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Random shortcuts induce phase synchronization in complex Chua systems*

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This paper studies how phase synchronization in complex networks depends on random shortcuts, using the piecewise-continuous chaotic Chua system as the nodes of the networks. It is found that for a given coupling strength, when the number of random shortcuts is greater than a threshold the phase synchronization is induced. Phase synchronization becomes evident and reaches its maximum as the number of random shortcuts is further increased. These phenomena imply that random shortcuts can induce and enhance the phase synchronization in complex Chua systems. Furthermore, the paper also investigates the effects of the coupling strength and it is found that stronger coupling makes it easier to obtain the complete phase synchronization.

Keywords: phase synchronization, random shortcuts, Chua systems
PACC: 0545

1. Introduction

Over the past few years, the analysis of complex systems from the viewpoint of networks has become an important interdisciplinary issue.[1−3] One such example is the small-world (SW) network which is characterized by a given fraction of the so-called shortcut links that connect distant units of the system, while all other units are coupled in a nearest-neighbor manner.[4] Nowadays, most of the studies on complex networks have focused on understanding how the random shortcuts influence the system's dynamic behaviors. For instance, it is found that the introduction of shortcuts can enhance chaos synchronization in discrete neural networks,[5] prevent the collapse of spatiotemporal chaos in excitable Gray–Scott dynamical elements[6] and induce transition from a highly fragmented phase to a uniform phase in a social systems model,[7] etc. All these works show that random shortcuts play a vital role in the system's dynamics.

In the past few years, the notion of synchronization has been extended to phase synchronization (PS) in a system composed of mutually coupled nonidentical self-sustained chaotic oscillators.[8] PS does not reveal itself directly from the trajectory, but as a boundedness of phase difference between the interacting oscillators. In such a synchronization the trajectories can be uncorrelated, and therefore, the oscillators present some independence of the amplitudes, but still preserving the collective behavior. Chaotic PS has been detected in many laboratory experiments and natural systems such as coupled neurons,[9] limit-cycle oscillators[10] and circuits.[11] In a good many of those previous works, the connectivity between the elements of the chaotic network was regular (local or global). However, recent experimental evidence shows that most real-world networks present the SW property. Hence, introducing the random shortcuts into networks investigations is a significant job. The main aim of this work is to investigate how PS of complex networks depends on topological randomness p, defined as the fraction of random shortcuts. The network element is described by a piecewise-continuous Chua canonical circuit,[12,13] which has emerged as a simple yet powerful experimental and analytical tool in studying chaotic behavior in nonlinear dynamics. The complex network is constructed by randomly adding shortcuts to an originally nearest-neighbor coupled network. In order to study the effect of the randomness p on the PS, each circuit in the network is chaotic and has nonidentical properties. It is found that for a given coupling strength, there exists a critical randomness p^*, such that if $p > p^*$ the PS, which is absent in the nearest-neighbor neural network, occurs.

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Further increasing the randomness p, the PS becomes evident and reaches its maximum. To quantitatively study the synchronization of the complex network, we introduce the standard order parameter r as a measure of the phase order. All these studies show that topological randomness can effectively enhance the PS of Chua networks. Furthermore, we investigate the effects of the coupling strength and found that stronger coupling will make it easier to obtain the complete PS.

2. The complex Chua network model

The complex Chua network studied in this paper is described by the following coupled 3-dimensional ordinary differential equations:[12−14]

$$\begin{aligned}
\frac{\mathrm{d}x_i}{\mathrm{d}t} &= \alpha_i(y_i - h(x_i)) + \frac{C}{N}\sum_{j=1}^{N} a_{ij}(x_j - x_i),\\
\frac{\mathrm{d}y_i}{\mathrm{d}t} &= x_i - y_i + z_i + \frac{C}{N}\sum_{j=1}^{N} a_{ij}(y_j - y_i),\\
\frac{\mathrm{d}z_i}{\mathrm{d}t} &= -\beta y_i + \frac{C}{N}\sum_{j=1}^{N} a_{ij}(z_j - z_i),\\
& i = 1, 2, 3 \ldots N, \qquad (1)
\end{aligned}$$

where the nonlinear function $h(x)$ is given by $h(x) = bx+((a-b)/2)(\mid x+1 \mid - \mid x-1 \mid)$. Some of the parameters are taken as $\beta = 100/7$, $a = -1/7$ and $b = 2/7$. The individual dynamics of the Chua circuit, which depends on the system parameters and nonlinear term $h(x)$, has been widely studied since the pioneer work of Chua.[14] It is shown that with different system parameters and $h(x)$, the individual Chua circuit is experiencing single or double scroll chaos attractors. In this paper, we generate randomly α_i which conform to a uniform distribution on the interval [8.5, 9.0], such that each neuron in the network is single scroll chaos attractors and has nonidentical initial properties. The bifurcation structure of the variable x versus the parameter α is shown in Fig.1, where the inset is the chaotic attractor of an arbitrary individual Chua circuit in the complex network. $\frac{C}{N}\sum_{j=1}^{N} a_{ij}(x_j - x_i)$, $\frac{C}{N}\sum_{j=1}^{N} a_{ij}(y_j - y_i)$ and $\frac{C}{N}\sum_{j=1}^{N} a_{ij}(z_j - z_i)$ are the coupling terms of the complex Chua circuits, where C is the coupling strength, the matrix a_{ij} encodes the connection topology: if a connection exists between neurons i and j, $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$. Also, $a_{ii} = 0$ for all i. The network patterns are achieved as follows:[15] It starts with a 3-dimensional cubic lattice which comprises $N = 500$ neurons. Each neuron is connected to its 4 nearest neighbors. Shortcuts are then randomly added between non-nearest vertices. In the limiting case where all neurons are coupled to each other, the network contains $N(N-1)/2$ edges. If M is the number of added random shortcuts, then the fraction of random shortcuts $p = M/[(N(N-1)/2]$ can be used to characterize the randomness of the network. In particular, for $p = 0$, the system reduces to the originally nearest-neighbor coupled network; for $p = 1$, it becomes a globally coupled network. Note that for a given p there are a lot of network realizations.

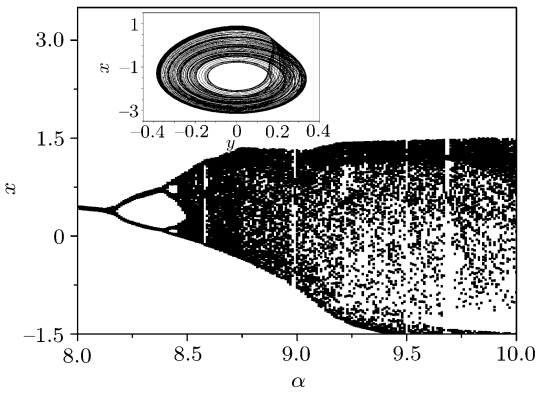


Fig.1. The bifurcation structure of the variable x versus the α of Chua circuit with parameters $\beta = 100/7$, $a = -1/7$ and $b = 2/7$. The inset shows the chaotic attractor of an arbitrary individual Chua circuit.

3. Numerical results and analysis

The results discussed hereafter were obtained by varying the topological randomness p. For each p, 100 network realizations were generated. In each network realization, the initial values of the 500 neurons' dynamical variable are newly chosen randomly. Numerical integration of Eq.(1) is performed by a fourth-order Runge–Kutta method with time steps of $\mathrm{d}t = 0.001$.

Firstly, we have to define the amplitude and the phase of the chaotic attractor. Since the chaotic attractor adopted in this paper is a simple single scroll, following the approach usually adopted in the literature,[16,17] the phases and the amplitudes of the chaotic Chua system for each time step can be determined from

$$\phi_j = \arctan\frac{\widetilde{y}_j}{\widetilde{x}_j}, \quad A_j = \sqrt{\widetilde{x}_j^2 + \widetilde{y}_j^2}, \qquad (2)$$

where $\widetilde{x}_j = x_j - \langle x_j \rangle$ and $\widetilde{y}_j = y_j - \langle y_j \rangle$, $\langle x_j \rangle = \frac{1}{\Delta t}\sum_{t_1}^{t_2} x_j$ is the time average over an interval $\Delta t = t_2 - t_1$ (and correspondingly for $\langle y_j \rangle$). The mean phase difference of the network is defined as: $H(t) = \frac{1}{N-1}\sum_{j=2}^{N} |\phi_j(t) - \phi_1(t)|$. For a phase synchronous state in the network, the mean phase difference among the chaotic oscillators does not grow with time, that is, $H <$ const. The time evolutions of the mean phase difference for various topological probability p with $C = 2.0$ are shown in Figs.2(a)–2(c). From Fig.2(a), it can be seen clearly that as $p = 0$, i.e., in the nearest-neighbor network, $H(t)$ shows sporadic and disorder oscillation, namely there is no phase order in systems. As the probability p increases, the slight regular phase can be observed which is a shown in Fig.2(b). As p increases to $p = 0.11$, the maximal PS of network appears (see Fig.2(c)). The phase plot of the amplitudes of two randomly selected chaotic oscillators (A_i and A_j) are shown in Fig. 3. When the network is chaotic phase synchronized, its amplitudes remain chaotic.

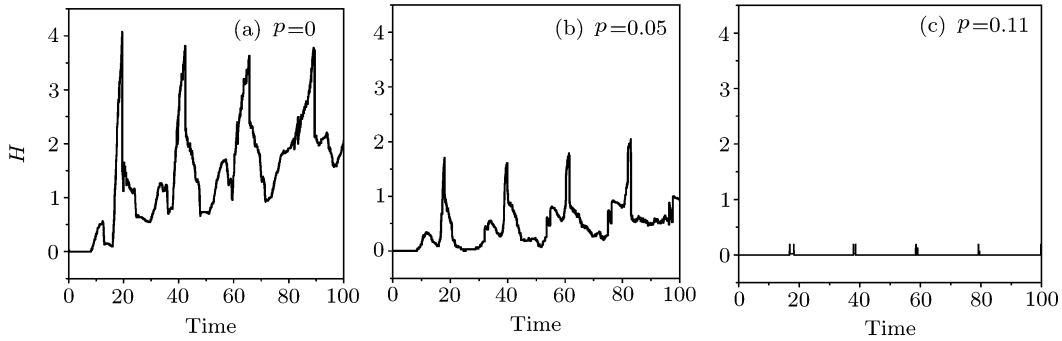


Fig.2. Time evolutions of mean phase difference $H(t)$ for different topological probability p with $C = 2.0$: (a) $p = 0.0$; (b) $p = 0.05$; (c) $p = 0.11$.

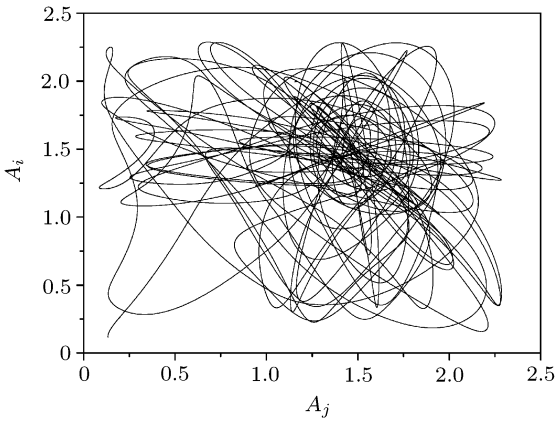


Fig.3. The phase plot of the amplitudes of two randomly selected chaotic oscillators with $p = 0.11$ and $C = 2.0$. In this case, the network is chaotic phase synchronized, and its amplitudes remain chaotic.

To quantitatively study the degree of PS, we introduce the standard order parameter to measure the phase order of the network:

$$r = \left\{ \left\langle \left| \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\phi_j} \right| \right\rangle \right\},$$

where $\langle\cdot\rangle$ and $\{\cdot\}$ denote the averages over time and the average over 100 different network realizations with the same p, respectively. ϕ_j has the same definition as above. The order parameter r is a normalized quantity whose values are in the range from 0 to 1, describing a continuum between a uniform distribution and a perfect PS.

The dependences of r on p with different C can be shown in Fig.4, from which one can see clearly that for a given coupling strength, when the value of p is smaller, there are no PS patterns in the network and there exists a critical randomness p^*, such that if $p > p^*$ the PS occurs; the network reaches the complete PS as p further increases. It is also found that the larger C number, the smaller the value of randomness p is needed for the system to achieve a maximal

PS.

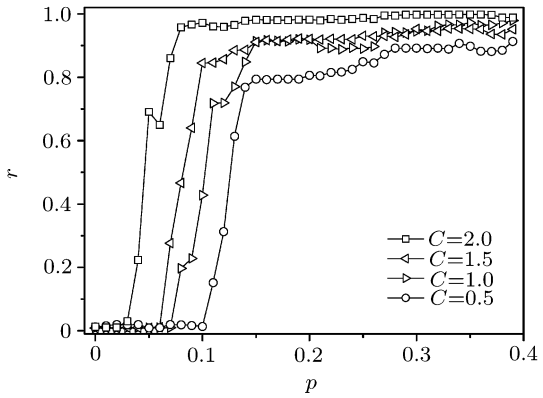


Fig.4. The measures of the phase synchronization vs. topological probability p with $C = 2.0$ ($\square$), $C = 1.5$ ($\triangle$), $C = 1.0$ ($\triangledown$), $C = 0.5$ ($\Diamond$).

4. Conclusion

In summary, PS of complex Chua systems is studied in this paper. In order to assess the effect of topological randomness p upon the PS, we have employed the standard order parameter r as a measure of the phase order. It is found that when the randomness p is small there is no PS in the networks. With the randomness p increases to a critical value, the PS occurs. PS becomes stronger and reaches its maximum when p is further increased. These phenomena imply that random shortcuts can induce and enhance the PS of complex Chua systems. The effect of the coupling strength is also studied and it is found that stronger coupling makes it easier to induce and enhance PS.

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