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Thermal effect and energy-level transition rule for a mesoscopic LC circuit with inductance–capacitance coupling*

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This paper reports that the mesoscopic inductance and capacitance coupling LC circuit is quantized by means of the canonical quantization method. Using the 'invariant eigen-operator' method, it deduces the energy-level transition rule when the system is disturbed by an external electromagnetic field. At the same time, the quantum fluctuations for the system at finite temperature are examined by virtue of the generalized Hellmann–Feynman theorem.

Keywords: mesoscopic, selection rule, quantum fluctuation, invariant eigen-operator
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1. Introduction

With the dramatic development of nanometer techniques and microelectronics, research work on mesoscopic physics has attracted more and more attention. Louisell is the pioneer who quantized the single LC circuit by denoting the charge as the generalized coordinate and the electric current as the generalized momentum.[1] From then on, the quantum effects of mesoscopic circuit were investigated widely by more and more physicists and distinct progress has been made,[2−10] due to its potential application in electronics and optoelectronic devices. For it is inevitable that practical mesoscopic circuits are disturbed by outside circumstance, e.g., radiated by the external electromagnetic field, transitions between different energy-levels of the system can occur. Thus, the energy-level transition rule[11] of the circuit system should be considered so as to obtain energy-level information. On the other hand, owing to the existence of dissipation and finite environmental temperature, it is necessary to consider the temperature influence on quantum effects of the mesoscopic circuit.[12−16]

In this paper, we shall investigate the mesoscopic inductance and capacitance coupling LC circuit from the following two respects: the energy-level transition rule by the 'invariant eigen-operator' (IEO) method,[17] and quantum fluctuations at finite temperature by the generalized Hellmann–Feynman theorem (HFT).[18] The procedures are as follows: Firstly, we introduce the IEO method and the generalized HFT. Secondly, we give the diagonalized Hamiltonian operator of this system. Thirdly, we discuss the energy-level transition rule with the IEO method when the system is affected by the external environment. Finally, we investigate the quantum fluctuations of the system at finite temperature by virtue of the generalized HFT.

2. The IEO method and the generalized HFT

For convenience of context, we introduce the IEO method[17] and the generalized HFT[18] respectively in this section.

2.1. The IEO method[17]

In the Heisenberg picture, the time evolution of operator $\hat{O}$ satisfies the equation of motion

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{O} = [\hat{O}, \hat{\mathcal{H}}], \tag{1}$$

which plays the same important role in quantum mechanics as the Schrödinger equation. However, it can

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hardly be directly employed to derive the energy-level formulas, since Eq.(1) has no direct relation with the wave functions or eigenvectors. If the operator $\hat{O}_{\rm e}$ (the subscript e means 'eigen') satisfies the following eigenvector-like equation:

$$\left(\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)^2\hat{O}_{\rm e}=\lambda\hat{O}_{\rm e},\quad \lambda>0, \tag{2}$$

the operator $\hat{O}_{\rm e}$ is 'invariant' under the action of $\left(\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)^2$. Using Eq.(1) we obtain

$$\left(\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)^2\hat{O}_{\rm e}=[[\hat{O}_{\rm e},\hat{\mathcal{H}}],\hat{\mathcal{H}}]=\lambda\hat{O}_{\rm e}. \tag{3}$$

From Eq.(3) we can judge that $\sqrt{\lambda}$ is one of the energy eigenvalues. Because we have known $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\leftrightarrow\hat{\mathcal{H}}$, $\left(\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)^2\leftrightarrow\hat{\mathcal{H}}^2$, Eq.(3) can be viewed as a parallel equation with the energy eigenvector equation $\hat{\mathcal{H}}^2\psi=E^2\psi$. Thus, once the operator $\hat{O}_{\rm e}$ is found, some information about energy eigenvalues can be obtained.

2.2. The generalized HFT[18]

Suppose that the Hamiltonian of a parameterized system is $\hat{\mathcal{H}}(\lambda)$, which possesses a series of normalized eigenstates $\psi_n(\lambda)$ with eigenvalues $E_n(\lambda)$,

$$\hat{\mathcal{H}}(\lambda)\psi_n(\lambda)=E_n(\lambda)\psi_n(\lambda). \tag{4}$$

Clearly, both $\psi_n(\lambda)$ and $E_n(\lambda)$ are λ-dependent. Using the HFT, we have

$$\frac{\partial E_n}{\partial\lambda}=\langle\psi_n|\frac{\partial\hat{\mathcal{H}}}{\partial\lambda}|\psi_n\rangle. \tag{5}$$

When $\psi_n(\lambda)$ is not a pure state, we have to deal with the problem with density operator instead. Let ρ be the density operator of the state,

$$\rho\equiv\sum_n \mathrm{e}^{-\beta\hat{\mathcal{H}}}|\psi_n\rangle\langle\psi_n|, \tag{6}$$

where $\beta=(kT)^{-1}$, k is the Boltzmann constant and T represents temperature. In this case the ensemble average energy $\left\langle\hat{\mathcal{H}}(\lambda)\right\rangle_{\rm e}$ (here the subscript e denotes ensemble average) is

$$\begin{aligned}\left\langle\hat{\mathcal{H}}(\lambda)\right\rangle_{\rm e}&=\frac{\mathrm{Tr}(\rho\hat{\mathcal{H}}(\lambda))}{Z(\lambda)}\\&=\frac{1}{Z(\lambda)}\sum_n \mathrm{e}^{-\beta E_n(\lambda)}E_n(\lambda)\\&\equiv\bar{E}(\lambda),\end{aligned} \tag{7}$$

where $Z(\lambda)=\mathrm{Tr}\rho$. Differentiating Eq.(7) with respect to the parameter λ, we obtain

$$\begin{aligned}&\frac{\partial\bar{E}(\lambda)}{\partial\lambda}\\&=\frac{1}{Z^2(\lambda)}\left\{Z(\lambda)\sum_n \mathrm{e}^{-\beta E_n(\lambda)}[-\beta E_n(\lambda)+1]\frac{\partial E_n(\lambda)}{\partial\lambda}-\left[\sum_n \mathrm{e}^{-\beta E_n(\lambda)}E_n(\lambda)\right]\left[\sum_n \mathrm{e}^{-\beta E_n(\lambda)}\frac{\partial E_n(\lambda)}{\partial\lambda}(-\beta)\right]\right\}\\&=\frac{1}{Z(\lambda)}\left\{\sum_n \mathrm{e}^{-\beta E_n(\lambda)}[-\beta E_n(\lambda)+\beta\bar{E}(\lambda)+1]\right\}.\end{aligned} \tag{8}$$

Then substituting Eq.(5) into Eq.(8) we obtain

$$\frac{\partial\left\langle\hat{\mathcal{H}}(\lambda)\right\rangle_{\rm e}}{\partial\lambda}=\frac{\partial\bar{E}(\lambda)}{\partial\lambda}=\left\langle[1+\beta\bar{E}(\lambda)-\beta\hat{\mathcal{H}}(\lambda)]\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_{\rm e}. \tag{9}$$

This, namely, is the generalized HFT.[18] When $\hat{\mathcal{H}}(\lambda)$ is independent of β, Eq.(9) can be rewritten as

$$\begin{aligned}\frac{\partial\bar{E}(\lambda)}{\partial\lambda}&=[1+\beta\bar{E}(\lambda)]\left\langle\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_{\rm e}-\beta\left[-\frac{\partial}{\partial\beta}\left\langle\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_{\rm e}+\left\langle\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_{\rm e}\bar{E}(\lambda)\right]\\&=\frac{\partial}{\partial\beta}\left[\beta\left\langle\frac{\partial\hat{\mathcal{H}}(\lambda)}{\partial\lambda}\right\rangle_{\rm e}\right].\end{aligned} \tag{10}$$

3. The quantization of the mesoscopic inductance and capacitance coupling LC circuit

The mesoscopic inductance and capacitance coupling LC circuit is shown in Fig.1, which consists of three inductors and three capacitors, where L_j $(j = 1, 2)$ is the self-inductance coefficient of the jth inductor, C_j the capacitance of the jth capacitor. $C_{\rm c}$, L are the coupling capacitance and the coupling inductance respectively. Provided that the circuit is excited by an instant impulse source, we define charge q_j of the jth $(j = 1, 2)$ subcircuit as the generalized coordinate, the potential energy of the system is

$$\mathcal{V} = \frac{1}{2}\frac{q_1^2}{C_1} + \frac{1}{2}\frac{q_2^2}{C_2} + \frac{1}{2}\frac{(q_1 - q_2)^2}{C_{\rm c}}, \tag{11}$$

and, correspondingly, the generalized kinetic energy is

$$\mathcal{T} = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 + \frac{1}{2}L(\dot{q}_1 - \dot{q}_2)^2 + M\dot{q}_1\dot{q}_2, \tag{12}$$

where $M = K\sqrt{L_1L_2}(0 \le K \le 1)$ is the mutual inductance coefficient between L_1 and L_2. Then the classical Lagrangian of the system is

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\sum_{j=1}^{2}\left[(L_j + L)\dot{q}_j^2 - \frac{C_j + C_{\rm c}}{C_jC_{\rm c}}q_j^2\right] \\ & + (M - L)\dot{q}_1\dot{q}_2 + \frac{q_1q_2}{C_{\rm c}}.\end{aligned} \tag{13}$$

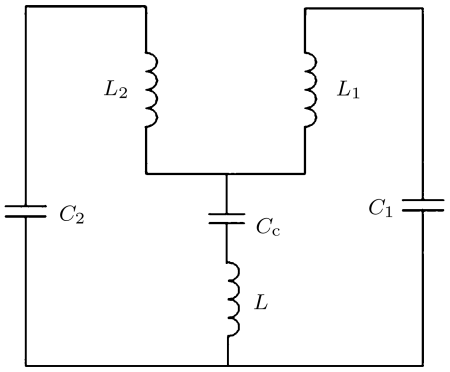


Fig.1. The mesoscopic inductance and capacitance coupling LC circuit.

From Eq.(13), the generalized momenta conjugated to q_1 and q_2 can be deduced respectively,

$$\begin{aligned}p_1 &\equiv \frac{\partial\mathcal{L}}{\partial\dot{q}_1} = (L_1 + L)\dot{q}_1 + (M - L)\dot{q}_2, \\ p_2 &\equiv \frac{\partial\mathcal{L}}{\partial\dot{q}_2} = (L_2 + L)\dot{q}_2 + (M - L)\dot{q}_1.\end{aligned} \tag{14}$$

So, we obtain the classical Hamiltonian of the system

$$\begin{aligned}\mathcal{H} = {} & \sum_{j=1}^{2}p_j\dot{q}_j - \mathcal{L} \\ = {} & \frac{1}{2}\sum_{j=1}^{2}\left[\frac{1}{\gamma(L_j + L)}p_j^2 + \frac{C_j + C_{\rm c}}{C_jC_{\rm c}}q_j^2\right] \\ & - \frac{(M - L)p_1p_2}{\gamma(L_1 + L)(L_2 + L)} - \frac{q_1q_2}{C_{\rm c}},\end{aligned} \tag{15}$$

where $\gamma = [(L_1+L)(L_2+L)-(M-L)^2]/[(L_1+L)(L_2+L)]$. According to the standard canonical quantization principle, a pair of canonical conjugate quantities q_j and p_j corresponds to a couple of Hermitian operators $\hat{q}_j$ and $\hat{p}_j$, which satisfy the commutation relation $[\hat{q}_j, \hat{p}_j] = {\rm i}\hbar$. Consequently, the Hamiltonian operator of the system is

$$\begin{aligned}\hat{\mathcal{H}} = {} & \frac{1}{2}\sum_{j=1}^{2}\left[\frac{1}{\gamma(L_j + L)}\hat{p}_j^2 + \frac{C_j + C_{\rm c}}{C_jC_{\rm c}}\hat{q}_j^2\right] \\ & - \frac{(M - L)\hat{p}_1\hat{p}_2}{\gamma(L_1 + L)(L_2 + L)} - \frac{\hat{q}_1\hat{q}_2}{C_{\rm c}}.\end{aligned} \tag{16}$$

In order to diagonalize the Hamiltonian operator $\hat{\mathcal{H}}$, we introduce a unitary operator[19]

$$\hat{U} = \iint_{-\infty}^{\infty}{\rm d}q_1{\rm d}q_2\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right|, \tag{17}$$

where A, B, C and D are real and satisfy the relation $AD - BC = 1$, $\left|\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right\rangle \equiv |q_1\rangle\,|q_2\rangle$ is the two mode coordinate eigenstate. In Fock space, $|q_j\rangle$ is expressed as

$$\begin{aligned}|q_j\rangle = {} & \left(\frac{m_j\omega_j}{\pi\hbar}\right)^{1/4} \\ & \times\exp\left(-\frac{m_j\omega_j}{2\hbar}q_j^2 + \sqrt{\frac{2m_j\omega_j}{\hbar}}q_j\hat{a}_j^\dagger - \frac{a_j^{\dagger 2}}{2}\right)|0\rangle\,.\end{aligned} \tag{18}$$

Here we have assigned

$$\begin{aligned}&A = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}}\cos\frac{\varphi}{2},\quad B = -\left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}}\sin\frac{\varphi}{2}, \\ &C = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}}\sin\frac{\varphi}{2},\quad D = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}}\cos\frac{\varphi}{2}, \\ &\tan\varphi = 2(M - L)\sqrt{\alpha\beta}\left[(L_1 + L)\beta - (L_2 + L)\alpha\right]^{-1},\end{aligned} \tag{19}$$

where $\alpha = \dfrac{L_1 + L}{M - L} + \dfrac{C_1 + C_{\rm c}}{C_1}$ and $\beta = \dfrac{L_2 + L}{M - L} + \dfrac{C_2 + C_{\rm c}}{C_2}$. By means of the following transformation

relations:[19]

$$\begin{aligned}
\hat{U}^{-1}\hat{q}_1\hat{U} &= A\hat{q}_1 + B\hat{q}_2,\\
\hat{U}^{-1}\hat{q}_2\hat{U} &= C\hat{q}_1 + D\hat{q}_2,\\
\hat{U}^{-1}\hat{p}_1\hat{U} &= D\hat{p}_1 - C\hat{p}_2,\\
\hat{U}^{-1}\hat{p}_2\hat{U} &= -B\hat{p}_1 + A\hat{p}_2,
\end{aligned} \tag{20}$$

we have

$$\begin{aligned}
\hat{\mathcal{H}}' &= \hat{U}^{-1}\hat{\mathcal{H}}\hat{U}\\
&= \frac{1}{2m_1}\hat{p}_1^2 + \frac{1}{2m_2}\hat{p}_2^2 + \frac{1}{2}m_1\omega_1^2\hat{q}_1^2 + \frac{1}{2}m_2\omega_2^2\hat{q}_2^2,
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\frac{1}{m_1} &= \frac{1}{\gamma}[(L_1+L)B^2 + (L_2+L)D^2\\
&\quad +2(M-L)BD],\\
\frac{1}{m_2} &= \frac{1}{\gamma}[(L_1+L)A^2 + (L_2+L)C^2\\
&\quad +2(M-L)AC],\\
\omega_1^2 &= \frac{1}{m_1}\left[\frac{C_1+C_\mathrm{c}}{C_1C_\mathrm{c}}A^2 + \frac{C_2+C_\mathrm{c}}{C_2C_\mathrm{c}}C^2 - \frac{2}{C_\mathrm{c}}AC\right],\\
\omega_2^2 &= \frac{1}{m_2}\left[\frac{C_2+C_\mathrm{c}}{C_2C_\mathrm{c}}B^2 + \frac{C_1+C_\mathrm{c}}{C_1C_\mathrm{c}}D^2 - \frac{2}{C_\mathrm{c}}BD\right].
\end{aligned} \tag{22}$$

Introducing the bosonic creation and annihilation operators $\hat{a}_j^\dagger$ and $\hat{a}_j$

$$\begin{aligned}
\hat{a}_j^\dagger &= \sqrt{\frac{m_j\omega_j}{2\hbar}}\left[\hat{q}_j + \frac{\mathrm{i}}{m_j\omega_j}\hat{p}_j\right],\\
\hat{a}_j &= \sqrt{\frac{m_j\omega_j}{2\hbar}}\left[\hat{q}_j - \frac{\mathrm{i}}{m_j\omega_j}\hat{p}_j\right],
\end{aligned} \tag{23}$$

where, based on the commutation relation $[\hat{q}_j, \hat{p}_j] = \mathrm{i}\hbar$, we find that the annihilation operator $\hat{a}_j$ and creation operator $\hat{a}_j^\dagger$ satisfy $[\hat{a}_j, \hat{a}_l^\dagger] = \delta_{jl}$, $[\hat{a}_j, \hat{a}_l] = [\hat{a}_j^\dagger, \hat{a}_l^\dagger] = 0$, $(j, l = 1, 2)$. Substituting Eq.(23) into Eq.(21) we have

$$\begin{aligned}
\hat{\mathcal{H}}' &= \hbar\omega_1\left(\hat{a}_1^\dagger\hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_2\left(\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\right)\\
&\equiv \hat{\mathcal{H}}_1' + \hat{\mathcal{H}}_2'.
\end{aligned} \tag{24}$$

It is obvious that, after the unitary transformation, the quantized mesoscopic system can be considered as two independent quantum mechanical harmonic oscillators. m_j and ω_j are respectively the generalized mass and oscillating frequency of the jth oscillator. Therefore, the complicated system is equivalent to a two mode non-coupling radiation field.

4. The energy-level transition rule

For a dynamical system, generally, it will remain in a certain stationary state so long as it is not disturbed by outside factors. The transitions between two states are restricted by the matrix element of the total electric displacement $\mathcal{D}$ in the Heisenberg picture for the system. Only when the matrix element referring to these two states is non-vanishing, the transitions are likely to occur. As is explained by Dirac,[11] for a series of energy eigenstates $|\xi\rangle$ of a dynamical system, there exists a relationship about the matrix elements of $\mathcal{D}$,

$$\sum_r f_r(\xi')\,\langle\xi'|\,\mathcal{D}\,|\xi''\rangle\,g_r(\xi'') = 0$$

or

$$\langle\xi'|\,\mathcal{D}\,|\xi''\rangle\sum_r f_r(\xi')g_r(\xi'') = 0, \tag{25}$$

where f_r, g_r are functions of the ξ only. Equation (25) shows that the relation $\sum_r f_r(\xi')g_r(\xi'') = 0$ must exist in order that $\langle\xi'|\,\mathcal{D}\,|\xi''\rangle$ may not vanish. Consequently, one can obtain the selection rules of energy-level transitions which connect with the component of $\mathcal{D}$.

Because it is difficult to construct an electric displacement vector operator of the coupling system, inspired by such a property that unitary transformation can not alter the eigenvalues of the operator, we will fulfill the work by employing the Hamiltonian operator after the unitary transformation.

From $[\hat{a}_j, \hat{a}_l^\dagger] = \delta_{jl}$, $[\hat{a}_j, \hat{a}_l] = [\hat{a}_j^\dagger, \hat{a}_l^\dagger] = 0$ $(j, l = 1, 2)$, we have

$$\begin{aligned}
[\hat{a}_j, \hat{\mathcal{H}}'] &= [\hat{a}_j, \hat{\mathcal{H}}_j'] = \hbar\omega_j\hat{a}_j,\\
[\hat{a}_j^\dagger, \hat{\mathcal{H}}'] &= [\hat{a}_j^\dagger, \hat{\mathcal{H}}_j'] = \hbar\omega_j\hat{a}_j^\dagger.
\end{aligned} \tag{26}$$

Furthermore, we obtain

$$\begin{aligned}
\left(\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)^2\hat{a}_j &= [[\hat{a}_j, \hat{\mathcal{H}}'], \hat{\mathcal{H}}'] = \hbar^2\omega_j^2\hat{a}_j,\\
\left(\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)^2\hat{a}_j^\dagger &= [[\hat{a}_j^\dagger, \hat{\mathcal{H}}'], \hat{\mathcal{H}}'] = \hbar^2\omega_j^2\hat{a}_j^\dagger,
\end{aligned} \tag{27}$$

which imply that the IEOs of the system are

$$\hat{O}_{\mathrm{e}j} = \hat{a}_j, \quad \hat{O}'_{\mathrm{e}j} = \hat{a}_j^\dagger. \tag{28}$$

Suppose $|b\rangle$ and $|c\rangle$ are two arbitrary eigenstates of the operator $\hat{\mathcal{H}}'$ with eigenvalues E_b and E_c, from Eq.(27) we have

$$\begin{aligned}
\left[(E_b - E_\mathrm{c})^2 - \hbar^2\omega_j^2\right]\langle c|\,\hat{a}_j\,|b\rangle &= 0,\\
\left[(E_b - E_\mathrm{c})^2 - \hbar^2\omega_j^2\right]\langle c|\,\hat{a}_j^\dagger\,|b\rangle &= 0.
\end{aligned} \tag{29}$$

The electric displacement vector operator[20] can be constructed as

$$\begin{aligned}\mathcal{D}_j &= \varepsilon_j \boldsymbol{E}_j \\ &= \mathrm{i}\varepsilon_j \left(\frac{\hbar\omega_j}{2\varepsilon_j}\right)^{1/2} \\ &\quad \times \left[\hat{a}_j \boldsymbol{u}_j(\boldsymbol{r}_j)\,\mathrm{e}^{-\mathrm{i}\omega_j t} - \hat{a}_j^{\dagger}\boldsymbol{u}_j^{*}(\boldsymbol{r}_j)\,\mathrm{e}^{\mathrm{i}\omega_j t}\right], \quad (30)\end{aligned}$$

where $\boldsymbol{u}_j(\boldsymbol{r}_j)$ is the vector mode functions which correspond to the frequency ω_j, and ε_j is the dielectric constant. Thus, using Eqs.(29) and (30), we deduce

$$\left[(E_b - E_c)^2 - \hbar^2\omega_j^2\right] \langle c|\,\mathcal{D}_j\,|b\rangle = 0. \quad (31)$$

Just as Eqs.(25) and (31) reveal that $\langle c|\,\mathcal{D}_j\,|b\rangle = 0$ unless $(E_b - E_c)^2 - \hbar^2\omega_j^2 = 0$, i.e., the energy-level transitions only possibly exist in the two states whose energy-level difference satisfies the selection rules

$$E_b - E_c = \pm\hbar\omega_j. \quad (32)$$

As mentioned above, the eigenvalues of the system cannot change along with a unitary transformation, as a result Eq.(32) also gives the selection rules between $|\psi_1\rangle = \hat{U}\,|b\rangle$ and $|\psi_2\rangle = \hat{U}\,|c\rangle$ for the original coupling circuit system. If the transition happens, the system should absorb or emit a phonon with a finite frequency, which equals ω_1 or ω_2. In a word, the energy-level transition rules can be obtained indirectly when the system is interfered with by outside circumstance.

5. The thermal effect of the system

From the Eq.(7), we can know that the diagonalization of the Hamiltonian operator does not alter the ensemble average energy. So we have

$$\left\langle\hat{\mathcal{H}}\right\rangle_{\mathrm{e}} = \left\langle\hat{\mathcal{H}}'\right\rangle_{\mathrm{e}} = \left\langle\hat{\mathcal{H}}_1'\right\rangle_{\mathrm{e}} + \left\langle\hat{\mathcal{H}}_2'\right\rangle_{\mathrm{e}}. \quad (33)$$

For a single quantum harmonic oscillator, the ensemble average energy had been known as[18]

$$\left\langle\hat{\mathcal{H}}_j\right\rangle_{\mathrm{e}} = \bar{E}_j = \frac{\hbar\omega_j}{2}\coth\frac{\beta\hbar\omega_j}{2}, \quad (j=1,2). \quad (34)$$

Substituting Eq.(34) into Eq.(33), we obtain

$$\left\langle\hat{\mathcal{H}}\right\rangle_{\mathrm{e}} = \bar{E} = \frac{\hbar\omega_1}{2}\coth\frac{\beta\hbar\omega_1}{2} + \frac{\hbar\omega_2}{2}\coth\frac{\beta\hbar\omega_2}{2}. \quad (35)$$

Because $\hat{\mathcal{H}}$ has nothing to do with β, using the Eq.(10), we obtain the integral form of the generalized HFT,

$$\beta\left\langle\frac{\partial\hat{\mathcal{H}}}{\partial\lambda}\right\rangle_{\mathrm{e}} = \int \mathrm{d}\beta\frac{\partial}{\partial\lambda}\bar{E}. \quad (36)$$

Substituting Eqs.(16) and (35) into Eq.(36) we obtain

$$\beta\left\langle\frac{\partial\hat{\mathcal{H}}}{\partial\lambda_1}\right\rangle_{\mathrm{e}} = \beta\left\langle\frac{1}{2}\hat{p}_1^2\right\rangle_{\mathrm{e}} = \int \mathrm{d}\beta\frac{\partial}{\partial\lambda_1}\bar{E}, \quad (37)$$

where we have assigned $\lambda_1 \equiv 1/[\gamma(L_1+L)]$. Then it follows,

$$\begin{aligned}\left\langle\hat{p}_1^2\right\rangle_{\mathrm{e}} &= \frac{2}{\beta}\frac{\partial}{\partial\lambda_1}\int \mathrm{d}\beta\bar{E} \\ &= \frac{m_1\hbar\omega_1}{2}D^2\coth\frac{\beta\hbar\omega_1}{2} \\ &\quad + \frac{m_2\hbar\omega_2}{2}C^2\coth\frac{\beta\hbar\omega_2}{2}, \quad (38)\end{aligned}$$

because of $\langle\hat{q}_j\rangle_{\mathrm{e}} = \langle\hat{p}_j\rangle_{\mathrm{e}} = 0$ $(j=1,2)$, the fluctuations of $\hat{p}_1$ is

$$\begin{aligned}\left\langle(\triangle\hat{p}_1)^2\right\rangle_{\mathrm{e}} &= \frac{m_1\hbar\omega_1}{2}D^2\coth\frac{\beta\hbar\omega_1}{2} \\ &\quad + \frac{m_2\hbar\omega_2}{2}C^2\coth\frac{\beta\hbar\omega_2}{2}. \quad (39)\end{aligned}$$

Similarly, when we assign $\lambda_2 \equiv 1/[\gamma(L_2+L)]$, $\lambda_3 \equiv (C_1+C_c)/(C_1C_c)$, and $\lambda_4 \equiv (C_2+C_c)/(C_2C_c)$, the quantum fluctuations of the system are as follows:

$$\begin{aligned}\left\langle(\triangle\hat{p}_2)^2\right\rangle_{\mathrm{e}} &= \frac{m_1\hbar\omega_1}{2}B^2\coth\frac{\beta\hbar\omega_1}{2} \\ &\quad + \frac{m_2\hbar\omega_2}{2}A^2\coth\frac{\beta\hbar\omega_2}{2}, \quad (40) \\ \left\langle(\triangle\hat{q}_1)^2\right\rangle_{\mathrm{e}} &= \frac{\hbar}{2m_1\omega_1}A^2\coth\frac{\beta\hbar\omega_1}{2} \\ &\quad + \frac{\hbar}{2m_2\omega_2}B^2\coth\frac{\beta\hbar\omega_2}{2}, \quad (41) \\ \left\langle(\triangle\hat{q}_2)^2\right\rangle_{\mathrm{e}} &= \frac{\hbar}{2m_1\omega_1}C^2\coth\frac{\beta\hbar\omega_1}{2} \\ &\quad + \frac{\hbar}{2m_2\omega_2}D^2\coth\frac{\beta\hbar\omega_2}{2}. \quad (42)\end{aligned}$$

Note that $\coth(\beta\hbar\omega_j/2)$ is a monotone increasing function which is always greater than 1 when $T>0$. If the temperature is very low, $\coth(\beta\hbar\omega_j/2)\to 1$, the fluctuations contain the Planck constant $\hbar$ visibly, and the quantum noise plays a major role; in contrast, $\coth(\beta\hbar\omega_j/2)\to 2/(\beta\hbar\omega_j)$, the Planck constant vanishes, the classical noise becomes dominant. On the other hand, from Eqs.(39)–(42) we can see that, besides the parameters of the circuit components, the fluctuations also depend on the environmental temperature, and this increases with the increase of temperature. Therefore, taking the temperature into consideration may be helpful to the design of low noise mesoscopic circuits.

6. Conclusion

In summary, for the mesoscopic inductance and capacitance coupling LC circuit, we obtain the quantized Hamiltonian by introducing a unitary operator. The energy-level transition rule of the system is deduced by the IEO method, and it is found that the transitions can only possibly exist in the two states whose energy-level difference satisfies the restrictive relations $E_b - E_c = \pm\hbar\omega_j$ $(j = 1, 2)$. In addition, by virtue of the generalized HFT, the thermal effects of the circuit are investigated. It is found that both the circuit cell parameters and the environmental temperature should be taken into account in designing practical circuits. The above conclusions will be constructive to enrich the theoretical research on mesoscopic circuits.

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