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New two-mode intermediate momentum-coordinate representation with quantum entanglement and its application*

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We construct four linear composite operators for a two-particle system and give common eigenvectors of those operators. The technique of integration within an ordered product (IWOP) of operators is employed to prove that those common eigenvectors are complete and orthonormal. Therefore, a new two-mode intermediate momentum-coordinate representation which involves quantum entanglement for a two-particle system is proposed and applied to some two-body dynamic problems. Moreover, the pure-state density matrix $|\xi_1,\xi_2\rangle_{C,D\ C,D}\langle\xi_1,\xi_2|$ is a Radon transform of Wigner operator.

Keywords: two-particle system, intermediate momentum-coordinate representation, IWOP technique

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1. Introduction

The representation theory and its transformation have been proved to be very useful in studying quantum mechanics, quantum optics and quantum statistics. As Dirac pointed out, "When one has a particular problem to work out in quantum mechanics, one can minimize the labour by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible".[1] The authors of Refs.[2]–[15] studied the representation theory and their applications widely. Those studies have led to the great development of representation theory. But to our knowledge, the intermediate momentum-coordinate representation for a two-particle system has not been constructed. In the present paper, we employ the technique of integration within an ordered product (IWOP) of operators[16,17] to make the proposition of a new intermediate momentum-coordinate representation $|\xi_1,\xi_2\rangle_{C,D}$ with quantum entanglement.

We construct four linear composite operators for a two-particle system as follows:

$$\hat{X}_1 = A\hat{x}_1 + B\hat{p}_2, \quad \hat{X}_2 = A\hat{x}_2 + B\hat{p}_1,$$
$$\hat{P}_1 = C\hat{x}_2 + D\hat{p}_1 \quad \text{and} \quad \hat{P}_2 = C\hat{x}_1 + D\hat{p}_2, \quad (1)$$

where A, B, C and D are all real. Obviously, $[\hat{X}_1, \hat{X}_2] = 0$ and $[\hat{P}_1, \hat{P}_2] = 0$. For guaranteeing $[\hat{X}_i, \hat{P}_j] = \mathrm{i}\delta_{ij}$, $(i, j = 1, 2)$, the condition

$$AD - BC = 1 \quad (2)$$

is required. We are challenged to search for the common eigenvectors $|\xi_1,\xi_2\rangle_{C,D}$ of $\hat{P}_1$ and $\hat{P}_2$. By using the IWOP technique, we show that $|\xi_1,\xi_2\rangle_{C,D}$ makes a complete representation and solves some two-body dynamic problems.

2. Common eigenvectors of $\hat{P}_1$ and $\hat{P}_2$

We begin with introducing the two-mode Fock space spanned by $|n_1, n_2\rangle = \frac{\hat{a}_1^{\dagger n_1}\hat{a}_2^{\dagger n_2}}{\sqrt{n_1!n_2!}}|00\rangle$, where

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$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, $|00\rangle$ is the ground state, $\hat{a}_i$ and $\hat{a}_i^\dagger$ are creation and annihilation operators, respectively, which are related to $\hat{x}_i$ and $\hat{p}_i$ by

$$\hat{x}_i = (\hat{a}_i + \hat{a}_i^\dagger)/\sqrt{2} \quad \text{and} \quad \hat{p}_i = (\hat{a}_i - \hat{a}_i^\dagger)/\mathrm{i}\sqrt{2}. \tag{3}$$

We shall prove that the common eigenvectors $|\xi_1, \xi_2\rangle_{C,D}$ are given in two-mode Fock space by

$$\begin{aligned} |\xi_1, \xi_2\rangle_{C,D} = & \frac{1}{\sqrt{\pi(C^2+D^2)}} \exp\left\{ -\frac{1}{2(C^2+D^2)}(\xi_1^2+\xi_2^2) + \frac{\sqrt{2}}{C^2+D^2}(\mathrm{i}D\xi_1 + C\xi_2)\hat{a}_1^\dagger \right. \\ & \left. + \frac{\sqrt{2}}{C^2+D^2}(C\xi_1 + \mathrm{i}D\xi_2)\hat{a}_2^\dagger - \frac{(\mathrm{i}D\hat{a}_1^\dagger + C\hat{a}_2^\dagger)^2}{2(C^2+D^2)} - \frac{(C\hat{a}_1^\dagger + \mathrm{i}D\hat{a}_2^\dagger)^2}{2(C^2+D^2)} + \mathrm{i}\frac{AC+BD}{C^2+D^2}\xi_1\xi_2 \right\} |00\rangle , \end{aligned} \tag{4}$$

where ξ_1 and ξ_2 are the eigenvalues of $\hat{P}_1$ and $\hat{P}_2$, respectively, and $AD - BC = 1$. In fact, performing the operator $\hat{a}_i$ on $|\xi_1, \xi_2\rangle_{C,D}$ and using the formulae[18] : $\dfrac{\partial}{\partial \hat{a}_i^\dagger} f(\hat{a}_i, \hat{a}_i^\dagger) : = [\hat{a}_i, : f(\hat{a}_i, \hat{a}_i^\dagger) :]$ and $\hat{a}_i |00\rangle = 0$, we have

$$\hat{a}_1 |\xi_1, \xi_2\rangle_{C,D} = \frac{\sqrt{2}(\mathrm{i}D\xi_1 + C\xi_2) - (C^2-D^2)\hat{a}_1^\dagger - \mathrm{i}2CD\hat{a}_2^\dagger}{C^2+D^2} |\xi_1, \xi_2\rangle_{C,D}$$

and

$$\hat{a}_2 |\xi_1, \xi_2\rangle_{C,D} = \frac{\sqrt{2}(C\xi_1 + \mathrm{i}D\xi_2) - (C^2-D^2)\hat{a}_2^\dagger - \mathrm{i}2CD\hat{a}_1^\dagger}{C^2+D^2} |\xi_1, \xi_2\rangle_{C,D} . \tag{5}$$

Expressions (1), (3) and (5) lead to

$$\hat{P}_i |\xi_1, \xi_2\rangle_{C,D} = \xi_i |\xi_1, \xi_2\rangle_{C,D} , \quad (i = 1, 2). \tag{6}$$

Thus we know that $|\xi_1, \xi_2\rangle_{C,D}$ is the eigenvector of $\hat{P}_1$ and $\hat{P}_2$.

We now examine whether $|\xi_1, \xi_2\rangle_{C,D}$ satisfies a complete relation. By using the IWOP technique and the ordering form of the two-mode vacuum state projector

$$|00\rangle \langle 00| = : \exp\{-\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\} : , \tag{7}$$

where the symbol : : denotes the normal product, we can easily perform the following integration

$$\begin{aligned} & \iint_{-\infty}^{\infty} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \, |\xi_1, \xi_2\rangle_{C,D\ C,D}\langle \xi_1, \xi_2| \\ = & \frac{1}{\pi(C^2+D^2)} \iint_{-\infty}^{\infty} \mathrm{d}\xi_1 \mathrm{d}\xi_2 : \exp\Big\{ -\frac{\xi_1^2+\xi_2^2}{(C^2+D^2)} + \frac{\sqrt{2}[\mathrm{i}D(\hat{a}_1^\dagger - \hat{a}_1) + C(\hat{a}_2^\dagger + \hat{a}_2)]}{(C^2+D^2)}\xi_1 \\ & + \frac{\sqrt{2}[C(\hat{a}_1^\dagger + \hat{a}_1) + \mathrm{i}D(\hat{a}_2^\dagger - \hat{a}_2)]}{(C^2+D^2)}\xi_2 - \frac{(\mathrm{i}D\hat{a}_1^\dagger + C\hat{a}_2^\dagger)^2}{2(C^2+D^2)} - \frac{(C\hat{a}_1^\dagger + \mathrm{i}D\hat{a}_2^\dagger)^2}{2(C^2+D^2)} \\ & - \frac{(-\mathrm{i}D\hat{a}_1 + C\hat{a}_2)^2}{2(C^2+D^2)} - \frac{(C\hat{a}_1 - \mathrm{i}D\hat{a}_2)^2}{2(C^2+D^2)} - \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\} : = : \exp\{0\} : = 1 , \end{aligned} \tag{8}$$

where we have used the mathematical formula $\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2 + \beta x} \mathrm{d}x = \sqrt{\pi/\alpha}\, \mathrm{e}^{\beta^2/4\alpha}$, Re $(\alpha) > 0$. From expression (6), we can also calculate

$$\begin{aligned} & {}_{C,D}\langle \xi_1', \xi_2'| \hat{P}_i |\xi_1, \xi_2\rangle_{C,D} \\ = & \xi_i' {}_{C,D}\langle \xi_1', \xi_2'| \xi_1, \xi_2\rangle_{C,D} \\ = & \xi_i {}_{C,D}\langle \xi_1', \xi_2'| \xi_1, \xi_2\rangle_{C,D} , \quad i = 1, 2 \end{aligned} \tag{9}$$

which means the overlap

$${}_{C,D}\langle \xi_1', \xi_2'| \xi_1, \xi_2\rangle_{C,D} = \delta(\xi_1' - \xi_1)\delta(\xi_2' - \xi_2), \tag{10}$$

thereby indicating that $|\xi_1, \xi_2\rangle_{C,D}$ is orthonormal.

We perform the Fourier transform

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\xi_1 \, |\xi_1, \xi_2\rangle_{C,D} \exp(\mathrm{i}u\xi_1) \\ = & |q_1 = -B\xi_2 - Du\rangle \otimes |p_2 = A\xi_2 + Cu\rangle, \end{aligned} \tag{11}$$

where

$$|q_1\rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{q_1^2}{2} + \sqrt{2}q_1\hat{a}_1^\dagger - \frac{\hat{a}_1^{\dagger 2}}{2}\right)|0\rangle_1$$

and

$$|p_2\rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{p_2^2}{2} + \mathrm{i}\sqrt{2}p_2\hat{a}_2^\dagger + \frac{\hat{a}_2^{\dagger 2}}{2}\right)|0\rangle_2$$

are the first-mode coordinate eigenstate and the second-mode momentum eigenstate, respectively. By taking the inverse Fourier transform of expression (11), we obtain

$$\begin{aligned} &|\xi_1, \xi_2\rangle_{C,D} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \exp(-\mathrm{i}u\xi_1)|q_1 \\ &= -B\xi_2 - Du\rangle \otimes |p_2 = A\xi_2 + Cu\rangle, \end{aligned} \tag{12}$$

which is the Schmidt decomposition of $|\xi_1, \xi_2\rangle_{C,D}$ and confirms that $|\xi_1, \xi_2\rangle_{C,D}$ itself is an entangled state.

Furthermore, from expressions (1)–(5), it is easy to derive $\hat{X}_1$'s and $\hat{X}_2$'s representatives in ${}_{C,D}\langle\xi_1, \xi_2|$ basis as follows:

$${}_{C,D}\langle\xi_1, \xi_2|\,\hat{X}_i = \mathrm{i}\frac{\partial}{\partial\xi_i}{}_{C,D}\langle\xi_1, \xi_2|\,, \quad (i = 1, 2). \tag{13}$$

In particular, when $A = 1$, $B = 0$, $C = 0$ and $D = 1$, the $|\xi_1, \xi_2\rangle_{C,D}$ evolves into a two-mode coordinate eigenstate

$$\begin{aligned} &|p_1, p_2\rangle \\ &= \pi^{-1/2} \exp\left\{-\frac{1}{2}(p_1^2 + p_2^2) + \mathrm{i}\sqrt{2}p_1\hat{a}_1^\dagger + \mathrm{i}\sqrt{2}p_2\hat{a}_2^\dagger \right. \\ &\left. + \frac{1}{2}(\hat{a}_1^{\dagger 2} + \hat{a}_2^{\dagger 2})\right\}|00\rangle\,. \end{aligned} \tag{14}$$

When $A = 0$, $B = -1$, $C = 1$ and $D = 0$, the $|\xi_1, \xi_2\rangle_{C,D}$ evolves into a two-mode momentum eigenstate

$$\begin{aligned} |x_1, x_2\rangle \;=\; &\pi^{-1/2} \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2) + \sqrt{2}x_1\hat{a}_1^\dagger \right. \\ &\left. + \sqrt{2}x_2\hat{a}_2^\dagger - \frac{1}{2}(\hat{a}_1^{\dagger 2} + \hat{a}_2^{\dagger 2})\right\}|00\rangle\,. \end{aligned} \tag{15}$$

From expressions (8), (10), (14) and (15), we see that $|\xi_1, \xi_2\rangle_{C,D}$ is qualified to make up a new intermediate momentum-coordinate representation involving quantum entanglement.

3. Simultaneously squeezing unitary transformation for $\hat{X}_i$ and $\hat{P}_i$

Squeezed states and squeezing operators have been an important topic in quantum optics since the 1970s. Now an interesting question thus naturally arises: if we simultaneously make classical transformation $\xi_1 \to \kappa_1\xi_1$ and $\xi_2 \to \kappa_2\xi_2$ in state $|\xi_1, \xi_2\rangle$, what is the corresponding squeezing operator? For this purpose we construct the following ket-bra integration:

$$\hat{U} = \iint_{-\infty}^{\infty} \mathrm{d}\xi_1\mathrm{d}\xi_2\sqrt{\kappa_1\kappa_2}\,|\kappa_1\xi_1, \kappa_2\xi_2\rangle\,\langle\xi_1, \xi_2|, \tag{16}$$

where κ_1 and κ_2 are two independent positive numbers. The unitarity of U can be obtained from expression (10) by calculating

$$\begin{aligned} \hat{U}\hat{U}^\dagger \;&=\; \kappa_1\kappa_2 \int\ldots\int_{-\infty}^{\infty} \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_1'\mathrm{d}\xi_2'\,|\kappa_1\xi_1, \kappa_2\xi_2\rangle\,\langle\xi_1, \xi_2|\,|\xi_1', \xi_2'\rangle\,\langle\kappa_1\xi_1', \kappa_2\xi_2'| \\ &= \kappa_1\kappa_2 \int\ldots\int_{-\infty}^{\infty} \mathrm{d}\xi_1\mathrm{d}\xi_2\,|\kappa_1\xi_1, \kappa_2\xi_2\rangle\,\langle\kappa_1\xi_1, \kappa_2\xi_2| =: \exp\{0\} := 1 = \hat{U}^\dagger\hat{U}. \end{aligned} \tag{17}$$

Using expression (6) and the IWOP technique, we can obtain

$$\begin{aligned} \hat{U}\hat{P}_1\hat{U}^\dagger \;&=\; \kappa_1\kappa_2 \int\ldots\int_{-\infty}^{\infty} \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_1'\mathrm{d}\xi_2'\,|\kappa_1\xi_1, \kappa_2\xi_2\rangle\,\langle\xi_1, \xi_2|\,\hat{P}_1\,|\xi_1', \xi_2'\rangle\,\langle\kappa_1\xi_1', \kappa_2\xi_2'| \\ &= \kappa_1\kappa_2 \int\ldots\int_{-\infty}^{\infty} \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_1'\mathrm{d}\xi_2'\,|\kappa_1\xi_1, \kappa_2\xi_2\rangle\,\xi_1'\delta(\xi_1' - \xi_1)\delta(\xi_2' - \xi_2)\,\langle\kappa_1\xi_1', \kappa_2\xi_2'| \\ &= \kappa_1\kappa_2 \int\ldots\int_{-\infty}^{\infty} \xi_1\mathrm{d}\xi_1\mathrm{d}\xi_2\,|\kappa_1\xi_1, \kappa_2\xi_2\rangle\,\langle\kappa_1\xi_1, \kappa_2\xi_2| \\ &= \kappa_1^{-1}\left\{\frac{C(\hat{a}_2 + \hat{a}_2^\dagger)}{\sqrt{2}} + \frac{D(\hat{a}_1 - \hat{a}_1^\dagger)}{\mathrm{i}\sqrt{2}}\right\} = \kappa_1^{-1}\hat{P}_1, \end{aligned} \tag{18}$$

where we have used the mathematical formula $\int_{-\infty}^{\infty} x\ \mathrm{e}^{-\alpha x^2+\beta x}\mathrm{d}x = \frac{\beta}{2\alpha}\sqrt{\frac{\pi}{\alpha}}\ \mathrm{e}^{\beta^2/4\alpha}$, $\mathrm{Re}\ (\alpha) > 0$.

Similarly, we have

$$\hat{U}\hat{P}_2\hat{U}^\dagger = \kappa_2^{-1}\hat{P}_2, \quad \hat{U}\hat{X}_1\hat{U}^\dagger = \kappa_1\hat{X}_1 \quad \text{and} \quad \hat{U}\hat{X}_2\hat{U}^\dagger = \kappa_2\hat{X}_2. \tag{19}$$

Comparing expression (18) with expression (19), we see that $\hat{U}$ is indeed an operator simultaneously squeezing $\hat{X}_1$, $\hat{X}_2$, $\hat{P}_1$ and $\hat{P}_2$. In particular, when $A = 1$, $B = 0$, $C = 0$ and $D = 1$, $\hat{U}$ is reduced into a usual two-mode squeezing operator

$$\hat{S} = \iint_{-\infty}^{\infty} \mathrm{d}p_1\mathrm{d}p_2\sqrt{\kappa_1\kappa_2}\,|\kappa_1 p_1, \kappa_2 p_2\rangle\langle p_1, p_2|,$$
$$\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = 1, \tag{20}$$

and its squeezing character of S is

$$\hat{S}\hat{x}_1\hat{S}^\dagger = \kappa_1\hat{x}_1, \quad \hat{S}\hat{x}_2\hat{S}^\dagger = \kappa_2\hat{x}_2,$$
$$\hat{S}\hat{p}_1\hat{S}^\dagger = \kappa_1^{-1}\hat{p}_1 \ \text{ and } \ \hat{S}\hat{p}_2\hat{S}^\dagger = \kappa_2^{-1}\hat{p}_2. \tag{21}$$

4. Solving the dynamic problem of two-mode harmonic oscillators with coordinate-momentum coupling by means of $|\boldsymbol{\xi_1}, \boldsymbol{\xi_2}\rangle_{C,D}$ representation

The Hamiltonian of two-mode harmonic oscillators with coordinate-momentum coupling reads

$$\begin{aligned}\hat{H} &= \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}m_1\omega_1^2\hat{x}_1^2 + \frac{1}{2}m_2\omega_2^2\hat{x}_2^2 - \lambda\hat{x}_1\hat{p}_2 \\ &= \frac{\hat{p}_1^2}{2m_1} + \frac{(m_2\omega_2\hat{x}_2)^2}{2m_2} + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}m_1\omega_1^2\hat{x}_1^2 - \lambda\hat{x}_1\hat{p}_2.\end{aligned} \tag{22}$$

This Hamiltonian involves a term describing a parametric down conversion process plus a Raman transition process from the point of view of quantum optics. Without loss of generality, we suppose $\omega_2 \geq \omega_1$ and $\lambda > 0$. Choosing the squeezing coefficients $\kappa_1 = 1$ and $\kappa_2 = \sqrt{m_2/m_1}$ in expression (20), we have

$$\begin{aligned}\hat{S}\hat{H}\hat{S}^\dagger &= \frac{\hat{p}_1^2}{2m_1} + \frac{(m_2\omega_2\hat{x}_2)^2}{2m_1} + \frac{m_1}{2m_2^2}\hat{p}_2^2 + \frac{1}{2}m_1\omega_1^2\hat{x}_1^2 - \lambda\sqrt{\frac{m_1}{m_2}}\hat{x}_1\hat{p}_2 \\ &= \frac{\hat{p}_1^2}{2m_1} + \frac{(m_2\omega_2\hat{x}_2)^2}{2m_1} + \begin{pmatrix}\hat{p}_2 & m_2\omega_2\hat{x}_1\end{pmatrix} M \begin{pmatrix}\hat{p}_2 \\ m_2\omega_2\hat{x}_1\end{pmatrix},\end{aligned} \tag{23}$$

where $M = \begin{pmatrix} \frac{m_1}{2m_2^2} & -\frac{\lambda m_1}{2\sqrt{m_1m_2}m_2\omega_2} \\ -\frac{\lambda m_1}{2\sqrt{m_1m_2}m_2\omega_2} & \frac{m_1\omega_1^2}{2m_2^2\omega_2^2} \end{pmatrix}$. We can derive two characteristic values of matrix M and corresponding orthonormal characteristic functions as follows:

$$\Lambda_1 = \frac{1}{4}\frac{m_1}{m_2^2}\left[1 + \left(\frac{\omega_1}{\omega_2}\right)^2 + \sqrt{\left(1 - \left(\frac{\omega_1}{\omega_2}\right)^2\right)^2 + \frac{m_2}{m_1}\left(\frac{2\lambda}{\omega_2}\right)^2}\right], \quad \boldsymbol{\beta}_1 = \begin{pmatrix}\frac{d_2}{\sqrt{2}} \\ \frac{-d_1}{\sqrt{2}}\end{pmatrix} \tag{24}$$

and

$$\Lambda_2 = \frac{1}{4}\frac{m_1}{m_2^2}\left[1 + \left(\frac{\omega_1}{\omega_2}\right)^2 - \sqrt{\left(1 - \left(\frac{\omega_1}{\omega_2}\right)^2\right)^2 + \frac{m_2}{m_1}\left(\frac{2\lambda}{\omega_2}\right)^2}\right], \quad \boldsymbol{\beta}_2 = \begin{pmatrix}\frac{d_1}{\sqrt{2}} \\ \frac{d_2}{\sqrt{2}}\end{pmatrix}, \tag{25}$$

where

$$d_1 = \sqrt{1 - \frac{1 - (\omega_1/\omega_2)^2}{\sqrt{[1 - (\omega_1/\omega_2)^2]^2 + (m_2/m_1)(2\lambda/\omega_2)^2}}}$$

and

$$d_2 = \sqrt{1 + \frac{1 - (\omega_1/\omega_2)^2}{\sqrt{[1 - (\omega_1/\omega_2)^2]^2 + (m_2/m_1)(2\lambda/\omega_2)^2}}}.$$

Letting $T=(\boldsymbol{\beta}_1\ \boldsymbol{\beta}_2)$, it is easy to see that T is a unitary matrix. We construct the following transformations:

$$\begin{pmatrix}\hat{p}_1\\ m_2\omega_2\hat{x}_2\end{pmatrix}=T\begin{pmatrix}\hat{P}_1\\ m_2\omega_2\hat{X}_2\end{pmatrix}=\begin{pmatrix}\frac{d_2}{\sqrt{2}} & \frac{d_1}{\sqrt{2}}\\ \frac{-d_1}{\sqrt{2}} & \frac{d_2}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\hat{P}_1\\ m_2\omega_2\hat{X}_2\end{pmatrix} \tag{26}$$

and

$$\begin{pmatrix}\hat{p}_2\\ m_2\omega_2\hat{x}_1\end{pmatrix}=T\begin{pmatrix}\hat{P}_2\\ m_2\omega_2\hat{X}_1\end{pmatrix}=\begin{pmatrix}\frac{d_2}{\sqrt{2}} & \frac{d_1}{\sqrt{2}}\\ \frac{-d_1}{\sqrt{2}} & \frac{d_2}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\hat{P}_2\\ m_2\omega_2\hat{X}_1\end{pmatrix}. \tag{27}$$

The inverse transformations of expressions (26) and (27) are

$$\hat{X}_1=\frac{d_2}{\sqrt{2}}\hat{x}_1+\frac{d_1}{\sqrt{2}m_2\omega_2}\hat{p}_2,\quad \hat{P}_1=\frac{-d_1m_2\omega_2}{\sqrt{2}}\hat{x}_2+\frac{d_2}{\sqrt{2}}\hat{p}_1 \tag{28}$$

and

$$\hat{X}_2=\frac{d_2}{\sqrt{2}}\hat{x}_2+\frac{d_1}{\sqrt{2}m_2\omega_2}\hat{p}_1,\quad \hat{P}_2=\frac{-d_1m_2\omega_2}{\sqrt{2}}\hat{x}_1+\frac{d_2}{\sqrt{2}}\hat{p}_2, \tag{29}$$

i.e. we have chosen $A=d_2/\sqrt{2}$, $B=d_1/\sqrt{2}m_2\omega_2$, $C=-d_1m_2\omega_2/\sqrt{2}$ and $D=d_2/\sqrt{2}$ to satisfy condition (2). From expressions (26), (27) and (23), the Hamiltonian can be written as

$$S\hat{H}S^{\dagger}=\frac{\hat{P}_1^2}{2m_1}+\frac{(m_2\omega_2\hat{X}_2)^2}{2m_1}+\Lambda_1\hat{P}_2^2+\Lambda_2(m_2\omega_2\hat{X}_1)^2=\sum_{i=1}^{2}\left(\frac{\hat{P}_i^2}{2\mu_i}+\frac{1}{2}\mu_i\Omega_i^2\hat{X}_i^2\right), \tag{30}$$

where

$$\begin{aligned}&\mu_1=m_1,\quad \Omega_1=\omega_1\sqrt{\frac{1}{2}\left[\frac{\omega_2^2}{\omega_1^2}+1-\sqrt{\left(\frac{\omega_2^2}{\omega_1^2}-1\right)^2+\frac{m_2}{m_1}\left(\frac{2\lambda}{\omega_1}\right)^2\left(\frac{\omega_2}{\omega_1}\right)^2}\right]},\\ &\mu_1=1/2\Lambda_1,\quad \Omega_2=\omega_2\sqrt{\frac{1}{2}\left[1+\frac{\omega_1^2}{\omega_2^2}+\sqrt{\left(1-\frac{\omega_1^2}{\omega_2^2}\right)^2+\frac{m_2}{m_1}\left(\frac{2\lambda}{\omega_2}\right)^2}\right]}.\end{aligned} \tag{31}$$

Sandwiching Eq.(30) between ${}_{C,D}\langle\xi_1,\xi_2|$ and $\hat{S}\,|E\rangle$, we obtain the Schrödinger equation in the ${}_{C,D}\langle\xi_1,\xi_2|$ representation

$${}_{C,D}\langle\xi_1,\xi_2|\,\hat{S}\,|E\rangle=\sum_{i=1}^{2}\left\{\frac{1}{2\mu_i}\xi_i^2\,{}_{C,D}\langle\xi_1,\xi_2|\,\hat{S}\,|E\rangle-\frac{1}{2}\hbar^2\mu_i\Omega_i^2\frac{\partial^2}{\partial\xi_i^2}\,{}_{C,D}\langle\xi_1,\xi_2|\,\hat{S}\,|E\rangle\right\}, \tag{32}$$

where $|E\rangle$ denotes the eigenstate of $\hat{H}$. Thus we can directly write down the eigenvalue and eigenfunction of Eq.(32) as

$$\begin{aligned}&E_{n_1n_2}=\sum_{i=1}^{2}\left(n_i+\frac{1}{2}\right)\hbar\Omega_i,\\ &{}_{C,D}\langle\xi_1,\xi_2|\,\hat{S}\,|E_{n_1n_2}\rangle=\prod_{i=1}^{2}\{N_{n_i}\exp(-\alpha_i^2\xi_i^2/2)H_{n_i}(\alpha_i\xi_i)\},\\ &\left(\alpha_i=\sqrt{1/\hbar\mu_i\Omega_i},\quad N_{n_i}=\sqrt{\alpha_i/(\sqrt{\pi}2^{n_i}n_i!)}\quad n_1,n_2=0,1,2,\ldots\right),\end{aligned} \tag{33}$$

where we have recovered the Planck constant $\hbar$.

To obtain the eigenfunction in $\langle p_1,p_2|$ representation, we calculate the overlap

$$\begin{aligned}\langle p_1,p_2\mid\xi_1,\xi_2\rangle_{C,D}&=\iint\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}\,\langle p_1,p_2|\,|z_1z_2\rangle\,\langle z_1z_2|\,|\xi_1,\xi_2\rangle\\ &=(2\pi\hbar C)^{-1}\exp\left\{\frac{\mathrm{i}}{\hbar}\frac{Dp_1p_2+A\xi_1\xi_2-p_1\xi_2-p_2\xi_1}{C}\right\},\end{aligned} \tag{34}$$

where we have used the over-completeness of two-mode coherent state $\iint \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} |z_1 z_2\rangle \langle z_1 z_2| = 1$ and the mathematical formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp\left\{\zeta |z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}\right\} = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left\{\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right\},$$

and recovered the Planck constant $\hbar$. Then we have

$$\begin{aligned}
\langle p_1, p_2| \hat{S} |E_{n_1 n_2}\rangle &= \iint_{-\infty}^{\infty} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \langle p_1, p_2| |\xi_1, \xi_2\rangle_{C,D\ C,D} \langle \xi_1, \xi_2| \hat{S} |E_{n_1 n_2}\rangle \\
&= \frac{N_{n_1} N_{n_2}}{\sqrt{A^2 + (C\hbar\alpha_1\alpha_2)^2}} \exp\left\{\frac{\mathrm{i}}{\hbar} \frac{AB + DC(\hbar\alpha_1\alpha_2)^2}{A^2 + (C\hbar\alpha_1\alpha_2)^2} p_1 p_2 - \frac{\alpha_1^2 p_1^2 + \alpha_2^2 p_2^2}{2[A^2 + (C\hbar\alpha_1\alpha_2)^2]}\right\} \cdot F_{n_1 n_2}(p_1, p_2) \\
&= \frac{\sqrt{2} N_{n_1} N_{n_2}}{\sqrt{d_2^2 + (d_1 m_2 \omega_2 / \sqrt{\mu_1 \Omega_1 \mu_2 \Omega_2})^2}} \cdot \exp\left\{\mathrm{i} \frac{1}{\hbar m_2 \omega_2} \frac{d_1 d_2 (\mu_1 \mu_2 \Omega_1 \Omega_2 - m_2^2 \omega_2^2)}{d_2^2 \mu_1 \mu_2 \Omega_1 \Omega_2 + d_1^2 m_2^2 \omega_2^2} p_1 p_2 \right. \\
&\quad \left. - \frac{\mu_1 \mu_2 \Omega_1 \Omega_2}{[d_2^2 \mu_1 \mu_2 \Omega_1 \Omega_2 + d_1^2 m_2^2 \omega_2^2]} (\alpha_1^2 p_1^2 + \alpha_2^2 p_2^2)\right\} \cdot F_{n_1 n_2}(p_1, p_2)\ , \qquad (35)
\end{aligned}$$

where we have used the mathematical formula

$$H_{n_1}(\zeta_1) H_{n_2}(\zeta_2) = \frac{\partial^{n_1 + n_2}}{\partial t_1^{n_1} \partial t_2^{n_2}} \exp\{2\zeta_1 t_1 - t_1^2 + 2\zeta_2 t_2 - t_2^2\}\Big|_{\substack{t_1 = 0 \\ t_2 = 0}}.$$

The function $F_{n_1 n_2}(p_1, p_2)$ is defined as

$$\begin{aligned}
&F_{n_1 n_2}(p_1, p_2) \\
&= \frac{\partial^{n_1 + n_2}}{\partial t_1^{n_1} \partial t_2^{n_2}} \exp\left\{\frac{2A}{A^2 + (C\hbar\alpha_1\alpha_2)^2} (\alpha_1 p_1 t_1 + \alpha_2 p_2 t_2) - \frac{A^2 - (C\hbar\alpha_1\alpha_2)^2}{A^2 + (C\hbar\alpha_1\alpha_2)^2} (t_1^2 + t_2^2) + \mathrm{i} \frac{4AC\hbar\alpha_1\alpha_2}{A^2 + (C\hbar\alpha_1\alpha_2)^2} t_1 t_2 \right. \\
&\quad \left. - \mathrm{i} \frac{2C\hbar\alpha_1\alpha_2}{A^2 + (C\hbar\alpha_1\alpha_2)^2} (\alpha_1 p_1 t_2 + \alpha_2 p_2 t_1) + \frac{2A}{A^2 + (C\hbar\alpha_1\alpha_2)^2} (\alpha_1 p_1 t_1 + \alpha_2 p_2 t_2)\right\}\Big|_{\substack{t_1 = 0 \\ t_2 = 0}}. \qquad (36)
\end{aligned}$$

Using expression (20) and $\langle p_1, p_2 \mid p_1', p_2'\rangle = \delta(p_1 - p_1')\delta(p_2 - p_2')$, we have

$$\begin{aligned}
&\langle p_1, p_2| E_{n_1 n_2}\rangle \\
&= \langle p_1, p_2| \hat{S}^\dagger \hat{S} |E_{n_1 n_2}\rangle \\
&= \sqrt{\kappa_1 \kappa_2} \langle \kappa_1 p_1, \kappa_2 p_2| \hat{S} |E_{n_1 n_2}\rangle \\
&= \frac{\sqrt{\kappa_1 \kappa_2} N_{n_1} N_{n_2}}{\sqrt{A^2 + (C\hbar\alpha_1\alpha_2)^2}} \exp\left\{\frac{\mathrm{i}}{\hbar} \frac{AB + DC(\hbar\alpha_1\alpha_2)^2}{A^2 + (C\hbar\alpha_1\alpha_2)^2} \kappa_1 \kappa_2 p_1 p_2 - \frac{\alpha_1^2 (\kappa_1 p_1)^2 + \alpha_2^2 (\kappa_2 p_2)^2}{2[A^2 + (C\hbar\alpha_1\alpha_2)^2]}\right\} \cdot F_{n_1 n_2}(\kappa_1 p_1, \kappa_2 p_2) \\
&= \left(\frac{m_2}{m_1}\right)^{1/4} \frac{N_{n_1} N_{n_2}}{\sqrt{A^2 + (BC\hbar\alpha_1\alpha_2)^2}} \exp\left\{-\frac{\mathrm{i}}{\hbar} \frac{AB + DC(\hbar\alpha_1\alpha_2)^2}{A^2 + (C\hbar\alpha_1\alpha_2)^2} p_1 \sqrt{\frac{m_1}{m_2}} p_2 - \frac{\alpha_1^2 p_1^2 + \alpha_2^2 p_2^2 m_2 / m_1}{2[A^2 + (C\hbar\alpha_1\alpha_2)^2]}\right\} \\
&\quad \times F_{n_1 n_2}\left(p_1, \sqrt{\frac{m_2}{m_1}} p_2\right). \qquad (37)
\end{aligned}$$

In particular, when $\lambda = 0$, then $A = 1$, $B = 0$, $C = 0$, $D = 1$, $\mu_1 = m_1$, $\mu_2 = m_2^2/m_1$, $\Omega_1 = \omega_1$, $\Omega_2 = \omega_2$, $\alpha_1 = \sqrt{1/\hbar m_1 \omega_1}$, and $\alpha_2 = \sqrt{m_1/\hbar m_2^2 \omega_2}$. From Eqs.(33), (36) and (37), we can obtain

$$\begin{aligned}
&E_{n_1 n_2} = \sum_{i=1}^{2} \left(n_i + \frac{1}{2}\right) \hbar\omega_i, \\
&\langle p_1, p_2| E_{n_1 n_2}\rangle = \prod_{i=1}^{2} \{N_{n_i}' \exp(-\alpha_i' 2 p_i 2/2) H_{n_i}(\alpha_i' p_i)\}, \\
&\left(\alpha_i' = \sqrt{1/\hbar m_i \omega_i}, \quad N_{n_i}' = \sqrt{\alpha_i'/(\sqrt{\pi} 2^{n_i} n_i!)}, \quad n_1, n_2 = 0, 1, 2, \ldots\right), \qquad (38)
\end{aligned}$$

which is just the case of two-mode quantum harmonic oscillator system without coupling.

5. Radon transform of Wigner operator

Given a density matrix $\hat{\rho}$, one constructs the corresponding Wigner function as $\mathrm{Tr}[\Delta(x,p)\hat{\rho}]$, where $\Delta(x,p)$ is the Wigner operator. The Wigner operator $\Delta(x,p)$ is the integral transformation kernel that transforms a classical function $h(x,p)$ into a quantum operator $\hat{H}(\hat{x},\hat{p})$,[19,20] i.e.

$$\iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}p h(x,p)\Delta(x,p) = \hat{H}(\hat{x},\hat{p}), \tag{39}$$

where

$$\Delta(x,p) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \mathrm{d}u\mathrm{d}v \cdot \exp[\mathrm{i}u(x-\hat{x}) + \mathrm{i}v(p-\hat{p})], \quad (\hbar = 1). \tag{40}$$

Using the normal ordered product of the one-mode vacuum projector $|0\rangle\langle 0| = :\exp[-\hat{a}^{\dagger}\hat{a}]:$ and the IWOP technique and performing the integration over u for expression (40), we have

$$\Delta(x,p) = \int_{-\infty}^{\infty} \frac{\mathrm{d}v}{2\pi}\mathrm{e}^{\mathrm{i}pv}\left|x+\frac{v}{2}\right\rangle\left\langle x-\frac{v}{2}\right|, \tag{41}$$

which is the expression of $\Delta(x,p)$ in $|x\rangle$ representation of the coordinate $\hat{x}$.

Similarly, by performing the integration over v for expression (40), we have

$$\Delta(x,p) = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi}\mathrm{e}^{\mathrm{i}xu}\left|p-\frac{u}{2}\right\rangle\left\langle p+\frac{u}{2}\right|, \tag{42}$$

which is the expression of $\Delta(x,p)$ in $|p\rangle$ representation of the momentum $\hat{p}$, then we can obtain

$$\mathrm{Tr}[\Delta(x,p)\Delta(x',p')] = (2\pi)^{-1}\delta(x-x')\delta(p-p'). \tag{43}$$

Generalizing expressions (42) and (43) from one-mode case to two-mode case, we have

$$\Delta(x_1,p_1;x_2,p_2) = \int_{-\infty}^{\infty} \frac{\mathrm{d}u_1\mathrm{d}u_2}{4\pi^2}\mathrm{e}^{\mathrm{i}x_1u_1+\mathrm{i}x_2u_2}\left|p_1-\frac{u_1}{2},p_2-\frac{u_2}{2}\right\rangle\left\langle p_1+\frac{u_1}{2},p_2+\frac{u_2}{2}\right|. \tag{44}$$

$$\mathrm{Tr}[\Delta(x_1,p_1;x_2,p_2)\Delta(x_1',p_1';x_2',p_2')] = (2\pi)^{-2}\delta(x_1-x_1')\delta(x_2-x_2')\delta(p_1-p_1')\delta(p_2-p_2'). \tag{45}$$

So we know that the classical function of a quantum operator $\hat{H}(\hat{x}_1,\hat{p}_1;\hat{x}_2,\hat{p}_2)$ is

$$h(x_1,p_1;x_2,p_2) = 4\pi^2\mathrm{Tr}[\Delta(x_1,p_1;x_2,p_2)\hat{H}(\hat{x}_1,\hat{p}_1;\hat{x}_2,\hat{p}_2)]. \tag{46}$$

Using expressions (34) and (46), we can obtain the corresponding Weyl classical function of $|\xi_1,\xi_2\rangle_{C,D\ C,D}\langle\xi_1,\xi_2|$

$$\begin{aligned}
&4\pi^2\ \mathrm{Tr}[|\xi_1,\xi_2\rangle_{C,D\ C,D}\langle\xi_1,\xi_2|\,\Delta(x_1,p_1;x_2,p_2)]\\
&= \iint_{-\infty}^{\infty}\mathrm{d}u_1\mathrm{d}u_2\ \mathrm{e}^{\mathrm{i}x_1u_1+\mathrm{i}x_2u_2}{}_{C,D}\langle\xi_1,\xi_2|\left|p_1-\frac{u_1}{2},p_2-\frac{u_2}{2}\right\rangle\left\langle p_1+\frac{u_1}{2},p_2+\frac{u_2}{2}\right||\xi_1,\xi_2\rangle_{C,D}\\
&= \delta(\xi_1-Cx_2-Dp_1)\delta(\xi_2-Cx_1-Dp_2).
\end{aligned} \tag{47}$$

Comparing expressions (39), (46) with Eq.(47), we have

$$\begin{aligned}
|\xi_1,\xi_2\rangle_{C,D\ C,D}\langle\xi_1,\xi_2| &= \int\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}p_1\mathrm{d}p_2\delta(\xi_1-Cx_2-Dp_1)\delta(\xi_2-Cx_1-Dp_2)\Delta(x_1,p_1;x_2,p_2)\\
&= \int\mathrm{d}\boldsymbol{r}_1\mathrm{d}\boldsymbol{r}_2\delta(\xi_1-\boldsymbol{\varsigma}\cdot\boldsymbol{r}_1)\delta(\xi_2-\boldsymbol{\varsigma}\cdot\boldsymbol{r}_2)\Delta(\boldsymbol{r}_1,\boldsymbol{r}_2),
\end{aligned} \tag{48}$$

where $\boldsymbol{r}_1 = (x_2,p_1)$, $\boldsymbol{r}_2 = (x_1,p_2)$ and $\boldsymbol{\varsigma} = (C,D)$. Expression (48) indicates that a transform of Wigner operator can be defined and the kernel of the transform is composed of delta functions for a group (ξ_1,ξ_2,C,D). This transform is referred to as the Radon transform[21] in mathematical theory. Expression (48) means that

the Radon transform of Wigner operator is just the pure-state density matrix $|\xi_1,\xi_2\rangle_{C,D\ C,D}\langle\xi_1,\xi_2|$, so it is necessary to introduce the common eigenvector $|\xi_1,\xi_2\rangle_{C,D}$ of $C\hat{x}_2+D\hat{p}_1$ and $C\hat{x}_1+D\hat{p}_2$. It then leads to

$$\left|\langle\psi|\ \xi_1,\xi_2\rangle_{C,D}\right|^2=\int\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\mathrm{d}p_1\mathrm{d}x_2\mathrm{d}p_2\delta(\xi_1-Cx_2-Dp_1)\delta(\xi_2-Cx_1-Dp_2)W(x_1,p_1;x_2,p_2),\qquad(49)$$

where $|\psi\rangle$ is a quantum state, and $W(x_1,p_1;x_2,p_2)={}_{C,D}\langle\xi_1,\xi_2|\,\Delta(x_1,p_1;x_2,p_2)\,|\xi_1,\xi_2\rangle_{C,D}$ is the Wigner function, from which it follows that $|\xi_1,\xi_2\rangle_{C,D\ C,D}\langle\xi_1,\xi_2|$ is the Radon transform of the Wigner operator**.** So the Radon transform of the Wigner function is the square module of the wave function of $|\psi\rangle$ in the ${}_{C,D}\langle\xi_1,\xi_2|$ representation, which represents a probability and is called margin distribution.

6. Summary

In the present paper, we have found a new intermediate momentum-coordinate representation $|\xi_1,\xi_2\rangle_{C,D}$ with quantum entanglement in two-mode Fock space, which possesses orthonormal and completeness relations and provides a new approach to solving some dynamic problems of a two-body system with momentum-coordinate coupling. Moreover, the pure-state density matrix $|\xi_1,\xi_2\rangle_{C,D\ C,D}\langle\xi_1,\xi_2|$ is a Radon transform of the Wigner operator. The IWOP technique plays an essential role in our derivations.

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