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Pinning control of complex Lur'e networks*

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This paper addresses the control problem of a class of complex dynamical networks with each node being a Lur'e system whose nonlinearity satisfies a sector condition, by applying local feedback injections to a small fraction of the nodes. The pinning control problem is reformulated in the framework of the absolute stability theory. It is shown that the global stability of the controlled network can be reduced to the test of a set of linear matrix inequalities, which in turn guarantee the absolute stability of the corresponding Lur'e systems whose dimensions are the same as that of a single node. A circle-type criterion in the frequency domain is further presented for checking the stability of the controlled network graphically. Finally, a network of Chua's oscillators is provided as a simulation example to illustrate the effectiveness of the theoretical results.

Keywords: complex dynamical network, Lur'e system, pinning control, linear matrix inequality
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1. Introduction

Complex dynamical networks have received increasing attention from various scientific communities in recent years, see Refs.[1]–[6] and references therein. The main reason is that many physical systems can be characterized by various complex network models, in which nodes are the elements of the system and the edges represent the interactions among them. Typical examples include the Internet, the World Wide Web, metabolic networks, scientific citation webs, and so forth.[7]

Traditionally, complex dynamics have been studied in the framework of random graph theory, introduced by Erdös and Rényi.[8] This model has dominated the research of complex networks for nearly half a century. The past few years have witnessed dramatic advances in this direction. Watts and Strogatz introduced the so-called small-world network model to represent the transition from a regular network to a random one.[5] Such networks exhibit a high degree of clustering along with a short average distance between nodes. Another even more important discovery is the scale-free network model, introduced by Barabási and Albert, of which the connectivity distribution has a power-law form.[9] A distinctive feature of scale-free networks is that most nodes have very few connections while a small number of nodes have many connections, leading to the interesting "robust yet fragile" characteristic.[10]

Recently, a large body of literature has focused on the research of dynamics and control problems of complex dynamical networks. The synchronization phenomenon in both small-world networks and scale-free networks is investigated in Refs.[11]–[17], and the synchronization of the general complex dynamical network with coupling delays is considered in Refs.[18]–[20]. The control problem of scale-free networks is investigated by applying local feedback injections only to a small fraction of nodes in Ref.[21], and the pinning control strategy is further used to study the stability of general complex networks with or without coupling delays in Refs.[1] and [22]–[25]. The controllability of complex networks via pinning is addressed in Ref.[26]. The synchronizabililty of complex dynamical networks is completely analyzed through a graph and control theoretic viewpoint in Refs.[27]–[29]. In these studies, a common approach is to linearize the nonlinear dynamical nodes around the network synchronized state, thereby obtaining some local results.

Lur'e systems refer to such systems that consist of a linear dynamical system and a nonlinear feedback

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loop satisfying a sector condition, which have many engineering applications.[30] Several chaotic and hyperchaotic systems, including Chua's circuit, n-scroll circuits, coupled Chua's circuits and others, can be represented by the Lur'e form. The master-slave synchronization scheme for Lur'e systems is examined in Refs.[31]–[33]. The stabilization of a class of dynamical complex networks with each node being a Lur'e system is addressed by a decentralized approach in Ref.[34]. Therefore, to study the synchronization and control problems of Lur'e systems is of significant importance.

This paper concerns the pinning control problem of a class of complex dynamical networks with each node being a Lur'e system. The pinning control problem is reformulated in the framework of the absolute stability theory, consequently it is reduced to the test of a set of linear matrix inequalities (LMIs), which guarantee the absolute stability of the corresponding Lur'e systems whose dimensions are the same as that of a single node. A circle-type criterion in the frequency domain is further presented, in virtue of which the stability of the controlled network can be checked graphically. A network of Chua's oscillators is provided as an application example to illustrate the effectiveness of the theoretical results. The novelty of this paper is two-fold. First, no linearization techniques are employed, thus the conditions for stability derived in this paper are global. Second, the pinning control strategy proposed here has more degrees of freedom to choose controlled nodes, thus more desirable, compared to other control methods, e.g., the decentralized approach used in Ref.[34].

The rest of this paper is organized as follows. In Section 2, a dynamical network with each node being a Lur'e system is proposed. The main results are presented in Section 3. A network of Chua's oscillators is provided as a simulation example in Section 4. Section 5 concludes the paper.

2. Dynamical network model

Consider a complex network which consists of N linearly and diffusively coupled identical nodes, with each node being an n-dimensional Lur'e system:

$$\dot{x}_i = Ax_i + Bf(Cx_i(t)) + c\sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}\Gamma(x_j - x_i),$$
$$i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = [x_{i1}, \ldots, x_{in}]^T \in \mathbb{R}^n$ is the state variable of node i, matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C = [C_1^T, \ldots, C_m^T]^T \in \mathbb{R}^{m\times n}$, $f(Cx_i(t)) = [f_1(C_1x_i(t)), \ldots, f_m(C_mx_i(t)]^T$ is a nonlinear function satisfying the sector conditions on $[0, \delta_k]$:

$$0 \leq \frac{f_k(b) - f_k(a)}{b - a} \leq \delta_k, \quad f_k(0) = 0,$$
$$\forall\ b \neq a \in \mathbb{R},\ k = 1, 2, \ldots, m. \tag{2}$$

In network (1), the constant $c > 0$ represents the coupling strength, $\Gamma = (\tau_{ij})_{n\times n}$ is the inner linking matrix, in which if some $\tau_{ij} \neq 0$, then it means two coupled nodes are linked through their i-th and j-th state variables, $i, j = 1, 2, \ldots, N$. The coupling matrix $G = (G_{ij})_{N\times N}$ stands for the coupling configuration of the network, with entries defined as follows: If there is a connection between node i and node j $(i \neq j)$, then $G_{ij} = G_{ji} = 1$; otherwise, $G_{ij} = G_{ji} = 0\,(i \neq j)$. The diagonal elements of the coupling matrix are defined by the diffusive relationship:

$$G_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}, \quad i = 1, 2, \ldots, N. \tag{3}$$

Under this condition, network (1) can be rewritten as

$$\dot{x}_i = Ax_i + Bf(Cx_i(t)) + c\sum_{j=1}^{N} G_{ij}\Gamma x_j,$$
$$i = 1, 2, \ldots, N. \tag{4}$$

Clearly, the rows of the coupling matrix G have zero sum, which implies that zero is an eigenvalue of G, and the corresponding eigenvector is all one vector 1^T. Assume, hereafter, that the network has no isolate clusters, i.e., the network is connected. Then, the coupling matrix G is symmetric and irreducible. In this case, it can be shown that zero is a simple eigenvalue of G and all the other eigenvalues of G are strictly negative.[35]

3. The main results

Suppose that one wants to stabilize network (4) onto a homogeneous state defined by

$$x_1 = x_2 = \ldots = x_N = \bar{x}, \quad t \to \infty, \tag{5}$$

where $\bar{x}$ is a solution of a single node, namely,

$$\dot{\bar{x}}(t) = A\bar{x} + Bf(C\bar{x}). \tag{6}$$

To achieve the control goal (5), one applies the pinning control strategy on a small fraction δ $(0 <$

$\delta \ll 1$) of the nodes in the dynamical network (4). Without loss of generality, one can rearrange, if necessary, the order of the nodes in the dynamical network (4) such that nodes $1, 2, \ldots, q$, in the rearranged network are selected to pin, where $q = [\delta N]$ stands for the smaller but nearest integer to the real number δN. Then, the controlled network can be described as

$$\begin{aligned}\dot{x}_i &= Ax_i + Bf(Cx_i(t)) + c\sum_{j=1}^{N} G_{ij}\Gamma x_j \\ &\quad - cd_i\Gamma(x_i - \bar{x}), \quad i = 1, 2, \ldots, q, \\ \dot{x}_i &= Ax_i + Bf(Cx_i(t)) + c\sum_{j=1}^{N} G_{ij}\Gamma x_j, \\ &\quad i = q+1, q+2, \ldots, N, \end{aligned} \tag{7}$$

where $d_k > 0$ for $k = 1, 2, \ldots, q$, are the feedback gains.

Let $e_i(t) = x_i(t) - \bar{x}, i = 1, \ldots, N$. Then, one can obtain the error dynamics from Eqs.(7) as follows:

$$\begin{aligned}\dot{e}_i &= Ae_i + B\phi(Ce_i; \bar{x}) + c\sum_{j=1}^{N} G_{ij}\Gamma e_j - cd_i\Gamma e_j, \\ &\quad i = 1, 2, \ldots, q, \\ \dot{e}_i &= Ae_i + B\phi(Ce_i; \bar{x}) + c\sum_{j=1}^{N} G_{ij}\Gamma e_j, \\ &\quad i = q+1, q+2, \ldots, N, \end{aligned} \tag{8}$$

where $\phi(Ce_i; \bar{x}) = [\phi_1(C_1e_i; \bar{x}), \ldots, \phi_m(C_me_i; \bar{x})]^m = f(Ce + C\bar{x}) - f(C\bar{x})$, with $\phi_k(C_ke_i; \bar{x})$ satisfying the sector conditions on $[0, \delta_k]$:

$$\begin{aligned}&0 \le \frac{\phi_k(C_kx_i; \bar{x})}{C_kx_i} \le \delta_k, \\ &\phi_k(0) = 0, \quad k = 1, 2, \ldots, m. \end{aligned} \tag{9}$$

Define the following matrix $\hat{D} = \mathrm{diag}(d_1, d_2, \ldots, d_N)$, in which $d_j = 0$ if $j \in \{q+1, q+2, \ldots, N\}$. By letting $e(t) = [e_1(t)^T, \ldots, e_N(t)^T]^T$, $\Phi((I_N \otimes C)e; \bar{x}) = [\phi(Ce_1; \bar{x})^T, \ldots, \phi(Ce_N; \bar{x})^T]^T$, one can write the error dynamics in a compact form, as

$$\begin{aligned}\dot{e} &= (I_N \otimes A + c\hat{G} \otimes \Gamma)e \\ &\quad + (I_N \otimes B)\Phi((I_N \otimes C)e; \bar{x}), \end{aligned} \tag{10}$$

where $\otimes$ denotes the Kronecker product, I_N means an $N \times N$ identity matrix, $\hat{G} = G - \hat{D}$. Hence, the absolute stability problem of controlled network (7) for all the nonlinear functions satisfying the sector conditions (2) is transferred to the absolute stability problem of the error system (8) for all the nonlinear functions satisfying the sector conditions (9).

Before moving forward, one introduces the following preliminary lemmas, which will be useful for future analysis.

Lemma 1.[36] System (6) is absolutely stable about zero for every function $f(s) = [f_1(s), \ldots, f_m(s)]^T$ satisfying $f_i(C_ks)(f_i(C_ks) - \delta_iC_ks) < 0, i = 1, \ldots, m$, if there exist matrices $P = P^T > 0$ and $T = \mathrm{diag}(\tau_1, \ldots, \tau_m) \ge 0$ such that the following LMI holds:

$$\begin{bmatrix} A^TP + PA & PB + C^T\Delta T \\ B^TP + T\Delta C & -2T \end{bmatrix} < 0, \tag{11}$$

where $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_m)$.

Lemma 2.[23] Suppose $M = (M_{ij})_{N\times N}$ is an irreducible and symmetric matrix satisfying $M_{ij} \ge 0$, if $i \ne j$, and $\sum_{j=1}^{N} M_{ij} = 0$, for $i = 1, 2, \ldots, N$, diagonal matrix $R = \mathrm{diag}(r_1, r_2, \ldots, r_N) \ge 0$ with at least one $r_i > 0$, for $i = 1, 2, \ldots, N$. Then, all the eigenvalues of $M - R$ are negative.

Clearly, $\hat{G} = G - \hat{D}$ is a matrix satisfying Lemma 2, therefore it is negative-definite. From Lemma 1, one obtains the following result for the absolute stability of the controlled network (7).

Theorem 1 Let $0 > \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_l, 1 \le l \le N$, be the distinct eigenvalues of $\hat{G}$. Then, the controlled network (7) is absolutely stable about zero for all nonlinear functions satisfying the sector conditions (2), if there exist matrices $M = \mathrm{diag}(m_1, \ldots, m_m) \ge 0$ and $Q_i > 0, i = 1, \ldots, l$, satisfying the following LMIs:

$$\begin{bmatrix} (A + c\lambda_i\Gamma)^TQ_i + Q_i(A + c\lambda_i\Gamma) & Q_iB + C^T\Delta M \\ B^TQ_i + M\Delta C & -2M \end{bmatrix} < 0, \quad i = 1, \ldots, l, \tag{12}$$

where Δ is defined in Lemma 1.

Proof From Lemma 1, the error system (10) is absolutely stable about zero for all nonlinear functions satisfying the sector conditions (9), if there exist a diagonal matrix $M = \mathrm{diag}(m_1, \ldots, m_m)$ and a symmetrical

matrix $Q > 0$ such that

$$\begin{bmatrix} (I_N \otimes A + c\hat{G} \otimes \Gamma)^T Q + Q(I_N \otimes A + c\hat{G} \otimes \Gamma) & \Omega \\ \Omega^T & -I_N \otimes 2M \end{bmatrix} < 0, \tag{13}$$

where $\Omega = Q(I_N \otimes B) + (I_N \otimes C^T)(I_N \otimes \Delta)(I_N \otimes M)$. Let U be a unitary matrix such that $U^T \hat{G} U = \Lambda = \mathrm{diag}(\underbrace{\lambda_1, \cdots, \lambda_1}_{p_1}, \ldots, \underbrace{\lambda_i, \cdots, \lambda_i}_{p_i}, \ldots, \underbrace{\lambda_l, \cdots, \lambda_l}_{p_l})$, where $\lambda_i, i = 1, \ldots, l$, are all the distinct eigenvalues of $\hat{G}$ with multiplicity p_i such that $\sum_{i=1}^{l} p_i = N$. Take $X = \mathrm{diag}(U \otimes I_n, U \otimes I_m)$. Then, multiplying (13) to the left by X^T and to the right by X gives

$$\begin{bmatrix} (I_N \otimes A + c\Lambda \otimes \Gamma)^T \tilde{Q} + \tilde{Q}(I_N \otimes A + c\Lambda \otimes \Gamma) & \tilde{\Omega} \\ \tilde{\Omega}^T & -I_N \otimes 2M \end{bmatrix} < 0, \tag{14}$$

where $\tilde{Q} = (U^T \otimes I_n)Q(U \otimes I_n), \tilde{\Omega} = \tilde{Q}(I_N \otimes B) + I_N \otimes C^T \Delta M$. Clearly, in inequality (14), all the matrices except $\tilde{Q}$ are block-diagonal. Therefore, if there exist diagonal matrices M and $Q_i, i = 1, \ldots, l$, and satisfy inequality (12), then such M and $\mathrm{diag}(\underbrace{Q_1, \cdots, Q_1}_{p_1}, \ldots, \underbrace{Q_i, \cdots, Q_i}_{p_i}, \ldots, \underbrace{Q_l, \cdots, Q_l}_{p_l})$ satisfy inequality (14) also.

Remark 1 When the number of nodes N is large, the absolute stability of system (10) about zero for all nonlinear functions satisfying the sector conditions (9) is very hard to test directly. Theorem 1 avoids this difficulty by reducing the absolute stability problem of system (10) to the test of a set of LMIs with the existence of a common matrix M. The number of LMIs in inequality (12) to be verified is equal to the number of distinct eigenvalues of the coupling matrix G, sometimes much less than N. Note that the LMIs in expression (12) can guarantee the absolute stability of the following Lur'e systems:

$$\dot{x}_i = (A + c\lambda_i \Gamma)x_i + B\psi(Cx_i), \quad i = 1, \ldots, l, \tag{15}$$

in which the nonlinear function $\psi(Cx_1) = [\psi_1(C_1 x_1), \ldots, \psi_m(C_m x_1)]^T$, with $\psi_k(C_k x_1)$ satisfying the following sector conditions:

$$0 \le \frac{\psi_k(C_k x_1)}{C_k x_1} \le \delta_k, \quad \psi_k(0) = 0, \quad k = 1, 2, \ldots, m. \tag{16}$$

Clearly, the Lur'e networks in Eq.(15) are of the same dimension as that of a single node in Eq.(6).

The criterion given in Theorem 1 for the absolute stability of the controlled network (7) can be further refined for some special inner linking matrix Γ, as follows.

Corollary 1 Let $\Gamma = \mathrm{diag}(r_1, \ldots, r_n)$ be a constant $0-1$ matrix with $r_i = 1$ for some $1 \le i \le n$, which means that the coupled nodes are linked through their i-th state variables. Then, network (7) is absolutely stable about zero for all the nonlinear functions satisfying the sector conditions (2), if there exist matrices $M = \mathrm{diag}(m_1, \ldots, m_m) \ge 0$ and $Q_1 > 0$ such that

$$\begin{bmatrix} (A + c\lambda_1 \Gamma)^T Q_1 + Q_1(A + c\lambda_1 \Gamma) & Q_1 B + C^T \Delta M \\ B^T Q_1 + M\Delta C & -2M \end{bmatrix} < 0, \tag{17}$$

where Δ is defined in Lemma 1.

Proof The LMIs in expression (12) can be rewritten as

$$\begin{bmatrix} A^T Q_i + Q_i A & Q_i B + C^T \Delta M \\ B^T Q_i + M\Delta C & -2M \end{bmatrix} + \begin{bmatrix} 2c\lambda_i \Gamma Q_i & 0 \\ 0 & 0 \end{bmatrix} < 0, \quad i = 1, \ldots, l. \tag{18}$$

Clearly, the second term on the left-hand side of expression (18) is semi-negative-definite. Therefore, if there exist matrices $Q_1 > 0$ and $M \ge 0$ such that expression (17) holds, then such Q_1 and M satisfy the LMIs in expression (12) for $i = 2, \ldots, l$.

The criterion for the stability of the controlled network (7) provided in Theorem 1 is in the time domain. In what follows, some criteria in the frequency domain are derived. The following lemma, known as the Kalman–Yakubovich–Popov (KYP) lemma, establishes the equivalence between the matrix inequalities and the frequency inequalities.

Lemma 3 (KYP lemma)[37] Given $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and symmetric $M \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(jwI - A) \neq 0$ for all $w \in \mathbb{R}$, and (A, B) controllable, the following two statements are equivalent:
(i)

$$\begin{bmatrix} (jwI - A)^{-1}B \\ I \end{bmatrix}^* \Psi \begin{bmatrix} (jwI - A)^{-1}B \\ I \end{bmatrix} \leq 0, \quad \forall\ w \in \mathbb{R} \cup \{\infty\}. \tag{19}$$

(ii) There exists a matrix $P = P^T$ such that

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} + \Psi \leq 0. \tag{20}$$

The corresponding equivalent for strict inequalities holds even if (A, B) is not controllable.

Define $G_i(s) = C(sI_n - A - c\lambda_i\Gamma)^{-1}B$, and denote by $\mathrm{Re}\{G_i(jw)\}$ the real part of $G_i(jw)$ and by $\mathrm{Im}\{G_i(jw)\}$ the imaginary part of $G_i(jw)$. The following results present frequency-domain interpretations of Theorem 1 and Corollary 1, respectively.

Theorem 2 (MIMO circle-type criterion) The controlled network (7) is absolutely stable about zero for all nonlinear functions satisfying the sector conditions (2), if $A+c\lambda_i\Gamma, i = 1, \ldots, l$, are stable, and there exists a diagonal matrix $M = \mathrm{diag}(m_1, \ldots, m_m) \geq 0$ such that the following inequalities hold:

$$\begin{aligned} &\mathrm{Re}\{M\Delta G_i(jw) - 2M\} < 0, \\ &i = 1, \ldots, l,\ \forall\ w \in R, \end{aligned} \tag{21}$$

where Δ is defined in Lemma 1.

Proof The LMIs in expression (12) can be recast as

$$\begin{bmatrix} \bar{A}^TQ_i + Q_i\bar{A} & Q_iB \\ B^TQ_i & 0 \end{bmatrix} + \Psi < 0, \quad i = 1, \ldots, l, \tag{22}$$

where

$$\bar{A} = A + c\lambda_i\Gamma, \quad \Psi = \begin{bmatrix} 0 & C^T\Delta M \\ M\Delta C & -2M \end{bmatrix}. \tag{23}$$

In light of Lemma 3, the inequalities in expression (22) are equivalent to

$$\begin{aligned} &\begin{bmatrix} (jwI - \bar{A})^{-1}B \\ I \end{bmatrix}^* \Psi \begin{bmatrix} (jwI - \bar{A})^{-1}B \\ I \end{bmatrix} \leq 0, \\ &\forall\ w \in \mathbb{R}, \end{aligned} \tag{24}$$

which after some calculations leads to the theorem.

Corollary 2 Let $\Gamma = \mathrm{diag}(r_1, \ldots, r_n)$ be a constant $0-1$ matrix with $r_i = 1$ for some $1 \leq i \leq n$. Then, network (7) is absolutely stable about zero for all nonlinear functions satisfying the sector conditions (2), if $A + c\lambda_1\Gamma$ is stable and there exists a diagonal matrix $M = \mathrm{diag}(m_1, \ldots, m_m) \geq 0$ such that the following inequality holds:

$$\mathrm{Re}\{M\Delta G_1(jw) - 2M\} < 0, \quad \forall\ w \in R, \tag{25}$$

where Δ is defined in Lemma 1.

The conditions given in Theorem 2 and Corollary 2 for the absolute stability of network (7) will be referred to as multi-input multi-output (MIMO) circle-type criteria.

For each node being a single-input single-output (SISO) Lur'e system, the conditions given in Theorem 2 and Corollary 2 can be tested graphically. Let $M = \gamma$ and $\Delta = \bar{\delta}$ in expression (21), so that expression (21) becomes

$$\begin{aligned} &\mathrm{Re}\{G_i(jw)\} - \bar{\delta}^{-1} < 0, \\ &i = 1, \ldots, l,\ \forall\ w \in \mathbb{R}, \end{aligned} \tag{26}$$

which is equivalent to that all the Nyquist plots of $G_i(jw), i = 1, \ldots, l$, lie in the left half-plane of the vertical line defined by $\mathrm{Re}\{s\} = \bar{\delta}^{-1}$. The Nyquist plot is a plot of $\mathrm{Re}\{G_i(jw)\}$ versus $\mathrm{Im}\{G_i(jw)\}$ with w as a parameter varying from $-\infty$ to ∞. The above discussions are summarized in the following corollary.

Theorem 3 (SISO circle-type criterion) The controlled network (7) with each node being an SISO Lur'e system is absolutely stable about zero for all nonlinear functions satisfying the sector conditions (2), if $A + c\lambda_i\Gamma, i = 1, \ldots, l$, are stable and all the Nyquist plots of $G_i(jw), i = 1, \ldots, l$, lie in the left half-plane of the vertical line defined by $\mathrm{Re}\{s\} = \bar{\delta}^{-1}$.

Similarly, one can immediately obtain the following result, corresponding to Corollary 2.

Corollary 3 The controlled network (7), with each node being an SISO Lur'e system and inner linking matrix Γ defined in Corollary 1, is absolutely stable about zero for all nonlinear functions satisfying the sector conditions (2), if $A + c\lambda_1\Gamma$, is stable and the Nyquist plot of $G_1(jw)$ lies in the left half-plane of the vertical line defined by $\mathrm{Re}\{s\} = \bar{\delta}^{-1}$.

4. Simulation examples

In this section, a network of Chua's oscillators, which are special Lur'e systems, is provided to illustrate the theoretic results.

The dimensionless state equations of a Chua's oscillator are given by

$$\begin{aligned}\dot{x}_1 &= \alpha(-x_1 + x_2 - f(x_1)),\\ \dot{x}_2 &= x_1 - x_2 + x_3,\\ \dot{x}_3 &= -\beta x_2 - \gamma x_3,\end{aligned} \tag{27}$$

where $f(x_1)$ is the piecewise-linear characteristic of Chua's diode, described as,

$$\begin{aligned}&f(x_1)\\ &= m_1 x_1 + \frac{1}{2}(m_2 - m_1)(|x_1 + 1| - |x_1 - 1|).\end{aligned} \tag{28}$$

One can rewrite Eqs.(27) in the Lur'e form as

$$\dot{x} = Ax + B\varphi(Cx) \tag{29}$$

where

$$\begin{aligned}x &= \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \quad A = \begin{bmatrix}-\alpha(m_1+1) & \alpha & 0\\ 1 & -1 & 1\\ 0 & -\beta & -\gamma\end{bmatrix},\\ B &= \begin{bmatrix}-\alpha(m_2 - m_1)\\ 0\\ 0\end{bmatrix}, \; C = \begin{bmatrix}1 & 0 & 0\end{bmatrix},\end{aligned} \tag{30}$$

and the nonlinear function $\varphi(Cx) = \frac{1}{2}(|x_1+1| - |x_1 - 1|)$ satisfies the sector condition on $[0, 1]$.

Taking parameters $\alpha = 9$, $\beta = 14$, $\gamma = 0.01$, $m_1 = -0.714$, $m_2 = -1.14$, Chua's oscillator (27) generates a double-scroll attractor,[38] as depicted in Fig.1.

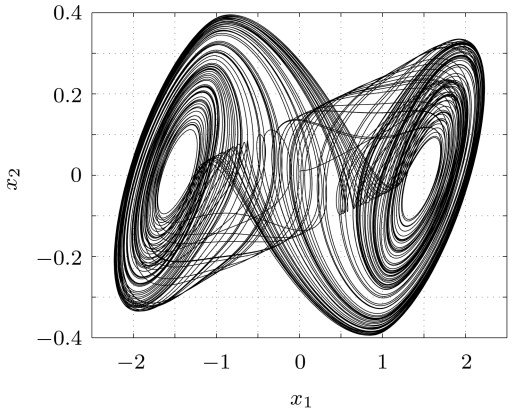


Fig.1. Attractor of Chua's oscillator (27) in the x_1–x_2 plane, where $\alpha = 9$, $\beta = 14$, $\gamma = 0.01$, $m_1 = -0.714$, $m_2 = -1.14$.

In this case, the Chua's oscillator (27) has three unstable equilibrium points:

$$\begin{aligned}S_1 &= [1.49324, 0.00107, -1.49217]^T,\\ S_2 &= [-1.49324, -0.00107, 1.49217]^T,\\ S_3 &= [0, 0, 0]^T.\end{aligned} \tag{31}$$

For simplicity, one considers below the pinning control problem of 200 Chua's oscillators coupled by a B-A scale-free network. Suppose the inner coupling matrix as $\Gamma = \mathrm{diag}(1, 1, 1)$. One wants to stabilize the dynamical network onto the equilibrium point $\bar{x} = S_1$, by applying local linear feedback injections to a small fraction of the nodes. The state equations of the controlled network are

$$\begin{aligned}\dot{x}_i &= Ax_i + B\varphi(Cx_i) + c\sum_{j=1}^{8} G_{1j}\, x_j - cd_i(x_i - \bar{x}),\\ &\quad i = 1, \ldots, l,\\ \dot{x}_i &= Ax_i + B\varphi(Cx_i) + c\sum_{j=1}^{8} G_{ij}\, x_j,\\ &\quad i = l+1, \ldots, 200.\end{aligned} \tag{32}$$

Consider here two different pinning control schemes, i.e., specifically pinning scheme and randomly pinning scheme. In the first case, the dynamical network (32) will be controlled by specifically pinning 4 "biggest" nodes, which have 41, 34, 35 and 32 degrees, respectively. Choose the feedback gain in (32) as $d = 10$. In this case, the largest eigenvalues of $\hat{G}$ is $\lambda_1 = -0.1464$. Hence, in virtue of the circle-type criterion (Corollary 3), the minimum coupling strength guaranteeing the global stability of the controlled network (32) is $c = 15.06$. This result can also be derived from Corollary 1 by using

the LMI Toolbox of Matlab.[39] In the second case, network (32) will be controlled by randomly pinning 4 nodes with the same feedback gain $d = 10$. The minimum coupling strength c in this case turns to $c = 35.22$. Figure 2 depicts the Nyquist plots of $G_1(jw) = C(jwI - A - c\lambda_1 I)^{-1}B$ for both the cases, when $c = 15.06$. Figure 3 visualizes the state variables of node $i, i = 1, \ldots, 200$, in network (32) for both the specifically and randomly pinning schemes when the coupling strength takes the value $c = 36$.

It can be observed that a much larger value of the coupling strength c is required to stabilize the whole network on the homogeneous state S_1 in the randomly pinning scheme than that of the specifically pinning scheme, as depicted in Figs.2 and 3. This essentially results from the heterogenous nature of scale-free networks, i.e., most nodes have very few connections while a small number of nodes have many connections.

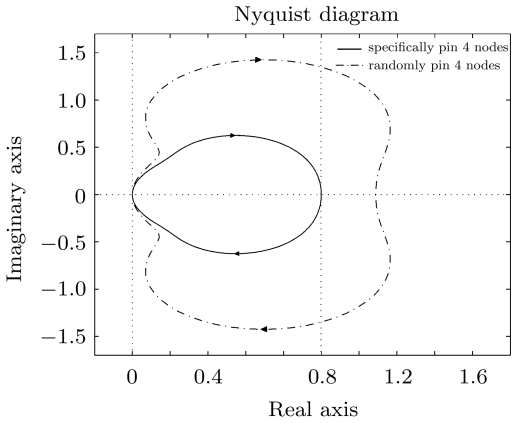


Fig.2. The Nyquist plots of $G_1(jw)$ with $c = 15.06$ for both the specifically and randomly pinning schemes.

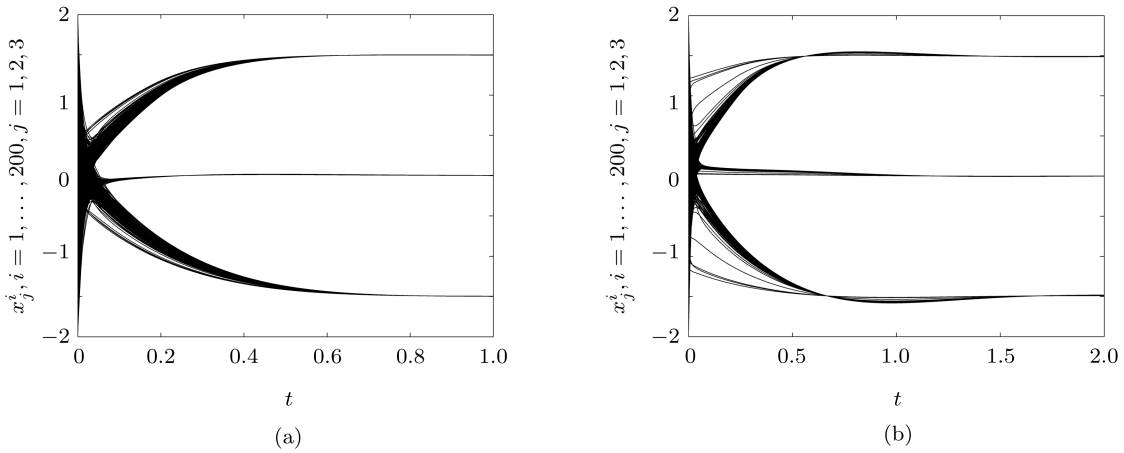


Fig.3. State variables of network (32) with $c = 36$, $d = 10$. (a) Specifically pin 4 "largest" nodes; (b) randomly pin 4 nodes.

5. Conclusions

In this paper, the pinning control problem of a class of complex dynamical networks with each node being a Lur'e system is investigated. The pinning control problem is reformulated in the framework of the absolute stability theory. The stability of the controlled network is consequently reduced to the test of a set of LMIs guaranteeing the absolute stability of the corresponding Lur'e systems whose dimensions are the same as that of a single node. A circle-type criterion in the frequency domain is further presented for checking the stability of the controlled network graphically. One desirable characteristic of the results derived in this paper is that no linearization techniques are employed, thus the conditions for stability are global. Another prominent feature is that the pinning control strategy applied here requires much fewer controllers to be designed, thus is less costly, compared to other control methods, such as the decentralized approach.

References

[1] Li X, Wang X F and Chen G R 2004 *IEEE Trans. Circ. Syst.* -I **51** 2074

[2] Lu X B, Wang X F and Fang J Q 2006 *Physica* A **371** 841

[3] Qiao B and Fang J Q 2007 *Physica* A **383** 753

[4] Wang W X, Wang B H, Hu B, Yan G and Ou Q 2005 *Phys. Rev. Lett.* **94** 188702

[5] Watts D J and Strogatz S H 1998 *Nature* **393** 440

[6] Xie Y B, Wang W X and Wang B H 2007 *Phys. Rev.* E **75** 026111

[7] Albert R and Barabasi A L 2002 *Rev. Mod. Phys.* **74** 47

[8] Erdös P and Rényi A 1960 *Publ. Math. Inst. Hung. Acas. Sci.* **5** 17

[9] Barabasi A L and Albert R 1999 *Science* **286** 509

[10] Albert R, Jeong H and Barabasi A L 2000 *Nature* **406** 387

[11] Gade P M and Hu C K 2000 *Phys. Rev.* E **62** 6409

[12] Barahona M and Pecora L M 2002 *Phys. Rev. Lett.* **89** 054101

[13] Lü J H and Chen G R 2005 *IEEE Trans. Circ. Syst.* -I **52** 841

[14] Wang X F and Chen G R 2002 *IEEE Trans. Circ. Syst.* -I **49** 54

[15] Li L X, Luo Q, Peng H P, Wu W and Yang Y X 2008 *Acta Phys. Sin.* **57** 1529 (in Chinese)

[16] Cui B T and Wu W 2007 *Chin. Phys.* **16** 1889

[17] Yu H J and Zheng N 2008 *Acta Phys. Sin.* **57** 4712 (in Chinese)

[18] Gao H, Lam J and Chen G R 2006 *Phys. Lett.* A **360** 263

[19] Li X, Jin Y and Chen G R 2004 *Physica* A **328** 367

[20] Zhang Q, Lu J, Lü J H and Tse C K 2008 *IEEE Trans. Circ. Syst.* -II **55** 183

[21] Wang X F and Chen G R 2002 *Physica* A **310** 521

[22] Chen T, Liu X and Lu W 2007 *IEEE Trans. Circ. Syst.* -I **54** 1317

[23] Liu Z X, Chen Z and Yuan Z 2007 *Physica* A **375** 345

[24] Min F H and Wang Z Q 2008 *Acta Phys. Sin.* **57** 31 (in Chinese)

[25] Li X C, Tang S P, Xiao Y Z and Xu W 2008 *Chin. Phys.* B **17** 80

[26] Sorrentino F, di Bernardo M, Garogalo F and Chen G R 2007 *Phys. Rev.* E **75** 046103

[27] Duan Z S, Chen G R and Huang L 2007 *Phys. Rev.* E **76** 056103

[28] Duan Z S, Chen G R and Huang L 2008 *Phys. Lett.* A **372** 3741

[29] Duan Z S, Liu C and Chen G R 2008 *Physica* D **237** 1006

[30] Khalil H K 1992 *Nonlinear Systems* (New York: Macmillan)

[31] Suykens J A K, Curran P F and Chua L O 1997 *Int. J. Bifur. Chaos* **7** 671

[32] Suykens J A K, Curran P F and Chua L O 1997 *IEEE Trans. Circ. Syst.* -I **44** 891

[33] Suykens J A K, Curran P F and Chua L O 1999 *IEEE Trans. Circ. Syst.* -I **46** 841

[34] Duan Z S, Wang J, Chen G R and Huang L 2008 *Automatica* **44** 1028

[35] Horn R and Johnson C 1985 *Matrix Analysis* (Cambridge: Cambridge Univ. Press)

[36] Boyd S, El Ghaoui L, Feron E and Balakrishnan V 1994 *Linear Matrix Inequalities in System and Control Theory* (Philadelphia MA: SIMA)

[37] Rantzer A 1996 *Sys. Contr. Lett.* **28** 7

[38] Wu C W 2002 *Synchronization in Coupled Chaotic Circuits and Systems* (Singapore: World Scientific)

[39] Gahinet P, Nemirovski A and Laub A J 1995 *LMI Control Toolbox* (Natick, Mass: The Math Works, Inc)