{15}\tanh\left[\frac{\sqrt{10}}{30k}(\xi-\xi_0)\right], \tag{39}$$

$$R_1=\sqrt{2}R_2=\pm\frac{\sqrt{2}}{15}\pm\frac{\sqrt{2}}{15}\coth\left[\frac{\sqrt{10}}{30k}(\xi-\xi_0)\right]. \tag{40}$$

3.3. $\gamma_4=0$, $\gamma_3\neq0$, $A_0=B_0$

We obtain

$$\gamma_3=-\frac{2D_0}{3},\quad \gamma_2=-B_0,\quad \gamma_1=0 \tag{41}$$

and γ_0 arbitrary. Again, equation (25) provides a two-parameter family of solutions, and the constraint is $\eta=4$, $\rho=-2$.

We obtain the solitary wave solutions

$$R_1=\sqrt{2}R_2=-\frac{3\sqrt{2}B_0}{2D_0}+\frac{3\sqrt{2}B_0}{2D_0}\times\tanh^2\left[\frac{\sqrt{-B_0}}{2}(\xi-\xi_0)\right], \tag{42}$$

$$R_1=\sqrt{2}R_2=-\frac{3\sqrt{2}B_0}{2D_0}+\frac{3\sqrt{2}B_0}{2D_0}\times\coth^2\left[\frac{\sqrt{-B_0}}{2}(\xi-\xi_0)\right]. \tag{43}$$

3.4. $\gamma_4=0$, $\gamma_3=0$, $\eta=0$

The equation (25) is reduced to $R_2'^2=-B_0\,R_2$, the solution is

$$\begin{aligned}
&R_1=0,\\
&R_2=\mathrm{e}^{\sqrt{-B_0}\,(\xi-\xi_0)},\qquad B_0<0,
\end{aligned} \tag{44}$$

while $B_0=0$, we have $R_2=\mu\,(\xi-\xi_0)$.

3.5. Phase locked solutions for $k=0$

We return to Eqs.(8)–(11) for $k=0$, from Eqs.(9)–(11), we see that ϕ_1, ϕ_2, R_1 and R_2 are constants when $k=0$.

3.6. Other explicit solutions

In Subsection 2.2. we presented solutions of system (1) satisfying $\omega\,\nu\neq0$ and such that $|\omega|$ and $|\nu|$ are not constant. Let us now discuss these previously rejected solutions.

3.6.1. Solutions with R_1 and R_2 constants

Equations (8)–(11) for constants $R_1 \neq 0$ and $R_2 \neq 0$ imply that we must have $\sin\phi = 0$, i.e., the solutions will be phase-locked, then we know

$$\phi_1 = \gamma_1\,\xi + \gamma_2, \quad \phi_2 = 2\gamma_1\,\xi + 2\gamma_2 + m\,\pi, \quad \epsilon = (-1)^m, \tag{45}$$

where γ_1 and γ_2 are arbitrary constants, m is an integer. The constants c_1, κ_1, R_1 and R_2 need to satisfy the following relations,

$$c_1 = -\frac{1}{\kappa_1}(\kappa_1^2 + R_1^2 + \rho R_2^2 - R_2\epsilon), \tag{46}$$

$$R_1 = \frac{2R_2(-\beta + (4\rho - \eta)R_2^2 - 4R_2\epsilon)}{2R_2\rho - 8R_2 - \epsilon}. \tag{47}$$

3.6.2. Solution with $\omega = 0$

When $\omega = 0$, system (1) can be rewritten as

$$2\mathrm{i}\frac{\partial\nu}{\partial t} + \frac{\partial^2\nu}{\partial x^2} - \beta\nu - \eta|\nu|^2\nu = 0. \tag{48}$$

Substituting $\nu = u(x,t)\exp\left(-\frac{1}{2}\mathrm{i}\beta t\right)$ into Eq.(48), we can obtain

$$2\mathrm{i}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - \eta|u|^2 u = 0. \tag{49}$$

System (1) for $\omega = 0$ reduces to the nonlinear Schrödinger equation (NLSE). equation (49) is a fundamental equation of nonlinear science and has been studied in the context of solitons.[17] When $\eta < 0$, the NLSE (49) has stable bright soliton solution

$$u = -\frac{2\sqrt{2\eta\beta}}{3\eta}\,\mathrm{sech}\left(\frac{2\sqrt{-\beta}}{3}x\right)\exp\left(\frac{2}{3}\mathrm{i}\beta t\right). \tag{50}$$

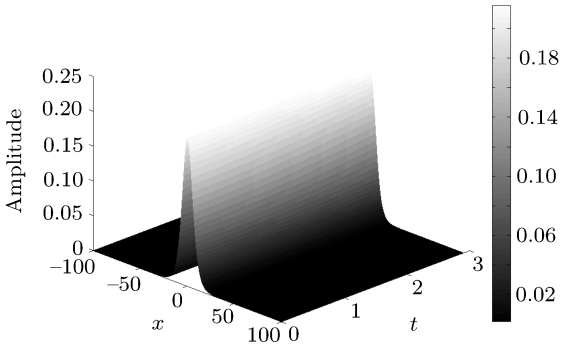
We also know its dark soliton solution when $\eta > 0$,

$$u = q_0\tanh\left(q_0\frac{\sqrt{2\eta}}{2}x\right)\exp\left(-\mathrm{i}\frac{\eta}{2}q_0^2 t\right), \tag{51}$$

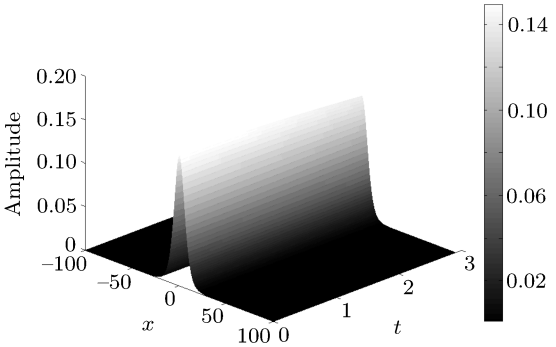
where parameter q_0 determines the amplitude and width of the pulse.

3.7. The stability of the exact solutions

It is necessary to know whether the optics waves can propagate stably or not in the non-integrable system (1), so we analyze the stability of the exact solutions by employing numerical simulation for different initial optical waves from solving system (1). The numerical method used is a split-step Fourier code.[1,18] The distance step is 1/150. To demonstrate their evolution of the exact solutions, we take solitary waves in case 3.3 as an example, figure 1 shows the evolution of the exact solutions obtained in case 3.3 over a distance $t = 3$.



(a) Evolution of the fundamental solitary wave



(b) Evolution of the second-harmonic solitary wave

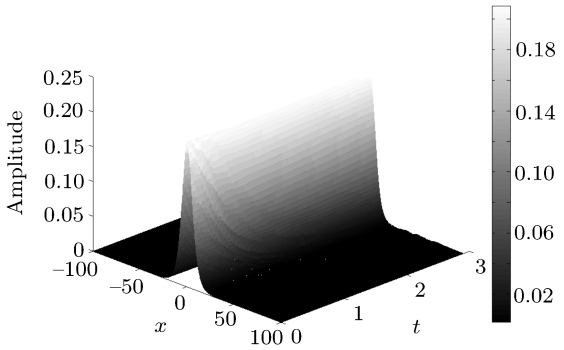
Fig.1. Typical evolution of the exact FH and SH solitary waves [Eq.(42)].

Obviously, when there is no perturbation the solitary solutions can propagate stably for a given distance. For the periodic and kink wave solutions, the same numerical method can be used, and from their evolutions, we can see that without perturbation the exact solutions can propagate stably.

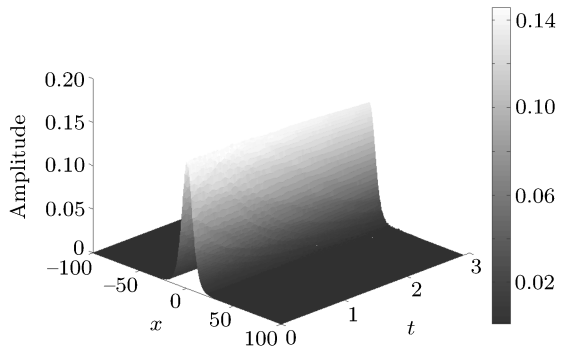
To check their stability with respect to finite perturbations, we have performed numerical experiments by adding amplitude noise whose maximal value is 0.08 on the exact solutions.

From Fig.2 we can see that, under the perturbation of finite initial amplitude noises, the FH and SH solitary waves are rather stable for a certain distance.

We perform the same calculations for the case 3.1. From Fig.3, we can see that the shapes of the periodic FH and its SH waves also propagate stably under the amplitude noise whose maximal value is 0.08, pictures for kink waves are quite similar, which can propagate stably under the amplitude noise.

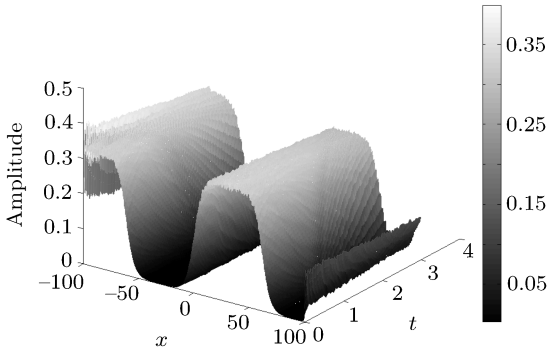


(a) Evolution of the fundamental wave under the perturbation of initial amplitude noise

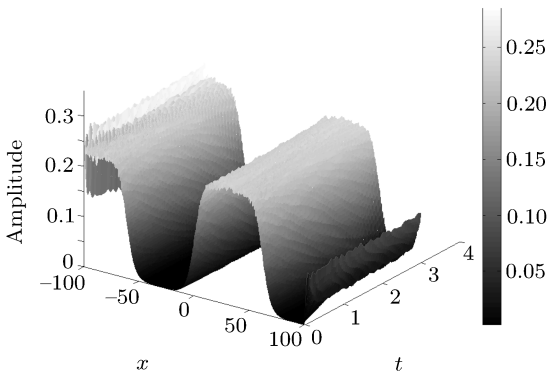


(b) Evolution of the second-harmonic wave under the perturbation of initial amplitude noise

Fig.2. Evolution of the solitary solution [Eq.(42)] under the perturbation of amplitude noises whose maximal value is 0.08.



(a) Evolution of the fundamental periodic wave under the perturbation



(b) Evolution of the SH periodic wave under the perturbation

Fig.3. Evolution of the periodic wave solutions [Eq.(34)] under the perturbation of amplitude noises whose maximal value is 0.08.

4. Conclusion

In this paper, we used the symmetry reduction method to solve the system (1), i.e., for all values of the constants ρ, η and β. This "generic" symmetry algebra is summed up in equation (5) representing space and distance translations and a shift in the phases of the FH and SH waves.

We reduced the system (1) to ordinary differential equations (8)–(11) by requiring that solution by invariant under the one-dimensional subgroup of the symmetry group, corresponding to the Lie algebra element (6). We decouple these equations on the phase-locked condition. The existence of the symmetry W in Eq.(5) guarantees that if the phases-locking condition (13) is imposed on the initial conditions, it will survive for all times. Thus, by the Jacobi elliptic function expansion, all the explicit solutions obtained in this paper are phase-locked traveling waves.

For certain values of the parameters ρ, β, and η, characterizing the material involved, and of the constants c, k, κ_1, and c_1, characterizing initial conditions, the elliptic function solutions reduced to exact periodic wave solutions. For other conditions, the elliptic function solutions reduced to kink wave solutions [(36) and (39)]. For Case 3.3, the elliptic function solutions reduce to solitary wave solutions (42) and (43), furthermore, we also obtained soliton solutions and plane wave solutions.

To sum up, using the group theoretical and Jacobi elliptic expansion methods, we have obtained the periodic, kink and solitary wave solutions of the Kerr media supported by cascaded nonlinearities. By the split-step method, we studied the evolution of the exact solutions under the perturbation, and we found that the exact solutions are rather stable for a certain distance.

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