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Synchronization speed of identical oscillators on community networks*

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Synchronizability of complex oscillators networks has attracted much research interest in recent years. In contrast, in this paper we investigate numerically the synchronization speed, rather than the synchronizability or synchronization stability, of identical oscillators on complex networks with communities. A new weighted community network model is employed here, in which the community strength could be tunable by one parameter δ. The results showed that the synchronization speed of identical oscillators on community networks could reach a maximal value when δ is around 0.1. We argue that this is induced by the competition between the community partition and the scale-free property of the networks. Moreover, we have given the corresponding analysis through the second least eigenvalue λ_2 of the Laplacian matrix of the network which supports the previous result that the synchronization speed is determined by the value of λ_2.

Keywords: complex networks, synchronization speed, community
PACC: 0565, 0547

The last few years have witnessed tremendous activity devoted to the characterization and understanding of complex network systems from physics, biology, sociology to engineering.[1−5] One of the central problems in the study of complex networks is to understand how the underlying topological structure affects dynamical processes. In this direction recent studies have considered several dynamics such as the evolutionary game,[6−10] synchronization of coupled oscillators,[11−20] traffic and congestion,[21−26] etc.

In this paper we study the converging speed of oscillators on complex networks with communities. And at first we give a brief introduction to previous related works, most of which focused on the stability of synchronization or synchronizability. Consider the coupled dynamical system as follows:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) + \sigma \sum_{j=1}^{N} \boldsymbol{G}_{ij} \boldsymbol{H}(\boldsymbol{x}_j), \tag{1}$$

where $\boldsymbol{x}_i$ with $i = 1, 2, \ldots, N$ are dynamical variables, $\boldsymbol{F}$ and $\boldsymbol{H}$ are evolution and coupling functions, respectively, σ is a constant. $G = (G_{ij})$ is the weighted Laplacian matrix defined by its elements $G_{ii} = s_i$ (the strength of node i, i.e., $s_i = \sum_j w_{ij}$), $G_{ij} = -w_{ij}$ if nodes i and j are connected, and $G_{ij} = 0$ otherwise where w_{ij} is the coupling strength between node i and node j. Noting the N eigenvalues of G by $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$, references [12] and [13] have determined that a synchronized state is linearly stable on a network if and only if $\alpha_1 < \sigma\lambda_2 \leq \ldots \leq \sigma\lambda_N < \alpha_2$ where α_1 and α_2 depend exclusively on the oscillator dynamics. It is straightforward to conclude that a synchronized state is linearly stable if and only if $\lambda_2/\lambda_N > \alpha_1/\alpha_2$. As the left-hand side depends only on the network topology and the interval $[\alpha_1, \alpha_2]$ in which the synchronized state is stable is larger for larger eigenratios λ_N/λ_2, the eigenratio is used to measure the synchronizability of a network. Since then there are many works to reveal the relationship between the synchronizability and the network properties,[15,20] and to enhance the synchronizability by optimizing the topology structure[19] or the edge weights.[17]

In order to investigate the converging speed of the oscillators network, we calculate the synchronization error averaged over the nodes that,

$$E(t) = \langle |\boldsymbol{x}(t) - \langle \boldsymbol{x}(t) \rangle| \rangle, \tag{2}$$

where $\langle \ldots \rangle$ means averaging over the whole network and $|\ldots|$ denotes the absolute value. It is obvious that E equals zero after the evolution of long time if the system could converge to a stable synchronized

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state. Moreover, in order to study the synchronization speed on networks with communities, a model is employed here which can generate weighted networks from module-free topology to ones with different strengths of communities by tuning only one parameter.[27] The model, starting from an initial state of a network of N nodes without any links, could be built up by three main processes, i.e., the local attachment (LA), the global attachment (GA) and the node deletion (ND). In the LA process, each node apart from the one with no neighbor begins to search for new neighbors. More specifically, node i chooses one of its neighbors, node j, with probability w_{ij}/s_i where w_{ij} is the strength of the link connecting i and j and $s_i = \sum_j w_{ij}$ is the strength of i. If the chosen node j has other neighbors except i, e.g., it chooses one of them, say k, with probability $w_{jk}/(s_j - w_{ij})$. If there is no link between i and k, a link with weight w_0 is established with probability $p_\Delta \Delta t$ where Δt is the time step, or otherwise its weight is increased by δ. In both situations, w_{ij} and w_{jk} are also increased by δ. In the GA process, if a node has no links, it creates a link of weight w_0 to a random node. If it has at least one neighbor, it also creates a link of weight w_0 randomly with probability p_r. In the ND process, any node can be removed with probability p_d whose adjacent links are also removed. The removed node is replaced by a new node with no neighbors so that the size of the network remains the same. We set the two probabilities p_r and p_d, the same as the value in the original model,[27] i.e., $p_r = 5 \times 10^{-4}$ and $p_d = 1 \times 10^{-3}$. At each time step, LA and GA processes are carried out using parallel update, followed by the ND step. In order to make the network reach steady state, we run the simulations for 25×10^3 time steps. The community strength of the networks is changed by tuning parameter δ and the larger the value of δ, the stronger the community structure. In the following studies on networks with different values of δ, we have made the average degree $\langle k \rangle \approx 9$ by adjusting probability p_Δ for each δ. Therefore, the total number of links has no apparent difference in all these generated networks and we can give a clear figure about how the strength of the community structure δ affects the converging speed of the synchronization.

In order to characterize the synchronization dynamics described in Eq.(1), in the studies below we use the Rössler oscillator: $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{F}(\boldsymbol{x}) = (-y - z, x + 0.2y, z(x - 2.5) + 0.2)$. And the coupling function $\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{x} = (x, y, z)$. At first, we generate the weighted network of size $N = 1 \times 10^4$ with different values of δ or different strength of the community structures according to the processes mentioned above. To get the unweighted networks, we could just set all the weights of the links equally to 1.0. And then we initialize the states of all nodes $\boldsymbol{x}_i(0)$ randomly in interval $[0, 1]$, i.e., distributed uniformly in $[0, 1]$. The dynamics described in Eq.(1) is simulated on the networks with different values of δ and the value of σ is set to 1.0 in all the simulations. Here, we use the simple Euler's method with step size $\Delta t = 0.01$ for the un-weighted networks and $\Delta t = 0.001$ for the weighted networks which ensure the convergence of the synchronization. Figure 1 displays the time evolution of the synchronization error $E(t)$ for $\delta = 0.1$ and $\delta = 0.5$. One can see that the value of $E(t)$ decays exponentially and the curve for $\delta = 0.1$ decays faster than that for $\delta = 0.5$. We use the slope of the decay curve in a semi-logarithmic diagram to represent the converging speed of the synchronization. And then figure 2 gives the complete behavior of the converging speed depending on the value of δ, i.e., the strength of the community structures. The results show that there is a maximal converging speed when δ is around 0.1 on both weighted and un-weighted networks, which means that the moderate community strength could induce higher synchronization speed. That is intelligible. As a matter of fact, there exist two main factors influencing the converging speed of the networks simultaneously. One is the heterogeneity of the nodes' degree or nodes' strength and the other is the strength of the community structure. For the model described above, in the LA process each node selects a neighbor according to their link weights and thus the selection favors strong links. Therefore, there exist some nodes with large degree or strength in each community. And according to Eq.(1), the nodes with large degrees have stronger influence than others and the states of the nodes with low degrees have a trend to converge to the same state associated with the large-degree node. In short, the existence of the large-degree nodes enhances the converging speed. As a result, when $\delta < 0.1$ the large-degree nodes take leading effect and the converging speed rises up quickly if the value of δ increases a little. On the other hand, when the value of δ becomes a little bit larger the community structures will take leading effect which separates the network into different parts or clusters. This partition will make the synchronization harder and thus it takes more time to converge. In a word, the maxi-

mal converging speed is induced by the balance of the competition between the heterogeneity of the nodes' degrees and the strength of the community structures. Moreover, the converging speeds on weighted networks are larger than that on un-weighted networks when there exist some community structures. This comparison indicates that heterogeneous link weights induce larger converging speed too.

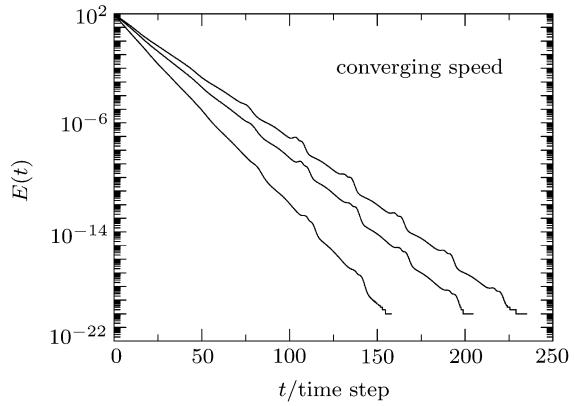


Fig.1. The synchronization error E defined in Eq.(2) as a function of time step t for the un-weighted community networks with $\delta = 0.1$ (bottom), $\delta = 0.5$ (top) and $\delta = 0.3$. It is obvious that the converging speed for $\delta = 0.1$ is larger than that for $\delta = 0.5$. All networks have size $N = 10^4$ and average degree $\langle k \rangle \approx 9$.

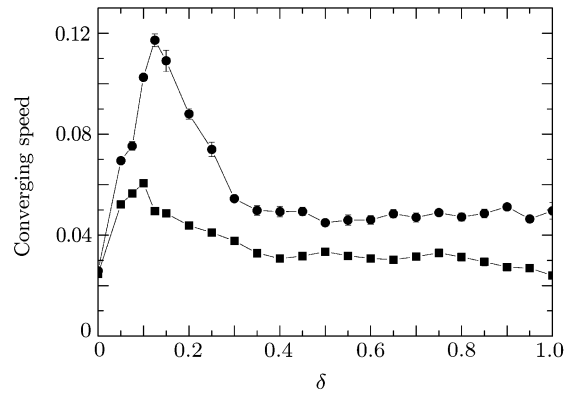


Fig.2. The synchronization speed as a function of the values of δ for un-weighted (■) and weighted (•) community networks. Each point is averaged for 20 times and the network parameters are the same as in Fig.1.

We now use the graph spectral theory to give a discussion for the results on un-weighted community networks presented above. The results on weighted networks could be analyzed similarly. As the previous works[28,29] have indicated that the converging speed is determined only by the second least eigenvalue (noted as λ_2) of the Laplacian matrix, here we calculate the values of λ_2 for different community strength, i.e., for different values of δ. Since the large complexity to calculate the eigenvalue of the networks of large size, we give the results for the networks with size $N = 1500$. The simplicity does not change the community structures in the networks and thus does not change the conclusion here. The corresponding results are shown in Fig.3. One can see that increasing the value of δ the λ_2 goes up sharply and then decreases when $\delta > 0.1$, which corresponds to the curve in Fig.2 for un-weighted networks. Furthermore, the behavior of λ_2 as a function of δ can also be understood by the balance of the competition between the node degree heterogeneity and the strength of the community structure. On one hand, the previous works have indicated that the large the heterogeneity of the nodes' degree the large the value of λ_2. That is to say, when δ increases in the interval [0,0.1], λ_2 will increase accordingly. On the other hand, when δ is large, i.e., the community structure is stronger, the networks will be separated into some obvious groups. And thus the algebraic connectivity, that is λ_2, is less. Therefore, the value of λ_2 will decrease when $\delta > 0.1$. As the previous works have pointed out that large λ_2 will induce large converging speed of synchronization of identical oscillators on networks, our analysis here supports the previous results.

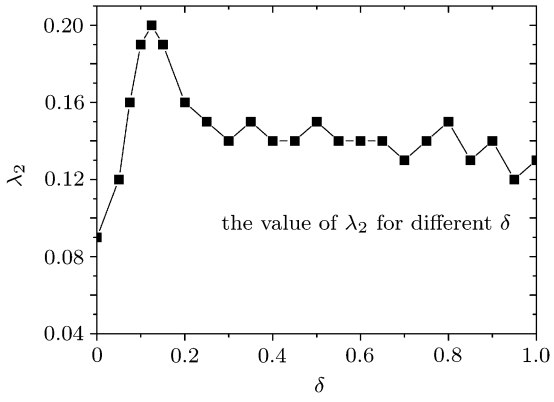


Fig.3. The second least eigenvalue λ_2 of the Laplacian matrix vs the values of δ. The network size $N = 1500$ and the average degree $\langle k \rangle \approx 9$.

In addition, it is worth while noting that in Ref.[30] the authors have studied the synchronization on community networks which focus on the relationship between the syncronizability and the community strength. In contrast, in the present paper we study one of the most important synchronization performances, the converging speed, of identical oscillators on un-weighted and weighted community networks which are based on a new model proposed recently. And we should point out that, although in this pa-

per we focus on the effect of the community structure strength on the synchronization speed, the synchronization speed for different general network topologies may also be very different as shown in previous works.[28,29] Thus, how to improve the synchronization performance rather than synchronizability (or stability of synchronization) is of both theory and application interests and remains a challenging problem.

In conclusion, we have studied an important synchronization performance, the converging speed, on un-weighted and weighted community networks with various community strength. The results have shown that when increasing the community strength, the converging speed will firstly increase to a peak and then decrease. And the analysis of the second least eigenvalue λ_2 of the Laplacian matrix associated with the community network exhibited that the converging speed is determined by λ_2 which supported the previous results. We argued that the relationship between the converging speed and the community strength was induced by the competition of the heterogeneity of nodes' degree or strength and the network partition.

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