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# A new method of researching fermion tunneling from the Vaidya–Bonner de Sitter black hole*

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Using the general tortoise coordinate transformation, we research the fermion tunneling of the Vaidya–Bonner de Sitter black hole via a semi-classical method and finally obtain the right surface gravity, Hawking temperature and tunneling rate near the event horizon and cosmical horizon.

**Keywords:** Vaidya–Bonner de Sitter black hole, Dirac equation, Hawking radiation, general tortoise coordinate transformation

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## 1. Introduction

Due to Hawking's research on quantum theory near the event horizon of black holes,[1,2] most theory physicists believe that the Hawking radiation could be created near the horizon of a black hole. Recently, Hawking radiation from several black holes has been studied, and several methods which can prove the Hawking radiation have been developed. The Hawking radiation theory could be related to the quantum gravity theory, so that the study of Hawking radiation is very interesting.[3−13] There are two equivalent viewpoints that can explain the origin of Hawking radiation. Due to the quantum vacuum fluctuation near the horizon of black holes, the positive-negative particles could be created or annihilated, and the negative energy particles could cross the black hole horizon since there is a negative energy orbit inside the black hole. Meanwhile, the positive energy particles cannot cross the horizon, but radiate to infinite finally. On the one hand, we can study the phenomenon from another point of view. The virtual particles inside black holes cross the horizon and become real particles, and it is no other than the Hawking radiation. The second viewpoint may be regarded as the theory base of tunneling phenomenon near the horizon. Recently, the semi-classical approximation tunneling method has been used to research Hawking radiation in scalar field theory. A semi-classical approximate method to study fermion tunneling of several black holes has been suggested in Refs.[14]–[20].

On the other hand, research on Hawking radiation from non-stationary black holes is very difficult. The case of non-stationary black holes is very different from the case of stationary ones. An excellent way to study the Hawking radiation via general tortoise coordinate transformation has been developed in Refs.[21]–[23]. In general tortoise coordinates, the scalar field equation can be simply in the wave equation form, and Zhao Zheng *et al* have used the Damour–Ruffini method to obtain the Hawking temperature and the surface gravity of non-stationary black holes.

## 2. General tortoise coordinate transformation in Vaidya–Bonner de Sitter spacetime

In this paper, we generalize the method in Vaidya–Bonner de Sitter spacetime. In the Dirac field, we introduce the general coordinate transformation and then obtain the tunneling rate and the Hawking temperature of the Vaidya–Bonner de Sitter black hole via the Dirac equation.

The metric of the Vaidya–Bonner de Sitter black

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hole is

$$\mathrm{d}s^2 = -\left(1-\frac{2M(v)}{r}+\frac{Q(v)^2}{r^2}-\chi^2 r^2\right)\mathrm{d}v^2 + 2\mathrm{d}r\mathrm{d}v + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right), \quad (1)$$

so the inverse metric is

$$\partial_s^2 = \left(1-\frac{2M(v)}{r}+\frac{Q(v)^2}{r^2}-\chi^2 r^2\right)\partial_r^2 + 2\partial_r\partial_v + r^{-2}\left(\partial_\theta^2+\sin^{-2}\theta\partial_\phi^2\right). \quad (2)$$

Before researching the tunneling radiation near the event horizon of the black hole, we introduce the general tortoise coordinate transformation as follows:

$$r_* = r + \frac{1}{2\kappa}\ln\left(r-r_{\mathrm{H}}(v)\right) \quad (3)$$

with

$$v_* = v - v_{\mathrm{H}},$$

where $\kappa$ is the surface gravity and $r_{\mathrm{H}}(v)$ is event horizon, so the inverse metric in tortoise coordinate is

$$\left(\frac{\partial}{\partial s}\right)^2 = f\left(\frac{\partial}{\partial r}\right)^2 + 2g\left(\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial r}\right) + r^{-2} \times\left(\left(\frac{\partial}{\partial\theta}\right)^2+\sin^{-2}\theta\left(\frac{\partial}{\partial\varphi}\right)^2\right) \quad (4)$$

and

$$f = \left(1-\frac{2M(v)}{r}+\frac{Q(v)^2}{r^2}-\chi^2 r^2\right)\times\left(1+\frac{1}{2\kappa(r-r_{\mathrm{H}})}\right)^2 - \frac{\dot{r}_{\mathrm{H}}}{\kappa(r-r_{\mathrm{H}})}\left(1+\frac{1}{2\kappa(r-r_{\mathrm{H}})}\right) \quad (5)$$

with

$$g = 1+\frac{1}{2\kappa(r-r_{\mathrm{H}})}. \quad (6)$$

In expression (5), $\dot{r}_{\mathrm{H}} = \partial r_{\mathrm{H}}/\partial v$. Therefore the Dirac equation may be written as

$$\mathrm{i}\boldsymbol{\gamma}^\mu D_\mu\boldsymbol{\Psi} + \frac{m}{\hbar}\boldsymbol{\Psi} = 0, \quad (7)$$

where

$$D_\mu = \partial_\mu + \boldsymbol{\Gamma}_\mu - \mathrm{i}e\boldsymbol{A}_\mu/\hbar \quad (8)$$

and

$$\boldsymbol{\Gamma}_\mu = -\frac{\Gamma_\mu^{\alpha\beta}}{8}\left[\boldsymbol{\gamma}^\alpha,\boldsymbol{\gamma}^\beta\right]. \quad (9)$$

In the spacetime $A_t = -Q/r$, and the $\boldsymbol{\gamma}^\mu$ matrixes must satisfy the relation

$$\left\{\boldsymbol{\gamma}^\alpha,\boldsymbol{\gamma}^\beta\right\} = 2g^{\alpha\beta}\boldsymbol{I}. \quad (10)$$

In Eq.(10), $g^{\alpha\beta}$ is the component of adverse metric. Considering the Vaidya–Bonner de Sitter spacetime in general tortoise coordinate, the $\boldsymbol{\gamma}^\mu$ matrixes are

$$\boldsymbol{\gamma}^{r_*} = \sqrt{f}\boldsymbol{\gamma}^a = \sqrt{f}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}, \quad (11)$$

$$\boldsymbol{\gamma}^{v_*} = \frac{g}{\sqrt{f}}\boldsymbol{\gamma}^b = \frac{g}{\sqrt{f}}\begin{pmatrix}\mathrm{i}&0&1&0\\0&\mathrm{i}&0&-1\\1&0&-\mathrm{i}&0\\0&-1&0&-\mathrm{i}\end{pmatrix}, \quad (12)$$

$$\boldsymbol{\gamma}^\theta = \frac{1}{r}\boldsymbol{\gamma}^c = \frac{1}{r}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}, \quad (13)$$

and

$$\boldsymbol{\gamma}^\phi = \frac{1}{r\sin\theta}\boldsymbol{\gamma}^d = \frac{1}{r\sin\theta}\begin{pmatrix}0&0&0&-\mathrm{i}\\0&0&\mathrm{i}&0\\0&-\mathrm{i}&0&0\\\mathrm{i}&0&0&0\end{pmatrix}. \quad (14)$$

From the $\boldsymbol{\gamma}^\mu$ matrixes, we obtain the Dirac equation

$$\mathrm{i}\sqrt{f}\boldsymbol{\gamma}^a D_{v_*}\boldsymbol{\Psi} + \mathrm{i}\frac{g}{\sqrt{f}}\boldsymbol{\gamma}^b D_{r_*}\boldsymbol{\Psi} + \mathrm{i}\frac{\boldsymbol{\gamma}^c}{r}D_\theta\boldsymbol{\Psi} + \mathrm{i}\frac{\boldsymbol{\gamma}^d}{r\sin\theta}D_\phi\boldsymbol{\Psi} + \frac{m}{\hbar}\boldsymbol{\Psi} = 0. \quad (15)$$

In the semi-classical approximation, we can obtain the conclusive expression in the approximate method

$$D_\mu \approx \partial_\mu - \mathrm{i}e\boldsymbol{A}_\mu/\hbar. \quad (16)$$

The conclusive expression will be used when the research on the tunneling equations is carried out, and we have applied the conclusive expression to Eq.(15). Considering the Dirac equation in the tortoise coordinate as expressed in the following form:

$$\mathrm{i}\sqrt{f}\boldsymbol{\gamma}^a D_{v_*}\boldsymbol{\Psi} + \mathrm{i}\sqrt{f}\boldsymbol{\gamma}^b D_{r_*}\boldsymbol{\Psi} + \boldsymbol{G} = 0, \quad (17)$$

where $G$ is the non-radial term of the Dirac equation, we can obtain the formula

$$1=\frac{f}{g}=\lim_{\substack{r\to r_{\rm H}(v_{\rm H})\\ v\to v_{\rm H}}}\frac{\left(r^2-2M\left(v\right)r+Q\left(v\right)^2-\chi^2r^4\right)\left(2\kappa\left(r-r_{\rm H}\right)+1\right)-2r\dot{r}_{\rm H}}{2\kappa r^2\left(r-r_{\rm H}\right)}. \tag{18}$$

Via the L'Hospital's rule, we can obtain the surface gravity of the black hole as

$$\kappa=\frac{\left(1-2\dot{r}_{\rm H}\right)r_{\rm H}-2\chi^2r_{\rm H}^3-M}{\chi^2r_{\rm H}^4+2Mr_{\rm H}-Q^2}. \tag{19}$$

Next, we study the tunneling rate of the Vaidya–Bonner de Sitter black hole.

## 3. Fermion tunneling of the Vaidya–Bonner de Sitter black hole

In fact, we can write the solutions of Eq.(15) as

$$\boldsymbol{\Psi}_{\uparrow}=\begin{bmatrix}A\\0\\B\\0\end{bmatrix}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\uparrow}}, \tag{20}$$

for the spin-up case, and

$$\boldsymbol{\Psi}_{\downarrow}=\begin{bmatrix}0\\C\\0\\D\end{bmatrix}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\downarrow}} \tag{21}$$

for the spin-down case.

To start with, we consider the spin-up case. Substituting expression (20) into Eq.(15), and dividing it by the exponential term and then multiplying the resulting equation by $\hbar$, we can obtain the Dirac equation as

$$\sqrt{f}B\frac{\partial S_{\uparrow}}{\partial r_*}+\sqrt{f}\left(\mathrm{i}\,A+B\right)\left(\frac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)-mA=0, \tag{22}$$

$$\frac{B}{r}\left(\frac{\partial S_{\uparrow}}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial S_{\uparrow}}{\partial\varphi}\right)=0, \tag{23}$$

$$\sqrt{f}A\frac{\partial S_{\uparrow}}{\partial r_*}+\sqrt{f}\left(A-\mathrm{i}\,B\right)\left(\frac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)-mB=0, \tag{24}$$

and

$$\frac{A}{r}\left(\frac{\partial S_{\uparrow}}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial S_{\uparrow}}{\partial\varphi}\right)=0. \tag{25}$$

From these equations, we can see that expression (16) holds true in the approximate method. Equations (23) and (25) describe the non-radial behaviour of the fermions near the horizon. As for the radial behaviour, we use Eqs.(22) and (24) to research the fermion tunneling from the event horizon of the Vaidya–Bonner de Sitter black hole.

In fact, the radical equations can be written as

$$\begin{pmatrix}\sqrt{f}\dfrac{\partial S_{\uparrow}}{\partial r_*}+\sqrt{f}\left(\dfrac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right) & -m-\mathrm{i}\sqrt{f}\left(\dfrac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)\\ \mathrm{i}\sqrt{f}\left(\dfrac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)-m & \sqrt{f}\dfrac{\partial S_{\uparrow}}{\partial r_*}+\sqrt{f}\left(\dfrac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0. \tag{26}$$

It can be easily seen that if and only if the determinant of the coefficient matrix vanishes, the equation does have a non-trivial solution for $A$ and $B$, so we have

$$\begin{vmatrix}\sqrt{f}\dfrac{\partial S_{\uparrow}}{\partial r_*}+\sqrt{f}\left(\dfrac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right) & -m-\mathrm{i}\sqrt{f}\left(\dfrac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)\\ \mathrm{i}\sqrt{f}\left(\dfrac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)-m & \sqrt{f}\dfrac{\partial S_{\uparrow}}{\partial r_*}+\sqrt{f}\left(\dfrac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)\end{vmatrix}=0, \tag{27}$$

namely

$$\left(\frac{\partial S_{\uparrow}}{\partial r_*}\right)^2+2\frac{\partial S_{\uparrow}}{\partial r_*}\left(\frac{\partial S_{\uparrow}}{\partial v_*}-eA_t\right)-\frac{m^2}{f}=0. \tag{28}$$

On the other hand, assume the function $\rho_\uparrow = C\mathrm{e}^{\frac{\mathrm{i}\,S_\uparrow}{\hbar}}$ such that it satisfies the following equation

$$\frac{\partial^2\rho_\uparrow}{\partial r_*^2} + 2\frac{\partial^2\rho_\uparrow}{\partial r_*\partial v_*} - \mathrm{i}\,2eA_t\frac{\partial\rho_\uparrow}{\partial r_*} = 0. \tag{29}$$

Near the event horizon $r_\mathrm{H}$ of the Vaidya black hole, $f^{-1}\to 0$, and equation (28) is an approximate equation in which the $\hbar$ order term is ignored, so the equation, near the horizon, is equivalent to Eq.(29). Solving Eq.(29) yields the solutions

$$\rho_{\uparrow\mathrm{in}} = \mathrm{e}^{-\frac{\mathrm{i}\,\omega v_*}{\hbar}} \tag{30}$$

for the ingoing case and

$$\rho_{\uparrow\mathrm{out}} = \exp\left[-\frac{\mathrm{i}\,\omega v_* - 2\,\mathrm{i}\,(\omega-\omega_0)\,r_*}{\hbar}\right] \tag{31}$$

for the outcoming case, where $\omega_0 = -eA_t(r_\mathrm{H}) = eQ/r_\mathrm{H}$. In any case, the following relation holds true

$$\frac{\partial S_\uparrow}{\partial v_*} = -\omega. \tag{32}$$

Of course, because equations (28) and (29) are equivalent in the approximate method, using the Damour–Ruffini method yields the tunneling rate

$$\Gamma = \exp\left(\frac{-2\pi\,(\omega-\omega_0)}{\kappa}\right). \tag{33}$$

However, we are to research the Hawking radiation via tunneling theory. From Eqs.(28) and (32), we obtain

$$\frac{\partial S_\uparrow}{\partial r} = \left[1+\frac{1}{2\kappa\,(r-r_\mathrm{H})}\right]\frac{\partial S_\uparrow}{\partial r_*} = \frac{[2\kappa\,(r-r_\mathrm{H})+1]\left[(\omega-\omega_0)\pm\sqrt{(\omega-\omega_0)^2-m^2f^{-1}}\right]}{2\kappa\,(r-r_\mathrm{H})}. \tag{34}$$

Via the Wentzel–Kramer–Brillouin (WKB) approximation, we consider the particle tunneling behaviour from inside to outside the event horizon. From Eq.(34), we can obtain the imaginary part of the action as

$$\mathrm{Im}\,S_\uparrow = \mathrm{Im}\,S_{\uparrow+} - \mathrm{Im}\,S_{\uparrow-} = \frac{\pi\,(\omega-\omega_0)}{\kappa} = \frac{\pi\,(\omega-\omega_0)\left(\chi^2r_\mathrm{H}^4+2Mr_\mathrm{H}-Q^2\right)}{(1-2\dot{r}_\mathrm{H})\,r_\mathrm{H}-2\chi^2r_\mathrm{H}^3-M}, \tag{35}$$

where $S_{\uparrow+}$ is part of the outcoming solution, and $S_{\uparrow-}$ is part of the ingoing solution. So the tunneling rate of the Vaidya–Bonner de Sitter black hole is

$$\Gamma = \mathrm{e}^{-2\mathrm{Im}S_\uparrow} = \exp\left(\frac{-2\pi\,(\omega-\omega_0)}{\kappa}\right), \tag{36}$$

and the Hawking temperature is

$$T_\mathrm{H} = \frac{\kappa}{2\pi} = \frac{(1-2\dot{r}_\mathrm{H})\,r_\mathrm{H}-2\chi^2r_\mathrm{H}^3-M}{2\pi\left(\chi^2r_\mathrm{H}^4+2Mr_\mathrm{H}-Q^2\right)}. \tag{37}$$

In a similar way, we can study the Hawking radiation for the spin-down case. The Hawking temperature and the tunneling rate are uniform for the spin-up case. Similarly, we can obtain the tunneling rate and the Hawking temperature near the cosmical horizon $r_\mathrm{C}$ as

$$\Gamma = \exp\left(\frac{-(\omega-\omega_0')}{T_\mathrm{C}}\right) \tag{38}$$

and

$$T_\mathrm{C} = \frac{\kappa}{2\pi} = \frac{(1-2\dot{r}_\mathrm{C})\,r_\mathrm{C}-2\chi^2r_\mathrm{C}^3-M}{2\pi\left(\chi^2r_\mathrm{C}^4+2Mr_\mathrm{C}-Q^2\right)} \tag{39}$$

respectively, where $\omega_0' = eQ/r_\mathrm{C}$. Obviously, the results are right, so our new method of studying Fermion tunneling is reasonable.

# 4. Conclusion

We introduced the general tortoise coordinate transformation into the research on fermion tunneling of the Vaidya–Bonner de Sitter black hole, so that the tunneling rate and the Hawking temperature near the event horizon can be obtained. Expressions (36)–(39) show that the tunneling spectrum is purely thermal, which means the information of the black hole should lose finally due to the Hawking radiation. However, in our study, the change of background spacetime and the self-gravitation interaction are ignored. With the effects above taken into account, the conclusion of the problem of the information loss in Vaidya–Bonner de Sitter spacetime could be another story.

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