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Symmetry and general symmetry groups of the coupled Kadomtsev–Petviashvili equation*

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In this paper, the Lie symmetry algebra of the coupled Kadomtsev–Petviashvili (cKP) equation is obtained by the classical Lie group method and this algebra is shown to have a Kac–Moody–Virasoro loop algebra structure. Then the general symmetry groups of the cKP equation is also obtained by the symmetry group direct method which is proposed by Lou *et al*。 From the general symmetry groups, the Lie symmetry group can be recovered and a group of discrete transformations can be derived simultaneously. Lastly, from a known simple solution of the cKP equation, we can easily obtain two new solutions by the general symmetry groups.

Keywords: symmetry, general symmetry groups, coupled KP equation
PACC: 0220, 1130L, 3310C, 0340K

1. Introduction

The coupled Kadomtsev–Petviashvili (cKP) equation first appeared in the paper of Jimbo and Miwa[1]

$$\begin{aligned}&(\mathrm{D}_x^4-4\mathrm{D}_x\mathrm{D}_t+3\mathrm{D}_y^2)\tau\cdot\tau=24\tilde{\sigma}\sigma,\\&(\mathrm{D}_x^3+2\mathrm{D}_t-3\mathrm{D}_x\mathrm{D}_y)\tilde{\sigma}\cdot\tau=0,\\&(\mathrm{D}_x^3+2\mathrm{D}_t+3\mathrm{D}_x\mathrm{D}_y)\sigma\cdot\tau=0,\end{aligned}\tag{1}$$

where the Hirota bilinear derivatives are defined by

$$\begin{aligned}\mathrm{D}_t^m\mathrm{D}_x^n\mathrm{D}_y^l f\cdot g=&\left(\frac{\partial}{\partial_{t_1}}-\frac{\partial}{\partial_{t_2}}\right)^m\left(\frac{\partial}{\partial_{x_1}}-\frac{\partial}{\partial_{x_2}}\right)^n\\&\left(\frac{\partial}{\partial_{y_1}}-\frac{\partial}{\partial_{y_2}}\right)^l f(x_1,y_1,t_1)\\&\cdot g(x_2,y_2,t_2)\Big|_{t_1=t_2,x_1=x_2,y_1=y_2}.\end{aligned}\tag{2}$$

Equation (1) can be transformed into the following coupled nonlinear partial differential equations (PDEs):

$$\begin{aligned}&(4u_t-12uu_x-u_{xxx})_x-3u_{yy}+12(v\tilde{v})_{xx}=0,\\&2v_t+6uv_x+v_{xxx}+3v_{xy}+6v\int^x u_y\mathrm{d}x=0,\\&2\tilde{v}_t+6u\tilde{v}_x+\tilde{v}_{xxx}-3\tilde{v}_{xy}-6\tilde{v}\int^x u_y\mathrm{d}x=0\end{aligned}\tag{3}$$

by means of the dependent variable transformation

$$u=(\ln\tau)_{xx},\quad v=\sigma/\tau,\quad \tilde{v}=\tilde{\sigma}/\tau.\tag{4}$$

Since then, the cKP equation has been reconstructed several times by other specialists, from different points of view. Hietarinta obtained it in his research of possible integrable equations.[2] Hirota and Ohta found it with the approach known as Pfaffinization.[3] Adler *et al* also introduced it in the context of random matrices.[4]

Recently, the cKP equation has been the subject of extensive studies. Kakei showed that the cKP hierarchy can be reformulated as a reduced case of a two-component KP hierarchy.[5] Isojima *et al* presented various solutions, especially spider-web solutions, for the cKP equation.[6,7] Using the spin-representation formulation, van de Leur constructed an elementary Darboux transformation for the cKP hierarchy.[8] Very recently, Yang *et al* also constructed a different elementary Darboux transformation for the cKP equation by means of an elementary approach.[9]

The purpose of this paper is to study the cKP equation from a group theoretical point of view. By the classical Lie group approach,[10−14] we obtain the

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Lie symmetry algebra of the cKP equation involving four arbitrary functions of time t and show that it possesses a Kac–Moody–Virasoro (KMV) loop algebra structure which is shared by the symmetry algebras of some known integrable equations in (2+1) dimensions such as the KP equation,[15−17] a new (2+1)-dimensional bilinear model equation,[18] the usual integrable DS equations,[19] the (2+1) dimensional AKNS system,[21] the DLW equation and the extended SDYM system.[21] When restricting the four arbitrary functions to some special forms, we reduce the cKP equation to a (1+1)-dimensional PDE.

However it is difficult for us to obtain the general symmetry group of the cKP equation by solving the characteristic line equations. So we turn to the symmetry group direct method recently developed by Lou *et al*,[20−23] which is an effective and valid method to obtain general symmetry groups for some PDEs.[24,25] Once the general symmetry group of a given system is obtained by the symmetry group direct method, the related Lie point symmetries can be obtained simply by restricting the arbitrary functions or arbitrary constants in infinitesimal forms. At the same time, the related group of discrete transformations of the given system can also be derived from the general symmetry groups. Furthermore, by the general symmetry groups and known simple solutions, one can easily obtain some new solutions.

The rest of this article is organized as follows. In Section 2, we obtain the Lie point symmetry transformation group of the cKP equation by the standard Lie approach, and reduce the cKP equation to (1+1)-dimensional PDEs by restricting all arbitrary functions to some special forms. In Section 3, we derive the general symmetry groups of the cKP equation from the symmetry group direct method and generate two new solutions of the cKP equation from a simple one. The last section is a short summary and discussion.

2. Lie symmetry and symmetry reduction of the cKP equation

Firstly, we set $u_y = w_x$ and substituting it to the cKP equation (3), we can obtain

$$\begin{aligned}
&(4u_t - 12uu_x - u_{xxx})_x - 3w_{xy} + 12(v\tilde{v})_{xx} = 0,\\
&2v_t + 6uv_x + v_{xxx} + 3v_{xy} + 6vw = 0,\\
&2\tilde{v}_t + 6u\tilde{v}_x + \tilde{v}_{xxx} - 3\tilde{v}_{xy} - 6\tilde{v}w = 0,\\
&u_y - w_x = 0.
\end{aligned} \tag{5}$$

We apply the standard infinitesimal procedure to find the infinitesimals of $x, y, t, u, v, \tilde{v}, w$ and hence the symmetry group of Eq.(5). We write equation (5) as $\Delta_1(x, y, t, u, v, \tilde{v}, w) = 0$, $\Delta_2(x, y, t, u, v, \tilde{v}, w) = 0$, $\Delta_3(x, y, t, u, v, \tilde{v}, w) = 0$, $\Delta_4(x, y, t, u, v, \tilde{v}, w) = 0$. We generally give the infinitesimal operator

$$\mathbb{V} = \xi\partial_x + \eta\partial_y + \tau\partial_t + \phi_1\partial_u + \phi_2\partial_v + \phi_3\partial_{\tilde{v}} + \phi_4\partial_w, \tag{6}$$

where the coefficients ξ, η, τ and ϕ_1, ϕ_2, ϕ_3, ϕ_4 are functions of x, y, t, u, v, $\tilde{v}$, w. According to the general theory for symmetries of differential equations, to find these functions we prolong the vector to fourth order derivatives and require that the fourth prolonged vector field annihilates Δ_1, Δ_2, Δ_3, Δ_4 on the solution manifold of equation (5), respectively, namely

$$\mathrm{pr}^{(4)}\mathbb{V}(\Delta_i(x, y, t, u, v, \tilde{v}, w))|_{\Delta_i} = 0, \quad i = 1, 2, 3, 4, \tag{7}$$

where $\mathrm{pr}^{(4)}\mathbb{V}$ is the fourth prolongation of the vector field $\mathbb{V}$. This condition provides us with quite a complicated system of determining equations (a system of linear partial differential equations) for the coefficients. With the help of symbolic computation system Maple, solving these huge numbers of determining equations, we can get

$$\begin{aligned}
\xi &= \frac{1}{3}f_t x + \frac{2}{9}f_{tt} + \frac{2}{3}g_t y + h(t),\\
\eta &= \frac{2}{3}f_t y + g(t), \quad \tau = f(t),\\
\phi_1 &= -\frac{1}{9}f_{tt}x - \frac{2}{27}f_{ttt}y^2 - \frac{2}{9}g_{tt}y - \frac{1}{3}h_t - \frac{2}{3}uf_t,\\
\phi_2 &= \frac{4}{9}\Big(- 3f_t + xf_{tt}y + \frac{3}{2}xg_t + \frac{2}{9}f_{ttt}y^3\\
&\quad + g_{tt}y^2 + 3h_t y - \frac{9}{4}n(t)\Big)v,\\
\phi_3 &= -\frac{4}{9}\Big(xyf_{tt} + \frac{3}{2}xg_t + \frac{2}{9}f_{ttt}y^3 + g_{tt}y^2\\
&\quad + 3h_t y - \frac{9}{4}n(t)\Big)\tilde{v},\\
\phi_4 &= -\frac{4}{9}uf_{tt}y - \frac{2}{3}ug_t + \frac{2}{9}f_{tt} - \frac{4}{27}xyf_{ttt} - \frac{2}{9}xg_{tt}\\
&\quad - \frac{8}{243}f_{tttt}y^3 - \frac{4}{27}g_{ttt}y^2 - \frac{4}{9}h_{tt}y + \frac{1}{3}n_t - wf_t,
\end{aligned} \tag{8}$$

where $f(t)$, $g(t)$, $h(t)$, $n(t)$ are arbitrary functions of time t and the subscripts denote derivatives of t. We find that the general element can be written as

$$\mathbb{V} = \sigma_1(f) + \sigma_2(g) + \sigma_3(h) + \sigma_4(n), \tag{9}$$

where

$$\sigma_1(f)=\begin{pmatrix} -\frac{1}{9}f_{tt}x-\frac{2}{27}y^2f_{ttt}-\frac{2}{3}uf_t \\ -\frac{4}{3}vf_t+\frac{4}{9}vxyf_{tt}+\frac{8}{81}vy^3f_{ttt} \\ -\frac{4}{9}\tilde{v}xyf_{tt}-\frac{1}{81}\tilde{v}y^3f_{ttt}-\frac{4}{3}\tilde{v}f_t \\ -\frac{2}{9}(2uy-1)f_{tt}-\frac{4}{27}xyf_{ttt} \\ -\frac{8}{243}y^3f_{tttt}-wf_t \end{pmatrix},$$

$$\sigma_2(g)=\begin{pmatrix} -\frac{2}{9}g_{tt}y \\ \frac{2}{3}vxg_t+\frac{4}{9}vg_{tt}y^2 \\ -\frac{4}{9}\tilde{v}g_{tt}y^2-\frac{2}{3}\tilde{v}xg_t \\ -\frac{2}{3}ug_t-\frac{2}{9}xg_{tt}-\frac{4}{27}y^2g_{ttt} \end{pmatrix},$$

$$\sigma_3(h)=\begin{pmatrix} -\frac{1}{3}h_t \\ \frac{4}{3}vyh_t \\ -\frac{4}{3}\tilde{v}yh_t \\ -\frac{4}{9}yh_{tt} \end{pmatrix},\quad \sigma_4(n)=\begin{pmatrix} 0 \\ -vn(t) \\ \tilde{v}n(t) \\ \frac{1}{3}n_t \end{pmatrix}.$$

The nonzero commutators among the general element are

$$\begin{aligned} &[\sigma_1(f_1),\sigma_1(f_2)]=\sigma_1(f_1f_{2t}-f_2f_{1t}),\\ &[\sigma_1(f),\sigma_2(g)]=\sigma_2\Big(fg_t-\frac{2}{3}gf_t\Big),\\ &[\sigma_1(f),\sigma_3(h)]=\sigma_3\Big(fg_t-\frac{2}{3}gf_t\Big),\\ &[\sigma_2(g_1),\sigma_2(g_2)]=\frac{2}{3}\sigma_2(g_1g_{2t}-g_2g_{1t}). \end{aligned}\tag{10}$$

From Eq.(10), we can see that the general element of the cKP equation possesses a KMV-type symmetry algebra.

To obtain the similarity reductions and some exact solution of the cKP equation, we have to solve the characteristic line equations

$$\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}y}{\eta}=\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}u}{\phi_1}=\frac{\mathrm{d}v}{\phi_2}=\frac{\mathrm{d}\tilde{v}}{\phi_3}=\frac{\mathrm{d}w}{\phi_4}.\tag{11}$$

When we restrict $f(t)=g(t)=h(t)=n(t)=t$ especially, one type of reduced equation can be obtained:

$$\begin{aligned} &12(m_{2s}m_{3s}-m_{1ss}m_1+m_3m_{2ss}+m_2m_{3ss}-m_{1s}^2)\\ &\quad-\frac{4}{3}(sm_{1ss}+rm_{1sr})-m_{1ssss}-4m_{1s}-3m_{4sr}=0,\\ &4rm_{2r}-3m_{2sss}+8m_2-18m_1m_{2s}-9m_{2sr}\\ &\quad-18m_2m_4+2sm_{2s}=0,\\ &9m_{3sr}+2sm_{3s}+18m_3m_4-3m_{3sss}+4rm_{3r}\\ &\quad-18m_1m_{3s}=0,\\ &m_{1r}-m_{4s}=0, \end{aligned}\tag{12}$$

where m_1, m_2, m_3, and m_4 are functions of the independent variables s and r, and

$$s=t^{-1/3}\Big(x+2y-\frac{3}{2}t\Big),\quad r=t^{-2/3}(y-3t).\tag{13}$$

Then by the same Lie symmetry method, we can further reduce Eq.(12) to an ordinary differential equation. Due to the complexity of the reduction process, we do not study it in this paper.

But when $f(t)$, $g(t)$, $h(t)$, $n(t)$ keep arbitrary, it is difficult for us to give out the general solutions of Eq.(11). So we shall turn to the symmetry group direct method recently developed by Lou *et al* for the general symmetry groups of the cKP equation (5).

3. General symmetry groups of the cKP equation

According to the symmetry group direct method developed by Lou *et al*,[20,21] one can find that the symmerty transformation should have the following form:

$$\begin{aligned} u&=\alpha_1+\beta_1U+\gamma_1V+\theta_1\tilde{V}+\rho_1W,\\ v&=\alpha_2+\beta_2U+\gamma_2V+\theta_2\tilde{V}+\rho_2W,\\ \tilde{v}&=\alpha_3+\beta_3U+\gamma_3V+\theta_3\tilde{V}+\rho_3W,\\ w&=\alpha_4+\beta_4U+\gamma_4V+\theta_4\tilde{V}+\rho_4W, \end{aligned}\tag{14}$$

where α_i, β_i, γ_i, θ_i, ρ_i $(i=1,2,3,4)$, ξ, η, and τ are functions of $\{x,y,t\}$ only, and $U\equiv U(\xi,\eta,\tau)$, $V\equiv V(\xi,\eta,\tau)$, $\tilde{V}\equiv\tilde{V}(\xi,\eta,\tau)$, $W\equiv W(\xi,\eta,\tau)$ satisfy the cKP equation (5) but with new independent variables $\{\xi,\eta,\tau\}$, i.e.,

$$\begin{aligned} &(4U_\tau-12UU_\xi-U_{\xi\xi\xi})_\xi-3W_{\xi\eta}+12(V\tilde{V})_{\xi\xi}=0,\\ &2V_\tau+6UV_\xi+V_{\xi\xi\xi}+3V_{\xi\eta}+6VW=0,\\ &2\tilde{V}_\tau+6U\tilde{V}_\xi+\tilde{V}_{\xi\xi\xi}-3\tilde{V}_{\xi\eta}-6\tilde{V}W=0,\\ &U_\eta-W_\xi=0. \end{aligned}\tag{15}$$

Substituting Eq.(14) into Eq.(5) and eliminating all the terms including $U_{\xi\xi\xi\xi}$, $V_{\xi\xi\xi}$, $\tilde{V}_{\xi\xi\xi}$ and W_ξ by Eq.(15), one can obtain four polynomial differential equations with respect to $\{U,V,\tilde{V},W\}$ and their derivatives. Then collecting their coefficients

of $\{U, V, W, \Phi\}$ and their derivatives, we can obtain a huge number of nonlinear PDEs with respect to differentiable functions: $\{\alpha_i, \beta_i, \gamma_i, \theta_i, \rho_i,\ (i = 1, 2, 3, 4), \xi, \eta,\ \tau\}$.

With the help of symbolic computation system Maple, after some thorough analyses and some quite tedious calculations, solving these huge numbers of determining equations, we obtain two types of solutions.

Case 1

$$
\begin{aligned}
&\xi = \delta_1 \tau_t^{1/3} x - \frac{2\delta_1^2 g_{1t} y}{3\tau_t^{1/3}} + \frac{2\delta_1 y^2 \tau_{tt}}{9\tau_t^{2/3}} + g_2,\\
&\eta = -\tau_t^{2/3} y/\delta_1 + g_1, \quad \rho_4 = -\tau_t, \quad \tau = \tau(t),\\
&\alpha_1 = \frac{1}{9}\frac{\tau_{tt} x}{\tau_t} + \frac{2}{9}\frac{\delta_1 y \tau_{tt} g_{1t}}{\tau_t^{5/3}} - \frac{8}{81}\frac{y^2\tau_{tt}^2}{\tau_t^2} - \frac{1}{9}\frac{g_{1t}^2}{\delta_1 \tau_t^{4/3}}\\
&\qquad - \frac{2}{9}\frac{\delta_1 y g_{1tt}}{\tau_t^{2/3}} + \frac{2}{27}\frac{y^2\tau_{ttt}}{\tau_t} + \frac{1}{3}\frac{g_{2t}}{\delta_1\tau_t^{1/3}},\\
&\alpha_4 = \frac{2}{9}\frac{x\delta_1 g_{1t}\tau_{tt}}{\tau_t^{5/3}} - \frac{16}{81}\frac{xy\tau_{tt}^2}{\tau_t^2} - \frac{2}{9}\frac{x\delta_1 g_{1tt}}{\tau_t^{2/3}}\\
&\qquad + \frac{20}{81}\frac{y\tau_{tt}g_{1t}^2}{\tau_t^{7/3}\delta_1} + \frac{2}{9}\frac{g_{1t}g_{2t}}{\tau_t} + \frac{2}{9}\frac{\tau_{tt}}{\tau_t} - \frac{4}{81}\frac{g_{1t}^3}{\tau_t^2}\\
&\qquad + \frac{224}{2187}\frac{y^3\tau_{tt}^3}{\tau_t^3} + \frac{8}{27}\frac{\delta_1 y^2\tau_{tt}g_{1tt}}{\tau_t^{5/3}} - \frac{1}{3}\frac{g_{3t}}{g_3}\\
&\qquad + \frac{4}{27}\frac{\delta_1 y^2\tau_{ttt}g_{1t}}{\tau_t^{5/3}} - \frac{32}{243}\frac{y^3\tau_{tt}\tau_{ttt}}{\tau_t^2} - \frac{8}{27}\frac{y\tau_{tt}g_{2t}}{\tau_t^{4/3}\delta_1}\\
&\qquad - \frac{4}{27}\frac{\delta_1 y^2 g_{1ttt}}{\tau_t^{2/3}} + \frac{8}{243}\frac{y^3\tau_{tttt}}{\tau_t} + \frac{4}{9}\frac{yg_{2tt}}{\tau_t^{1/3}\delta_1}\\
&\qquad - \frac{8}{27}\frac{g_{1t}g_{1tt}y}{\tau_t^{4/3}\delta_1} - \frac{8}{27}\frac{\delta_1 y^2\tau_{tt}^2 g_{1t}}{\tau_t^{8/3}} + \frac{4}{27}\frac{xy\tau_{ttt}}{\tau_t},
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&\beta_1 = \tau_t^{2/3}/\delta_1, \beta_4 = 2(2\tau_t^{2/3} y\tau_{tt} - 3\tau_t g_{1t}\delta_1)/(9\tau_t\delta_1),\\
&\theta_2 = g_3 \exp\Big((-4\tau_t^{2/3} y\tau_{tt} + 6\tau_t g_{1t}\delta_1)x/(9\tau_t^{5/3})\\
&\qquad + \frac{2}{81} y(-12\delta_1^2 y\tau_{tt}g_{1t}\tau_t^2 + 18\delta_1^2 g_{1tt} y\tau_t^3\\
&\qquad + 4y^2\tau_{tt}^2\tau_t^{5/3}\delta_1 - 4y^2\tau_{ttt}\tau_t^{8/3}\delta_1 - 54g_{2t}\tau_t^{10/3}\\
&\qquad + 9g_{1t}^2\tau_t^{7/3})/(\delta_1\tau_t^{11/3})\Big),\\
&\gamma_3 = \frac{\tau_t^{4/3}}{g_3\delta_1^2}\exp\Big((4\tau_t^{2/3} y\tau_{tt} - 6\tau_t g_{1t}\delta_1)x/(9\tau_t^{5/3})\\
&\qquad - \frac{2}{81} y(-12\delta_1^2 y\tau_{tt}g_{1t}\tau_t^2 + 9g_{1t}^2\tau_t^{7/3}\\
&\qquad + 18\delta_1^2 g_{1tt} y\tau_t^3 + 4y^2\tau_{tt}^2\tau_t^{5/3}\delta_1 - 4y^2\tau_{ttt}\tau_t^{8/3}\delta_1\\
&\qquad - 54g_{2t}\tau_t^{10/3})/(\tau_t^{11/3}\delta_1)\Big).
\end{aligned}
$$

Case 2

$$
\begin{aligned}
&\xi = \delta_2\tau_t^{1/3} x + \frac{2}{3}\frac{\delta_2^2 G_{1t} y}{\tau_t^{1/3}} + \frac{2}{9}\frac{\delta_2 y^2\tau_{tt}}{\tau_t^{2/3}} + G_2,\\
&\eta = \tau_t^{2/3} y/\delta_2 + G_1, \quad \tau = \tau(t), \quad \rho_4 = \tau_t,\\
&\alpha_1 = \frac{1}{9}\frac{\tau_{tt}x}{\tau_t} - \frac{2}{9}\frac{\delta_2 y\tau_{tt}G_{1t}}{\tau_t^{5/3}} + \frac{2}{9}\frac{\delta_2 y G_{1tt}}{\tau_t^{2/3}} - \frac{8}{81}\frac{y^2\tau_{tt}^2}{\tau_t^2}\\
&\qquad + \frac{2}{27}\frac{y^2\tau_{ttt}}{\tau_t} + \frac{1}{3}\frac{G_{2t}}{\delta_2\tau_t^{1/3}} - \frac{1}{9}\frac{G_{1t}^2}{\delta_2\tau_t^{4/3}},\\
&\beta_2 = \tau_t^{2/3}/\delta_2, \quad \beta_4 = 2(2\tau_t^{2/3} y\tau_{tt} + 3\tau_t G_{1t}\delta_2)/(9\tau_t\delta_2),\\
&\gamma_2 = G_3\exp\Big((-4\tau_t^{2/3} y\tau_{tt} - 6\tau_t G_{1t}\delta_2)x/(9\tau_t^{5/3})\\
&\qquad - \frac{2}{81} y(-12\delta_2^2 y\tau_{tt}G_{1t}\tau_t^2 + 18\delta_2^2 G_{1tt} y\tau_t^3\\
&\qquad - 4y^2\tau_{tt}^2\tau_t^{5/3}\delta_2 + 4y^2\tau_{ttt}\tau_t^{8/3}\delta_2 - 9G_{1t}^2\tau_t^{7/3}\\
&\qquad + 54G_{2t}\tau_t^{10/3})/(\delta_2\tau_t^{11/3})\Big),
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&\alpha_4 = -\frac{2}{9}\frac{x\delta_2 G_{1t}\tau_{tt}}{\tau_t^{5/3}} + \frac{2}{9}\frac{x\delta_2 G_{1tt}}{\tau_t^{2/3}} - \frac{16}{81}\frac{xy\tau_{tt}^2}{\tau_t^2}\\
&\qquad + \frac{4}{27}\frac{\delta_2 y^2 G_{1ttt}}{\tau_t^{2/3}} + \frac{8}{243}\frac{y^3\tau_{tttt}}{\tau_t} + \frac{20}{81}\frac{y\tau_{tt}G_{1t}^2}{\tau_t^{7/3}\delta_2}\\
&\qquad + \frac{4}{81}\frac{G_{1t}^3}{\tau_t^2} - \frac{1}{3}\frac{G_{3t}}{G_3} + \frac{4}{9}\frac{yG_{2tt}}{\tau_t^{1/3}\delta_2} - \frac{8}{27}\frac{\delta_2 y^2\tau_{tt}G_{1t}}{\tau_t^{5/3}}\\
&\qquad + \frac{224}{2187}\frac{y^3\tau_{tt}^3}{\tau_t^3} - \frac{32}{243}\frac{y^3\tau_{tt}\tau_{ttt}}{\tau_t^2} - \frac{8}{27}\frac{y\tau_{tt}G_{2t}}{\tau_t^{4/3}\delta_2}\\
&\qquad - \frac{4}{27}\frac{\delta_2 y^2\tau_{ttt}G_{1t}}{\tau_t^{5/3}} - \frac{2}{9}\frac{G_{1t}G_{2t}}{\tau_t} + \frac{2}{9}\frac{\tau_{tt}}{\tau_t}\\
&\qquad - \frac{8}{27}\frac{G_{1t}G_{1tt}y}{\tau_t^{4/3}\delta_2} + \frac{8}{27}\frac{y^2\tau_{tt}^2 G_{1t}\delta_2}{\tau_t^{8/3}} + \frac{4}{27}\frac{xy\tau_{ttt}}{\tau_t},\\
&\theta_3 = \frac{\tau_t^{4/3}}{G_3\delta_2^2}\exp\Big((4\tau_t^{2/3} y\tau_{tt} + 6\tau_t G_{1t}\delta_2)x/(9\tau_t^{5/3})\\
&\qquad + \frac{2}{81} y(-12\delta_2^2 y\tau_{tt}G_{1t}\tau_t^2 + 18\delta_2^2 G_{1tt} y\tau_t^3\\
&\qquad - 4y^2\tau_{tt}^2\tau_t^{5/3}\delta_2 - 9G_{1t}^2\tau_t^{7/3} + 54G_{2t}\tau_t^{10/3}\\
&\qquad + 4y^2\tau_{ttt}\tau_t^{8/3}\delta_2)/(\tau_t^{11/3}\delta_2)\Big),
\end{aligned}
$$

where $g_1, g_2, g_3, G_1, G_2, G_3$ and τ are all arbitrary functions of t, and the subscripts denote derivatives of t and

$$\delta_1^3 = 1, \quad \delta_2^3 = 1. \tag{18}$$

We can arrive at the following final transformation group theorem of the cKP equation.

Theorem 1 If $\{U(x, y, t),\ V(x, y, t),\ \tilde{V}(x, y, t),\ W(x, y, t)\}$ is a solution of the cKP equation (5), so are $\{u_1, v_1, \tilde{v}_1, w_1\}$ and $\{u_2, v_2, \tilde{v}_2, w_2\}$ with

Type I

$$
\begin{aligned}
&u_1 = \alpha_1 + \beta_1 U(\xi,\eta,\tau),\\
&v_1 = \theta_2 \tilde{V}(\xi,\eta,\tau),\\
&\tilde{v}_1 = \gamma_3 V(\xi,\eta,\tau),\\
&w_1 = \alpha_4 + \beta_4 U(\xi,\eta,\tau) + \rho_4 W(\xi,\eta,\tau),
\end{aligned} \tag{19}
$$

where α_1, β_1, θ_2, γ_3, α_4, β_4, ρ_4, ξ, η, τ are determined by Eq.(16), and δ_1, δ_2 are determined by Eq.(18).

Type II

$$
\begin{aligned}
&u_2 = \alpha_1 + \beta_1 U(\xi,\eta,\tau),\\
&v_2 = \gamma_2 V(\xi,\eta,\tau),\\
&\tilde{v}_2 = \theta_3 \tilde{V}(\xi,\eta,\tau),\\
&w_2 = \alpha_4 + \beta_4 U(\xi,\eta,\tau) + \rho_4 W(\xi,\eta,\tau),
\end{aligned} \tag{20}
$$

where α_1, β_1, γ_2, θ_3, α_4, β_4, ρ_4, ξ, η, τ are determined by Eq.(17), and δ_1, δ_2 are determined by Eq.(18).

From theorem 1, it is easy to see that the symmetry group, $\mathcal{G}_{\rm cKP}$, of the cKP equation is divided into four sectors which can be considered as a product of the discrete $\mathcal{D}$ group and the usual Lie point symmetry group $\mathcal{S}$ which is related to transformations $\{u, v, \tilde{v}, w\} \to \{u_2, v_2, \tilde{v}_2, w_2\}$ with $\delta_2 = 1$. That means

$$\mathcal{G}_{\rm cKP} = \mathcal{D} \otimes \mathcal{S}, \tag{21}$$

$$\mathcal{D} \equiv \big\{I, \quad R_1, \quad R_2, \quad \Sigma\big\}, \tag{22}$$

where I is the identity transformation, R_1, R_2 and Σ are as follows:

$$
\begin{aligned}
&R_1 : \{x, y, t, u, v, \tilde{v}, w\} \to \{\delta_1 x, -\delta_1^2 y, t, \delta_1^2 u, v, \delta_1 \tilde{v}, w\},\\
&\qquad \delta_1 = \frac{-1+\sqrt{3}\mathrm{i}}{2},\\
&R_2 : \{x, y, t, u, v, \tilde{v}, w\} \to \{\delta_2 x, -\delta_2^2 y, t, \delta_2^2 u, v, \delta_2 \tilde{v}, w\},\\
&\qquad \delta_2 = \frac{-1-\sqrt{3}\mathrm{i}}{2},\\
&\Sigma : \{x, y, t, u, v, \tilde{v}, w\} \to \{x, -y, t, u, \tilde{v}, v, -w\}.
\end{aligned}
$$

It should be mentioned that some other types of reflection transformation can be read off from Eq.(21).

From Theorem 1, the Lie point symmetries (6), (8) and (9) of cKP equation (5) can be reproduced by restricting

$$
\begin{aligned}
&\tau = t + \epsilon f(t), \quad G_1 = \epsilon g(t), \quad G_2 = \epsilon h(t),\\
&G_3 = 1 + \epsilon n(t), \quad \delta_2 = 1,
\end{aligned} \tag{23}
$$

in $\{u_2, v_2, \tilde{v}_2, w_2\}$, where $f(t)$, $g(t)$, $h(t)$ and $n(t)$ are all arbitrary functions of t, and ϵ is an infinitesimal parameter.

By means of Theorem 1, we can obtain many kinds of group-invariant solutions starting from some simple solutions for the cKP equation. For example, one soliton solution has been obtained for the cKP equation (15) in Ref.[6].

$$U = 2(\ln\zeta)_{\xi\xi}, \quad V = \sigma/\zeta, \quad \tilde{V} = \hat{\sigma}/\zeta, \tag{24}$$

where

$$
\begin{aligned}
&\theta_{p_i} = p_i\xi + p_i^2\eta + p_i^3\tau,\\
&\theta_{q_i} = q_i\xi + q_i^2\eta + q_i^3\tau, \quad (i = 1, 2)\\
&\zeta = 1 + \frac{a_1 b_1 (p_1 - p_2)(q_1 - q_2)}{(p_1 - q_1)(p_1 - q_2)(p_2 - q_1)(p_2 - q_2)}\\
&\qquad \times \exp(\theta_{p_1} + \theta_{p_2} - \theta_{q_1} - \theta_{q_2}),\\
&\sigma = b_1(q_1 - q_2)\exp(-\theta_{q_1} - \theta_{q_2}),\\
&\hat{\sigma} = a_1(p_1 - p_2)\exp(\theta_{p_1} - \theta_{p_2}),
\end{aligned} \tag{25}
$$

here p_i and q_i $(i = 1, 2)$ will represent arbitrary parameters which characterize the behaviour of the solitons; a_1 and b_1 are (equally arbitrary) parameters which will play the role of the so-called phase constants for the individual solitons (i.e. fixing the relative positions of the solitons). Because of the transformation $U_\eta = W_\xi$, the expression of w can be obtained:

$$W = 2\int \frac{\zeta_{\xi\xi\eta}\zeta^2 - \zeta_{\xi\xi}\zeta_\eta\zeta - 2\zeta_\xi\zeta_{\xi\eta}\zeta + 2\zeta_\xi^2\zeta_\eta}{\zeta^3}\mathrm{d}\xi. \tag{26}$$

Substituting $U, V, \tilde{V}, W$ into Eqs.(25) and (27) for $U(\xi,\eta,\tau)$, $V(\xi,\eta,\tau)$, $\tilde{V}(\xi,\eta,\tau)$ and $W(\xi,\eta,\tau)$ into Eqs.(19) and (20), two new solutions can easily be obtained for the cKP equation (5). For simplification, we do not list them in this paper.

4. Summary and discussion

In summary, we obtain the Lie symmetry algebra of the coupled Kadomtsev–Petviashvili (cKP) involving four arbitrary functions of t by the classical Lie group approach and show that this algebra has a Kac–Moody–Virasoro loop algebra structure. Then the cKP equation is reduced to four (1+1)-dimensional PDEs by restricting four arbitrary functions to some special forms. But we find it is difficult to obtain the general symmetry groups of the cKP equation by solving the corresponding characteristic line equations. So we turn to the symmetry group direct method developed by Lou *et al* and obtain the general symmetry groups of the cKP equation by solving the determining equations, a huge number of nonlinear PDEs. From the general symmetry groups, the Lie symmetry group

can be recovered and a group of discrete transformations can be obtained simultaneously. If we obtain a solution of the cKP equation by other methods, we can derive a type of new solution from the general symmetry group transformation, and lastly we give an example for that.

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