Vol 18 No 5, May 2009
1674-1056/2009/18(05)/1906-11

Chinese Physics B

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Third-order Stokes wave solutions for interfacial internal waves in a three-layer density-stratified fluid*

Chen Xiao-Gang(陈小刚)[a)b)†], Guo Zhi-Ping(郭志萍)[a)c)],
Song Jin-Bao(宋金宝)[b)], He Xiao-Dong(赫晓东)[d)],
Guo Jun-Ming(郭军明)[a)], Bao Shu-Hong(包曙红)[a)], and Cui Wei(崔 巍)[a)]

[a)] *College of Science, Inner Mongolia University of Technology, Hohhot* 010051, *China*
[b)] *Institute of Oceanology, Chinese Academy of Sciences, Qingdao* 266071, *China*
[c)] *Teaching Department of Basic Sciences, Shanxi Water Technical Professional College, Shanxi* 044004, *China*
[d)] *Chemical Engineering College, Inner Mongolia University of Technology, Hohhot* 010051, *China*

(Received 18 August 2008; revised manuscript received 31 October 2008)

Interfacial internal waves in a three-layer density-stratified fluid are investigated using a singular perturbation method, and third-order asymptotic solutions of the velocity potentials and third-order Stokes wave solutions of the associated elevations of the interfacial waves are presented based on the small amplitude wave theory. As expected, the third-order solutions describe the third-order nonlinear modification and the third-order nonlinear interactions between the interfacial waves. The wave velocity depends on not only the wave number and the depth of each layer but also on the wave amplitude.

Keywords: three-layer density-stratified fluid, interfacial internal waves, third-order Stokes wave solutions, small amplitude wave theory

PACC: 4735, 0340G

1. Introduction

Interfacial internal waves, which travel in the interfaces between density-stratified fluids, are wave phenomena which occur frequently in oceans and lakes. Investigation of them is not only important but is also helpful to the understanding of three-dimensional internal waves in the ocean, to find out the actions of these waves on the exchange process of mass, momentum, energy, etc in the ocean, and thus increase the knowledge about the interactions between internal waves and many other multi-scale ocean waves (for example, internal waves–surface waves, internal waves–current, internal waves–eddy, etc), about ocean engineering, military affairs and so on. In the research of interfacial internal waves, we usually adopt a two-layer model. For example, Umeyama[1,2] conducted a study on second-order and third-order Stokes internal waves for a density-stratified two-layer fluid of finite depth. Song[3] derived second-order random wave solutions for internal waves in a two-layer fluid. Cheng[4] studied the evolution equation for two-dimensional nonlinear interfacial waves in a two-layer fluid. You *et al*[5] discussed the reflection and transmission of water waves that pass a vertical thin barrier in a two-layer fluid. Wei *et al*[6] established the reflection and transmission of an internal solitary wave over a step in a stratified fluid. The above papers investigated the waves in a two-layer water mass. But in the real ocean, density-stratified phenomena sometimes are very complicated. So it is necessary to develop a relative study on interfacial internal waves in multilayer density-stratified fluids. Zhang and Yang[7] studied internal waves in a three-layer stratified fluid. Chen *et al*[8] derived second-order Stokes solutions for internal waves in a three-layer density-stratified fluid. Chen and Song[9] obtained second-order random wave solutions for interfacial internal waves in an N-layer density-stratified fluid. Pang *et al*[10] studied second-order Stokes wave solutions for interfacial waves in a three-layer stratified fluid with a background current.

The equations describing interfacial internal

*Project supported by the Natural Science Foundation of Inner Mongolia, China (Grant No 200711020116), and Open Fund of the Key Laboratory of Ocean Circulation and Waves, Chinese Academy of Sciences (Grant No KLOCAW0805), and the Key Program of the Scientific Research Plan of Inner Mongolia University of Technology, China(Grant No ZD200608) and National Science Fund for Distinguished Young Scholars of China (Grant No 40425015).
†E-mail: xiaogang_chen@imut.edu.cn

waves in multilayer stratified fluids are nonlinear wave equations and finding their solutions is important for nonlinear problems. Currently, there are many methods for solving nonlinear wave equations. For example, the perturbation method,[1−3,8−12] the homogeneous balance principle,[13−15] the expansion method for the Jacobi elliptic function,[16] various tanh methods,[17] the general projective Riccati method[18] and so on.

In this paper, based on small amplitude wave theory and the assumption of irrotational, inviscid and incompressible conditions, third-order Stokes waves solutions of interfacial internal waves in a three-layer density-stratified fluid are investigated by using a singular perturbation method. The results show that third-order solutions can describe the third-order nonlinear modification and the third-order nonlinear interactions between interfacial waves. The wave velocity depends not only on the wave number and the depth of each layer but also on the wave amplitude.

2. Basic equations and boundary conditions

Consider interfacial waves in a three-layer density-stratified fluid contained between two rigid walls at the top and the bottom as shown in Fig.1. The horizontal coordinate x is fixed along the undisturbed interface and the vertical axis z is positive upward; z_i (i = 1, 2, 3) is the coordinate of depth of the undisturbed ith layer of the fluid, in which $z_0 = 0$ is the upper boundary and z_3 is the bottom. $\rho^{(i)}$ (i = 1, 2, 3) is the density of the ith layer of the fluid, $h_i = z_{i-1} - z_i$ (i = 1, 2, 3) is the thickness of the undisturbed ith layer, and $\eta^{(i)}$ (i = 1, 2) is the vertical displacement of the ith density interface.

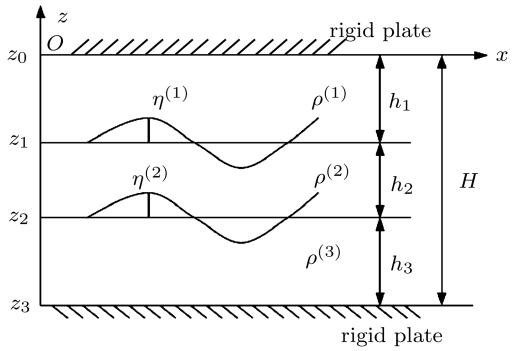


Fig.1. Schematic of the three-layer fluid.

Assuming that the fluid in each layer is homogeneous, inviscid, incompressible, and irrotational, the velocity potential $\Phi^{(i)}$ (i = 1, 2, 3) satisfies the Laplace equations:

$$\frac{\partial^2 \Phi^{(i)}}{\partial x^2} + \frac{\partial^2 \Phi^{(i)}}{\partial z^2} = 0, \quad \left(z_i + \eta^{(i)} \le z \le z_{i-1} + \eta^{(i-1)}, \quad i = 1, 2, 3\right), \tag{1}$$

where $\eta^{(0)}(x,t) = 0$, and $\eta^{(3)}(x,t) = 0$.

The top and bottom boundary conditions are

$$\frac{\partial \Phi^{(1)}}{\partial z} = 0 \quad (z = z_0 = 0), \tag{2}$$

$$\frac{\partial \Phi^{(3)}}{\partial z} = 0 \quad (z = z_3). \tag{3}$$

At the density interfaces, the kinematic boundary conditions are:

$$\frac{\partial \Phi^{(i)}}{\partial z} = \frac{\partial \eta^{(i)}}{\partial t} + \frac{\partial \eta^{(i)}}{\partial x}\frac{\partial \Phi^{(i)}}{\partial x} \quad (z = z_i + \eta^{(i)}, \quad i = 1, 2), \tag{4}$$

$$\frac{\partial \Phi^{(i)}}{\partial z} = \frac{\partial \eta^{(i-1)}}{\partial t} + \frac{\partial \eta^{(i-1)}}{\partial x}\frac{\partial \Phi^{(i)}}{\partial x} \quad (z = z_{i-1} + \eta^{(i-1)}, \quad i = 2, 3), \tag{5}$$

and the dynamic boundary conditions are:

$$\rho^{(i)}\left\{g\eta^{(i)} + \frac{\partial \Phi^{(i)}}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi^{(i)}}{\partial x}\right)^2 + \left(\frac{\partial \Phi^{(i)}}{\partial z}\right)^2\right]\right\} = \rho^{(i+1)}\left\{g\eta^{(i)} + \frac{\partial \Phi^{(i+1)}}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi^{(i+1)}}{\partial x}\right)^2 + \left(\frac{\partial \Phi^{(i+1)}}{\partial z}\right)^2\right]\right\}, \quad (z = z_i + \eta^{(i)}), \quad (i = 1, 2). \tag{6}$$

3. Third-order Stokes interfacial internal waves by the perturbation method

As Umeyama[1,2] and Song[3] did in their studies of internal waves and random internal waves in a two-layer density-stratified fluid, the perturbation method is used to solve the above basic equations and boundary conditions. But the third-order Stokes wave solutions for interfacial internal waves will have so-called secular terms when using the regular perturbation method.[11] It is also noticed that the nonlinearity of the interfacial internal waves may lead to differences between the period of nonlinear waves and that of linear waves.[11] So we use the singular perturbation method to avoid the appearance of these secular terms.

We introduce nondimensional variables as follows:

$$
\begin{aligned}
&x' = kx, \quad z' = kz, \quad t' = \omega t,\\
&\Phi^{(i)'}(x', z', t') = \frac{2}{H}\sqrt{\frac{k}{g}}\,\Phi^{(i)}(x, z, t),\\
&h_i' = kh_i, \quad i = 1, 2, 3,\\
&\eta^{(i)'}(x', t') = \frac{2}{H}\eta^{(i)}(x, t), \quad i = 1, 2,\\
&\omega' = \frac{\omega}{\sqrt{gk}},
\end{aligned}
\tag{7}
$$

where k is the wave number, ω is the angular frequency, and H is the wave amplitude.

By using Eqs.(7), the governing equations and boundary conditions (1)–(6) can then be expressed in nondimensional form as:

$$
\frac{\partial^2 \Phi^{(i)}}{\partial x^2} + \frac{\partial^2 \Phi^{(i)}}{\partial z^2} = 0 \quad (z_i + \varepsilon\eta^{(i)} \le z \le z_{i-1} + \varepsilon\eta^{(i-1)}, \quad i = 1, 2, 3), \tag{8}
$$

$$
\frac{\partial \Phi^{(1)}}{\partial z} = 0, \qquad (z = z_0 = 0), \tag{9}
$$

$$
\frac{\partial \Phi^{(i)}}{\partial z} = \omega\frac{\partial \eta^{(i)}}{\partial t} + \varepsilon\frac{\partial \eta^{(i)}}{\partial x}\frac{\partial \Phi^{(i)}}{\partial x}, \qquad (z = z_i + \varepsilon\eta^{(i)}, \quad i = 1, 2), \tag{10}
$$

$$
\frac{\partial \Phi^{(i)}}{\partial z} = \omega\frac{\partial \eta^{(i-1)}}{\partial t} + \varepsilon\frac{\partial \eta^{(i-1)}}{\partial x}\frac{\partial \Phi^{(i)}}{\partial x}, \quad (z = z_{i-1} + \varepsilon\eta^{(i-1)}, \quad i = 2, 3), \tag{11}
$$

$$
\frac{\partial \Phi^{(3)}}{\partial z} = 0, \qquad (z = z_3), \tag{12}
$$

$$
\begin{aligned}
&\rho^{(i)}\left\{\eta^{(i)} + \omega\frac{\partial \Phi^{(i)}}{\partial t} + \frac{1}{2}\varepsilon\left[\left(\frac{\partial \Phi^{(i)}}{\partial x}\right)^2 + \left(\frac{\partial \Phi^{(i)}}{\partial z}\right)^2\right]\right\}\\
&= \rho^{(i+1)}\left\{\eta^{(i)} + \omega\frac{\partial \Phi^{(i+1)}}{\partial t} + \frac{1}{2}\varepsilon\left[\left(\frac{\partial \Phi^{(i+1)}}{\partial x}\right)^2 + \left(\frac{\partial \Phi^{(i+1)}}{\partial z}\right)^2\right]\right\}, \qquad (z = z_i + \varepsilon\eta^{(i)}, \quad i = 1, 2),
\end{aligned}
\tag{13}
$$

where $\varepsilon = kH/2$ can be chosen as a small parameter and we have dropped the primes for nondimensional variables.

The velocity potential $\Phi^{(i)}(i = 1, 2, 3)$ and the vertical displacements of the density interfaces $\eta^{(i)}$ $(i = 1, 2)$ can be expressed as:

$$
\Phi^{(i)} = \Phi_{\mathrm{I}}^{(i)} + \varepsilon\,\Phi_{\mathrm{II}}^{(i)} + \varepsilon^2\,\Phi_{\mathrm{III}}^{(i)} + O(\varepsilon^3), \quad i = 1, 2, 3, \tag{14}
$$

$$
\eta^{(i)} = \eta_{\mathrm{I}}^{(i)} + \varepsilon\eta_{\mathrm{II}}^{(i)} + \varepsilon^2\eta_{\mathrm{III}}^{(i)} + O(\varepsilon^3), \quad i = 1, 2, \tag{15}
$$

$$
\omega = \omega_{\mathrm{I}} + \varepsilon\omega_{\mathrm{II}} + \varepsilon^2\omega_{\mathrm{III}} + O(\varepsilon^3), \tag{16}
$$

where O is the order symbol, and the subscripts I, II and III denote quantities corresponding to the first-order, second-order and third-order perturbation solutions, respectively. Also, the velocity potential $\Phi^{(i)}$ $(i = 1, 2, 3)$ at the density interfaces $z = z_i + \eta^{(i-1)}$ $(i = 2, 3)$ and $z = z_i + \eta^{(i)} (i = 1, 2)$ can be expanded in a Taylor series about $z = z_i$ $(i = 1, 2)$. So we can replace the density interface boundary conditions in Eqs.(8), (10), (11) and (12) by the conditions to be satisfied for $z = z_i (i = 1, 2)$ instead of $z = z_i + \eta^{(i)}$. Substituting Eqs.(14)–(16) into Eqs.(8)–(13), we derive the governing equations for the first-order, second-order and third-order powers as follows.

The first-order equations and boundary conditions are:

$$\frac{\partial^2 \Phi_{\mathrm{I}}^{(i)}}{\partial x^2} + \frac{\partial^2 \Phi_{\mathrm{I}}^{(i)}}{\partial z^2} = 0, \quad (z_i + \varepsilon\eta^{(i)} \le z \le z_{i-1} + \varepsilon\eta^{(i-1)}, \quad i = 1, 2, 3), \tag{17}$$

$$\frac{\partial \Phi_{\mathrm{I}}^{(1)}}{\partial z} = 0, \quad (z = z_0 = 0), \tag{18}$$

$$\frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial z} = \omega_{\mathrm{I}} \frac{\partial \eta_{\mathrm{I}}^{(i)}}{\partial t}, \quad (z = z_i, \quad i = 1, 2) \tag{19}$$

$$\frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial z} = \omega_{\mathrm{I}} \frac{\partial \eta_{\mathrm{I}}^{(i-1)}}{\partial t}, \quad (z = z_{i-1}, \quad i = 2, 3), \tag{20}$$

$$\frac{\partial \Phi_{\mathrm{I}}^{(3)}}{\partial z} = 0, \quad (z = z_3), \tag{21}$$

$$\rho^{(i)} \left\{ \eta_{\mathrm{I}}^{(i)} + \omega_{\mathrm{I}} \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial t} \right\} = \rho^{(i+1)} \left\{ \eta_{\mathrm{I}}^{(i)} + \omega_{\mathrm{I}} \frac{\partial \Phi_{\mathrm{I}}^{(i+1)}}{\partial t} \right\}, \quad (z = z_i, i = 1, 2). \tag{22}$$

The second-order equations and boundary conditions are:

$$\frac{\partial^2 \Phi_{\mathrm{II}}^{(i)}}{\partial x^2} + \frac{\partial^2 \Phi_{\mathrm{II}}^{(i)}}{\partial z^2} = 0, \quad (z_i + \varepsilon\eta^{(i)} \le z \le z_{i-1} + \varepsilon\eta^{(i-1)}, \quad i = 1, 2, 3), \tag{23}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(1)}}{\partial z} = 0, \quad (z = z_0 = 0), \tag{24}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(i)}}{\partial z} + \eta_{\mathrm{I}}^{(i)} \frac{\partial^2 \Phi_{\mathrm{I}}^{(i)}}{\partial z^2} = \omega_{\mathrm{I}} \frac{\partial \eta_{\mathrm{II}}^{(i)}}{\partial t} + \omega_{\mathrm{II}} \frac{\partial \eta_{\mathrm{I}}^{(i)}}{\partial t} + \frac{\partial \eta_{\mathrm{I}}^{(i)}}{\partial x} \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial x}, \quad (z = z_i, i = 1, 2), \tag{25}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(i)}}{\partial z} + \eta_{\mathrm{I}}^{(i-1)} \frac{\partial^2 \Phi_{\mathrm{I}}^{(i)}}{\partial z^2}, = \omega_{\mathrm{I}} \frac{\partial \eta_{\mathrm{II}}^{(i-1)}}{\partial t} + \omega_{\mathrm{II}} \frac{\partial \eta_{\mathrm{I}}^{(i-1)}}{\partial t} + \frac{\partial \eta_{\mathrm{I}}^{(i-1)}}{\partial x} \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial x}, \quad (z = z_i, i = 2, 3), \tag{26}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(3)}}{\partial z} = 0, \quad (z = z_3), \tag{27}$$

$$\rho^{(i)} \left\{ \eta_{\mathrm{II}}^{(i)} + \omega_{\mathrm{I}} \frac{\partial \Phi_{\mathrm{II}}^{(i)}}{\partial t} + \omega_{\mathrm{II}} \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial t} + \omega_{\mathrm{I}} \eta_{\mathrm{I}}^{(i)} \frac{\partial^2 \Phi_{\mathrm{I}}^{(i)}}{\partial t \partial z} + \frac{1}{2} \left[\left(\frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial x} \right)^2 + \left(\frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial z} \right)^2 \right] \right\} = \rho^{(i+1)} \left\{ \eta_{\mathrm{II}}^{(i)} + \omega_{\mathrm{I}} \frac{\partial \Phi_{\mathrm{II}}^{(i+1)}}{\partial t} + \omega_{\mathrm{II}} \frac{\partial \Phi_{\mathrm{I}}^{(i+1)}}{\partial t} + \omega_{\mathrm{I}} \eta_{\mathrm{I}}^{(i)} \frac{\partial^2 \Phi_{\mathrm{I}}^{(i+1)}}{\partial t \partial z} + \frac{1}{2} \left[\left(\frac{\partial \Phi_{\mathrm{I}}^{(i+1)}}{\partial x} \right)^2 + \left(\frac{\partial \Phi_{\mathrm{I}}^{(i+1)}}{\partial z} \right)^2 \right] \right\}, \quad (z = z_i, i = 1, 2). \tag{28}$$

The third-order equations and boundary conditions are:

$$\frac{\partial \Phi_{\mathrm{III}}^{(i)}}{\partial x^2} + \frac{\partial \Phi_{\mathrm{III}}^{(i)}}{\partial z^2} = 0, \quad (z_i + \varepsilon\eta^{(i)} \le z \le z_{i-1} + \varepsilon\eta^{(i-1)}), \quad i = 1, 2, 3, \tag{29}$$

$$\frac{\partial \Phi_{\mathrm{III}}^{(1)}}{\partial z} = 0, \quad (z = z_0 = 0), \tag{30}$$

$$\begin{aligned}&\frac{\partial \Phi_{\rm III}^{(i)}}{\partial z}+\eta_{\rm I}^{(i)}\frac{\partial^2 \Phi_{\rm II}^{(i)}}{\partial z^2}+\eta_{\rm II}^{(i)}\frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial z^2}+\frac{1}{2}\eta_{\rm I}^{(i)^2}\frac{\partial^3 \Phi_{\rm I}^{(i)}}{\partial z^3}\\ &=\omega_{\rm III}\frac{\partial \eta_{\rm I}^{(i)}}{\partial t}+\omega_{\rm II}\frac{\partial \eta_{\rm II}^{(i)}}{\partial t}+\omega_{\rm I}\frac{\partial \eta_{\rm III}^{(i)}}{\partial t}+\frac{\partial \eta_{\rm I}^{(i)}}{\partial x}\cdot\frac{\partial \Phi_{\rm II}^{(i)}}{\partial x}+\frac{\partial \eta_{\rm II}^{(i)}}{\partial x}\cdot\frac{\partial \Phi_{\rm I}^{(i)}}{\partial x}\\ &+\eta_{\rm I}^{(i)}\frac{\partial \eta_{\rm I}^{(i)}}{\partial x}\cdot\frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial x\partial z},\quad (z=z_i,\quad i=1,2),\end{aligned}\tag{31}$$

$$\begin{aligned}&\frac{\partial \Phi_{\rm III}^{(i)}}{\partial z}+\eta_{\rm I}^{(i-1)}\frac{\partial^2 \Phi_{\rm II}^{(i)}}{\partial z^2}+\eta_{\rm II}^{(i-1)}\frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial z^2}+\frac{1}{2}\eta_{\rm I}^{(i-1)^2}\frac{\partial^3 \Phi_{\rm I}^{(i)}}{\partial z^3}\\ &=\omega_{\rm III}\frac{\partial \eta_{\rm I}^{(i-1)}}{\partial t}+\omega_{\rm II}\frac{\partial \eta_{\rm II}^{(i-1)}}{\partial t}+\omega_{\rm I}\frac{\partial \eta_{\rm III}^{(i-1)}}{\partial t}+\frac{\partial \eta_{\rm I}^{(i-1)}}{\partial x}\cdot\frac{\partial \Phi_{\rm II}^{(i)}}{\partial x}+\frac{\partial \eta_{\rm II}^{(i)}}{\partial x}\cdot\frac{\partial \Phi_{\rm I}^{(i)}}{\partial x}\\ &+\eta_{\rm I}^{(i-1)}\frac{\partial \eta_{\rm I}^{(i-1)}}{\partial x}\cdot\frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial x\partial z},\quad (z=z_i,\quad i=2,3),\end{aligned}\tag{32}$$

$$\frac{\partial \Phi_{\rm III}^{(3)}}{\partial z}=0,\quad (z=z_3),\tag{33}$$

$$\begin{aligned}&\rho^{(i)}\left\{\eta_{\rm III}^{(i)}+\omega_{\rm I}\frac{\partial \Phi_{\rm III}^{(i)}}{\partial t}+\omega_{\rm I}\left(\eta_{\rm I}^{(i)}\frac{\partial^2 \Phi_{\rm II}^{(i)}}{\partial t\partial z}+\eta_{\rm II}^{(i)}\frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial t\partial z}+\frac{1}{2}\eta_{\rm I}^{(i)^2}\frac{\partial^3 \Phi_{\rm I}^{(i)}}{\partial t\partial z^2}\right)+\omega_{\rm III}\frac{\partial \Phi_{\rm I}^{(i)}}{\partial t}\right.\\ &\left.+\omega_{\rm II}\left(\frac{\partial \Phi_{\rm II}^{(i)}}{\partial t}+\eta_{\rm I}^{(i)}\frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial t\partial z}\right)+\frac{\partial \Phi_{\rm I}^{(i)}}{\partial x}\cdot\frac{\partial \Phi_{\rm II}^{(i)}}{\partial x}+\frac{\partial \Phi_{\rm I}^{(i)}}{\partial z}\cdot\frac{\partial \Phi_{\rm II}^{(i)}}{\partial z}+\eta_{\rm I}^{(i)}\left(\frac{\partial \Phi_{\rm I}^{(i)}}{\partial x}\frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial x\partial z}+\frac{\partial \Phi_{\rm I}^{(i)}}{\partial z}\frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial z^2}\right)\right\}\\ &=\rho^{(i+1)}\left\{\eta_{\rm III}^{(i)}+\omega_{\rm I}\frac{\partial \Phi_{\rm III}^{(i+1)}}{\partial t}+\omega_{\rm I}\left(\eta_{\rm I}^{(i)}\frac{\partial^2 \Phi_{\rm II}^{(i+1)}}{\partial t\partial z}+\eta_{\rm II}^{(i)}\frac{\partial^2 \Phi_{\rm I}^{(i+1)}}{\partial t\partial z}+\frac{1}{2}\eta_{\rm I}^{(i)^2}\frac{\partial^3 \Phi_{\rm I}^{(i+1)}}{\partial t\partial z^2}\right)\right.\\ &+\omega_{\rm II}\left(\frac{\partial \Phi_{\rm II}^{(i+1)}}{\partial t}+\eta_{\rm I}^{(i)}\frac{\partial^2 \Phi_{\rm I}^{(i+1)}}{\partial t\partial z}\right)+\omega_{\rm III}\frac{\partial \Phi_{\rm I}^{(i+1)}}{\partial t}+\frac{\partial \Phi_{\rm I}^{(i+1)}}{\partial x}\cdot\frac{\partial \Phi_{\rm II}^{(i+1)}}{\partial x}\\ &\left.+\frac{\partial \Phi_{\rm I}^{(i+1)}}{\partial z}\cdot\frac{\partial \Phi_{\rm II}^{(i+1)}}{\partial z}+\eta_{\rm I}^{(i)}\left(\frac{\partial \Phi_{\rm I}^{(i+1)}}{\partial x}\frac{\partial^2 \Phi_{\rm I}^{(i+1)}}{\partial x\partial z}+\frac{\partial \Phi_{\rm I}^{(i)}}{\partial z}\frac{\partial^2 \Phi_{\rm I}^{(i+1)}}{\partial z^2}\right)\right\},\quad (z=z_i,\quad i=1,2).\end{aligned}\tag{34}$$

The results of the first-order solutions are as follows:

$$\Phi_{\rm I}^{(1)}=-\omega_{\rm I}\frac{b^{(1)}\cosh z}{\sinh h_1}\sin(x-t),\tag{35}$$

$$\Phi_{\rm I}^{(2)}=\omega_{\rm I}\frac{b^{(1)}\cosh(z-z_2)-b^{(2)}\cosh(z-z_1)}{\sinh h_2}\sin(x-t),\tag{36}$$

$$\Phi_{\rm I}^{(3)}=\omega_{\rm I}\frac{b^{(2)}\cosh(z-z_3)}{\sinh h_3}\sin(x-t),\tag{37}$$

$$\eta_{\rm I}^{(1)}=b^{(1)}\cos(x-t),\tag{38}$$

$$\eta_{\rm I}^{(2)}=b^{(2)}\cos(x-t),\tag{39}$$

Substituting Eqs.(35)–(39) into Eq.(22), we obtain the dispersion relation:

$$\begin{aligned}&\omega_{\rm I}^4\left[(\rho^{(1)}\coth h_1+\rho^{(2)}\coth h_2)(\rho^{(2)}\coth h_2+\rho^{(3)}\coth h_3)-\frac{\rho^{(2)^2}}{(\sinh h_2)^2}\right]\\ &+\omega_{\rm I}^2[(\rho^{(1)}-\rho^{(2)})(\rho^{(2)}\coth h_2+\rho^{(3)}\coth h_3)+g(\rho^{(2)}-\rho^{(3)})(\rho^{(1)}\coth h_1+\rho^{(2)}\coth h_2)]\\ &+(\rho^{(1)}-\rho^{(2)})(\rho^{(2)}-\rho^{(3)})=0,\end{aligned}\tag{40}$$

in which

$$b^{(2)}=\alpha b^{(1)},\tag{41}$$

where

$$\alpha = \frac{\left[\left(\rho^{(1)} - \rho^{(2)}\right) + \omega_{\mathrm{I}}^2 \left(\rho^{(1)} \coth h_1 + \rho^{(2)} \coth h_2\right)\right] \sinh h_2}{\rho^{(2)} \omega_{\mathrm{I}}^2}. \tag{42}$$

The results of the second-order solutions are as follows:

$$\Phi_{\mathrm{II}}^{(1)} = -\omega_{\mathrm{I}} \left[d^{(1)} + \frac{1}{2} b^{(1)^2} \coth h_1\right] \frac{\cosh(2z)}{\sinh(2h_1)} \sin(2(x - t)), \tag{43}$$

$$\begin{aligned} \Phi_{\mathrm{II}}^{(2)} = \omega_{\mathrm{I}} \Bigg[& \left(d^{(1)} - \frac{1}{2} b^{(1)} \frac{b^{(1)} \cosh h_2 - b^{(2)}}{\sinh h_2}\right) \frac{\cosh(2(z - z_2))}{\sinh(2h_2)} \\ & - \left(d^{(2)} - \frac{1}{2} b^{(2)} \frac{b^{(1)} - b^{(2)} \cosh h_2}{\sinh h_2}\right) \frac{\cosh(2(z - z_1))}{\sinh(2h_2)} \Bigg] \sin(2(x - t)), \end{aligned} \tag{44}$$

$$\Phi_{\mathrm{II}}^{(3)} = \omega_{\mathrm{I}} \left[d^{(2)} - \frac{1}{2} b^{(2)^2} \coth h_3\right] \frac{\cosh(2(z - z_3))}{\sinh(2h_3)} \sin(2(x - t)), \tag{45}$$

$$\eta_{\mathrm{II}}^{(1)} = d^{(1)} \cos(2(x - t)) + A_1, \tag{46}$$

$$\eta_{\mathrm{II}}^{(2)} = d^{(2)} \cos(2(x - t)) + A_2, \tag{47}$$

$$\omega_{\mathrm{II}} = 0, \tag{48}$$

where

$$d^{(1)} = \frac{l_1 a_{22} - l_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}}, \tag{49}$$

$$d^{(2)} = \frac{l_2 a_{11} - l_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}, \tag{50}$$

$$\begin{aligned} a_{11} = & \left(\rho^{(1)} - \rho^{(2)}\right) \\ & + 2\omega_{\mathrm{I}}^2 \left(\rho^{(1)} \coth(2h_1) + \rho^{(2)} \coth(2h_2)\right), \end{aligned} \tag{51}$$

$$a_{12} = a_{21} = -\frac{2\rho^{(2)} \omega_{\mathrm{I}}^2}{\sinh(2h_2)}, \tag{52}$$

$$\begin{aligned} a_{22} = & \left(\rho^{(2)} - \rho^{(3)}\right) \\ & + 2\omega_{\mathrm{I}}^2 \left(\rho^{(2)} \coth(2h_2) + \rho^{(3)} \coth(2h_3)\right), \end{aligned} \tag{53}$$

$$\begin{aligned} l_1 = & -\rho^{(1)} \omega_{\mathrm{I}}^2 b^{(1)^2} \left[\frac{1}{4} (\coth h_1)^2 + \coth h_1 \cdot \coth(2h_1) - \frac{3}{4}\right] \\ & + \rho^{(2)} \omega_{\mathrm{I}}^2 \left\{ b^{(1)} \frac{b^{(1)} \cosh h_2 - b^{(2)}}{\sinh h_2 \tanh(2h_2)} - b^{(2)} \frac{b^{(1)} - b^{(2)} \cosh h_2}{\sinh h_2 \sinh(2h_2)} + \frac{1}{4} \left(\frac{b^{(1)} \cosh h_2 - b^{(2)}}{\sinh h_2}\right)^2 - \frac{3}{4} b^{(1)^2} \right\}, \end{aligned} \tag{54}$$

$$\begin{aligned} l_2 = & -\rho^{(2)} \omega_{\mathrm{I}}^2 \left\{ b^{(1)} \frac{b^{(1)} \cosh h_2 - b^{(2)}}{\sinh h_2 \sinh(2h_2)} - b^{(2)} \frac{b^{(1)} - b^{(2)} \cosh h_2}{\sinh h_2 \tanh(2h_2)} + \frac{1}{4} \left(\frac{b^{(1)} - b^{(2)} \cosh h_2}{\sinh h_2}\right)^2 - \frac{3}{4} b^{(2)^2} \right\} \\ & + \rho^{(3)} \omega_{\mathrm{I}}^2 b^{(2)^2} \left[\frac{1}{4} (\coth h_3)^2 + \coth h_3 \coth(2h_3) - \frac{3}{4}\right], \end{aligned} \tag{55}$$

$$A_1 = \frac{1}{4} \omega_{\mathrm{I}}^2 b^{(1)^2} \frac{\rho^{(1)} \left[(\coth h_1)^2 - 1\right] - \rho^{(2)} \left[(\coth h_2 - \alpha/\sinh h_2)^2 - 1\right]}{\rho^{(2)} - \rho^{(1)}}, \tag{56}$$

$$A_2 = \frac{1}{4} \omega_{\mathrm{I}}^2 b^{(2)^2} \frac{\rho^{(2)} \left[(1/(\alpha \sinh h_2) - \coth h_2)^2 - 1\right] - \rho^{(3)} \left[(\coth h_3)^2 - 1\right]}{\rho^{(3)} - \rho^{(2)}}. \tag{57}$$

The results of the third-order solutions are as follows:

$$\Phi_{\mathrm{III}}^{(1)} = -\omega_{\mathrm{I}} \frac{l_{11}\cosh(3z)}{\sinh(3h_1)} \sin(3(x-t)), \tag{58}$$

$$\Phi_{\mathrm{III}}^{(2)} = \omega_{\mathrm{I}} \frac{l_{21}\cosh(3(z-z_2)) - l_{22}\cosh(3(z-z_1))}{\sinh(3h_2)} \sin(3(x-t)), \tag{59}$$

$$\Phi_{\mathrm{III}}^{(3)} = \omega_{\mathrm{I}} \frac{l_{31}\cosh(3(z-z_3))}{\sinh(3h_3)} \sin(3(x-t)), \tag{60}$$

$$\eta_{\mathrm{III}}^{(1)} = B_1 \cos(x-t) + p^{(1)} \cos(3(x-t)), \tag{61}$$

$$\eta_{\mathrm{III}}^{(2)} = B_2 \cos(x-t) + p^{(2)} \cos(3(x-t)), \tag{62}$$

$$\omega_{\mathrm{III}} = \omega_{\mathrm{I}}\beta, \tag{63}$$

$$l_{11} = p^{(1)} + b^{(1)} \left[\left(d^{(1)} + \frac{1}{2} b^{(1)^2} \coth h_1 \right) \coth(2h_1) + \frac{1}{2} d^{(1)} \coth h_1 - \frac{1}{8} b^{(1)^2} \right], \tag{64}$$

$$\begin{aligned} l_{21} = {} & p^{(1)} - b^{(1)} \left[\left(d^{(1)} - \frac{1}{2} b^{(1)} \frac{b^{(1)}\cosh h_2 - b^{(2)}}{\sinh h_2} \right) \coth(2h_2) - \left(d^{(2)} - \frac{1}{2} b^{(2)} \frac{b^{(1)} - b^{(2)}\cosh h_2}{\sinh h_2} \right) \frac{1}{\sinh(2h_2)} \right] \\ & - \frac{1}{2} d^{(1)} \frac{b^{(1)}\cosh h_2 - b^{(2)}}{\sinh h_2} - \frac{1}{8} b^{(1)^3}, \end{aligned} \tag{65}$$

$$\begin{aligned} l_{22} = {} & p^{(2)} - b^{(2)} \left[\left(d^{(1)} - \frac{1}{2} b^{(1)} \frac{b^{(1)}\cosh h_2 - b^{(2)}}{\sinh h_2} \right) \frac{1}{\sinh(2h_2)} - \left(d^{(2)} - \frac{1}{2} b^{(2)} \frac{b^{(1)} - b^{(2)}\cosh h_2}{\sinh h_2} \right) \coth(2h_2) \right] \\ & - \frac{1}{2} d^{(2)} \frac{b^{(1)} - b^{(2)}\cosh h_2}{\sinh h_2} - \frac{1}{8} b^{(2)^3}, \end{aligned} \tag{66}$$

$$l_{31} = p^{(2)} - b^{(2)} \left[\left(d^{(2)} - \frac{1}{2} b^{(2)^2} \coth h_3 \right) \coth(2h_3) + \frac{1}{2} d^{(2)} \coth h_3 + \frac{1}{8} b^{(2)^2} \right], \tag{67}$$

$$p^{(1)} = \frac{L_1 b_{22} - L_2 b_{12}}{b_{11} b_{22} - b_{12} b_{21}}, \tag{68}$$

$$p^{(2)} = \frac{L_2 b_{11} - L_1 b_{21}}{b_{11} b_{22} - b_{12} b_{21}}, \tag{69}$$

$$B_1 = \omega_{\mathrm{I}} \frac{\rho^{(2)} c_{12} - \rho^{(1)} c_{11}}{\rho^{(1)} - \rho^{(2)}}, \tag{70}$$

$$B_2 = \omega_{\mathrm{I}} \frac{\rho^{(3)} c_{22} - \rho^{(2)} c_{21}}{\rho^{(2)} - \rho^{(3)}}, \tag{71}$$

$$\beta = \frac{c_{31}}{(\rho^{(2)} - \rho^{(1)}) b^{(1)} + \omega_{\mathrm{I}}^2 [\rho^{(1)} b^{(1)} \coth h_1 + \rho^{(2)} (b^{(1)} \coth h_2 - b^{(2)} / \sinh h_2)]}, \tag{72}$$

$$b_{11}=\left(\rho^{(1)}-\rho^{(2)}\right)+3\omega_{\mathrm{I}}^{2}\left(\rho^{(1)}\coth(3h_1)+\rho^{(2)}\coth(3h_2)\right), \tag{73}$$

$$b_{12}=b_{21}=-\frac{3\rho^{(2)}\omega_{\mathrm{I}}^{2}}{\sinh(3h_2)}, \tag{74}$$

$$b_{22}=\left(\rho^{(2)}-\rho^{(3)}\right)+3\omega_{\mathrm{I}}^{2}\left(\rho^{(2)}\coth(3h_2)+\rho^{(3)}\coth(3h_3)\right), \tag{75}$$

$$\begin{aligned}L_1=&-\rho^{(1)}\omega_{\mathrm{I}}^{2}b^{(1)}\Bigg\{3\left[\left(d^{(1)}+\frac{1}{2}b^{(1)^2}\coth h_1\right)\coth(2h_1)+\frac{1}{2}d^{(1)}\coth h_1-\frac{1}{8}b^{(1)^2}\right]\coth(3h_1)\\&-\frac{7}{2}d^{(1)}+d^{(1)}\coth h_1\coth(2h_1)+\frac{1}{2}b^{(1)^2}(\coth h_1)^2\coth(2h_1)-\frac{11}{8}b^{(1)^2}(\coth h_1)^2\Bigg\}\\&+\rho^{(2)}\omega_{\mathrm{I}}^{2}\Bigg\{\Bigg(3b^{(1)}\Bigg[\left(d^{(1)}-b^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{2\sinh h_2}\right)\coth(2h_2)\\&-\left(d^{(2)}-b^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{2\sinh h_2}\right)\cdot\frac{1}{\sinh(2h_2)}\Bigg]+\frac{3}{2}d^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}+\frac{3}{8}b^{(1)^3}\Bigg)\coth(3h_2)\\&-\Bigg(3b^{(2)}\Bigg[\left(-b^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{2\sinh h_2}+d^{(1)}\right)\frac{1}{\sinh(2h_2)}-\left(d^{(2)}-b^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{2\sinh h_2}\right)\coth(2h_2)\Bigg]\\&+\frac{3}{2}d^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}+\frac{3}{8}b^{(2)^3}\Bigg)\frac{1}{\sinh(3h_2)}\\&+\Bigg[-\frac{7}{2}b^{(1)}d^{(1)}+\frac{11}{8}b^{(1)^2}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}+\frac{1}{2}b^{(2)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{(\sinh h_2)^2}\cdot\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh(2h_2)}\\&+d^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2\tanh(2h_2)}-d^{(2)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2\sinh(2h_2)}-\frac{1}{2}b^{(1)}\cdot\left(\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}\right)^2\coth(2h_2)\Bigg]\Bigg\},\end{aligned} \tag{76}$$

$$\begin{aligned}L_2=&-\rho^{(2)}\omega_{\mathrm{I}}^{2}\Bigg\{\Bigg(3b^{(1)}\left[\left(d^{(1)}-b^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{2\sinh h_2}\right)\coth(2h_2)+\left(b^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{2\sinh h_2}-d^{(2)}\right)\frac{1}{\sinh(2h_2)}\right]\\&+\frac{3}{2}d^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}+\frac{3}{8}b^{(1)^3}\Bigg)\frac{1}{\sinh(3h_2)}-\Bigg(3b^{(2)}\Bigg[\left(d^{(1)}-\frac{1}{2}b^{(1)}\cdot\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}\right)\frac{1}{\sinh(2h_2)}\\&-\left(d^{(2)}-b^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{2\sinh h_2}\right)\coth(2h_2)\Bigg]+\frac{3}{8}b^{(2)^3}+\frac{3}{2}d^{(2)}\cdot\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}\Bigg)\coth(3h_2)\\&+\Bigg[-\frac{7}{2}b^{(2)}d^{(2)}-\frac{1}{2}b^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{(\sinh h_2)^2}\cdot\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh(2h_2)}+\frac{11}{8}b^{(2)^2}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}\\&+d^{(1)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2\sinh(2h_2)}-d^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2\tanh(2h_2)}+\frac{1}{2}b^{(2)}\cdot\left(\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}\right)^2\coth(2h_2)\Bigg]\Bigg\}\\&+\rho^{(3)}\omega_{\mathrm{I}}^{2}b^{(2)}\Bigg\{3\left[\left(d^{(2)}-\frac{1}{2}b^{(2)^2}\coth h_3\right)\coth(2h_3)+\frac{1}{2}d^{(2)}\coth h_3+\frac{1}{8}b^{(2)^2}\right]\coth(3h_3)-\frac{7}{2}d^{(2)}\\&+d^{(2)}\coth h_3\coth(2h_3)+\frac{11}{8}b^{(2)^2}(\coth h_3)^2-\frac{1}{2}b^{(2)^2}(\coth h_3)^2\coth(2h_3)\Bigg\},\end{aligned} \tag{77}$$

$$c_{11}=b^{(1)}\Bigg[\omega_{\mathrm{I}}d^{(1)}\left(-\frac{3}{2}+\coth h_1\coth(2h_1)\right)-\frac{9}{8}\omega_{\mathrm{I}}b^{(1)^2}\coth h_1-\omega_{\mathrm{I}}A_1+\omega_{\mathrm{III}}\coth h_1$$

$$+\frac{1}{2}\omega_{\rm I}b^{(1)^2}(\coth h_1)^2\coth(2h_1)\Bigg], \tag{78}$$

$$\begin{aligned} c_{12} =& -\frac{3}{2}\omega_{\rm I}b^{(1)}d^{(1)}+\omega_{\rm I}d^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2\tanh(2h_2)}-\omega_{\rm I}d^{(2)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2\sinh(2h_2)}-\omega_{\rm I}b^{(1)}A_1\\ &-\omega_{\rm III}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}+\frac{9}{8}\omega_{\rm I}b^{(1)^2}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}\\ &-\frac{1}{2}\omega_{\rm I}b^{(1)}\left(\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}\right)^2\cdot\coth(2h_2)+\frac{1}{2}\omega_{\rm I}b^{(2)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{(\sinh h_2)^2}\cdot\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh(2h_2)}, \end{aligned} \tag{79}$$

$$\begin{aligned} c_{21} =& -\frac{3}{2}\omega_{\rm I}b^{(2)}d^{(2)}+\omega_{\rm I}d^{(1)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2\sinh(2h_2)}-\omega_{\rm I}d^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2\tanh(2h_2)}-\omega_{\rm I}b^{(2)}A_2\\ &-\omega_{\rm III}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}+\frac{9}{8}\omega_{\rm I}b^{(2)^2}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}+\frac{1}{2}\omega_{\rm I}b^{(2)}\left(\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}\right)^2\cdot\coth(2h_2)\\ &-\frac{1}{2}\omega_{\rm I}b^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{(\sinh h_2)^2}\cdot\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh(2h_2)}, \end{aligned} \tag{80}$$

$$\begin{aligned} c_{22} =& b^{(2)}\Bigg[\omega_{\rm I}d^{(2)}\left(-\frac{3}{2}+\coth h_3\coth(2h_3)\right)+\frac{9}{8}\omega_{\rm I}b^{(2)^2}\coth h_3-\omega_{\rm I}A_2-\omega_{\rm III}\coth h_3\\ &-\frac{1}{2}\omega_{\rm I}b^{(2)^2}(\coth h_3)^2\coth(2h_3)\Bigg], \end{aligned} \tag{81}$$

$$\begin{aligned} c_{31} =& \omega_{\rm I}^2\Bigg\{\rho^{(2)}\Bigg[-\frac{3}{2}b^{(1)}d^{(1)}+d^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2\tanh(2h_2)}-d^{(2)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2\sinh(2h_2)}-b^{(1)}A_1\\ &+\frac{9}{8}b^{(1)^2}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}-\frac{1}{2}b^{(1)}\left(\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}\right)^2\coth(2h_2)\\ &+b^{(2)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{2(\sinh h_2)^2}\cdot\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh(2h_2)}\Bigg]-\rho^{(1)}b^{(1)}\Bigg[d^{(1)}\left(-\frac{3}{2}+\coth h_1\coth(2h_1)\right)\\ &-\frac{9}{8}b^{(1)^2}\coth h_1-A_1+\frac{1}{2}b^{(1)^2}\cdot(\coth h_1)^2\coth(2h_1)\Bigg]\Bigg\}\\ &+\left(\rho^{(1)}-\rho^{(2)}\right)b^{(1)}\left[A_1\coth h_1+\left(d^{(1)}+\frac{1}{2}b^{(1)^2}\coth h_1\right)\coth(2h_1)+\frac{1}{2}d^{(1)}\coth h_1-\frac{3}{8}b^{(1)^2}\right]. \end{aligned} \tag{82}$$

4. Discussion

It is noted that the first-order solutions (35)–(39) and the second-order solutions (43)–(47) are exactly the same as the expressions derived by Chen[5] for second-order Stokes wave solutions in a three-layer density-stratified fluid. In addition, we find that the angular frequency of the second-order Stoke waves is the same as that of the first-order Stoke waves by using Eqs.(48), so the perturbation expansion at the angular frequency is not considered for the second-order Stokes wave solutions. However, according to Eqs.(63), we have $\omega_{\rm III}\neq 0$, then the angular frequency of the third-order Stokes wave is different from that of the first-order and second-order Stokes waves. Substituting Eqs.(16), (40), (48) and (63), we have the total angular frequency expression:

$$\omega=\omega_{\rm I}(1+\varepsilon^2\beta) \tag{83}$$

and the wave velocity of the third-order Stokes wave is

$$c=\frac{\omega}{k}=c_{\rm I}(1+\varepsilon^2\beta), \tag{84}$$

where $c_{\rm I}=\dfrac{\omega_{\rm I}}{k}$.

Equation (84) indicates that the wave velocity in the third-order solutions depends not only on the wave number and the depth of each layer but also on the wave amplitude. This is a characteristic of a nonlinear wave, but it can be obtained until it is approximated to the third-order.[11]

Equations (61) and (62) show that the third-order vertical displacement $\eta_{\mathrm{III}}^{(i)}$ $(i = 1, 2)$ contains not only a $\cos(3(x - t))$ term but also a $\cos(x - t)$ term. The latter's phase is equal to the linear wave, while the wave amplitude is ε^2 times as small as the wave amplitude of the linear wave. This term is similar to the constants A_1 and A_2 that the second-order vertical displacement $\eta_{\mathrm{II}}^{(i)}$ $(i = 1, 2)$ contains. This indicates that the nonlinear influence appears not only in the $n(n \geq 2)$ times high-frequency wave components, but also in the wave frequency or the stationary low-frequency components. The first, second and third solutions of the displacements of the upper and lower interfacial waves are shown in Figs.2 and 3, respectively.

In the figures η_{11}, η_{21}, η_{31} are the first, second and third solutions of the displacements of the upper interfacial wave, and η_{12}, η_{22}, η_{32} are the first, second and third solutions of the displacements of the lower interfacial wave; the parameters are $\rho^{(1)} = 1000$ mg/cm^3, $\rho^{(2)} = 1050$ mg/cm^3, $\rho^{(3)} = 1090$ mg/cm^3, $\varepsilon = \dfrac{kH}{2} = 0.1$, $h_1 = h_2 = h_3 = 10$ cm and $\theta = kx - \omega t$.

In addition, because expressions of the third-order solutions contain terms ${b^{(1)}}^3$, ${b^{(2)}}^3$, ${b^{(1)}}^2$, ${b^{(2)}}^2$, ${b^{(1)}}^2 \cdot b^{(2)}$, ${b^{(2)}}^2 \cdot b^{(1)}$, ${b^{(2)}}^2 \cdot d^{(2)}$, $b^{(1)} \cdot b^{(2)}$, $b^{(1)} \cdot d^{(1)}$, $b^{(1)} \cdot d^{(2)}$, $b^{(2)} \cdot d^{(1)}$, $b^{(2)} \cdot d^{(2)}$, the third-order solutions can describe the third-order nonlinear modification and the third-order nonlinear interactions between two interfacial waves. The results obtained also show that the third-order solutions consist of the first-order solutions, the second-order solutions and the third-order nonlinear modification and the third-order nonlinear interactions between the interfacial waves, and depend on the density and depth of each layer.

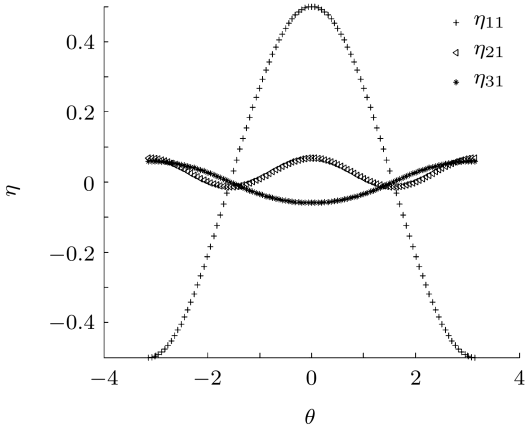


Fig.2. Displacements of the upper interfacial wave.

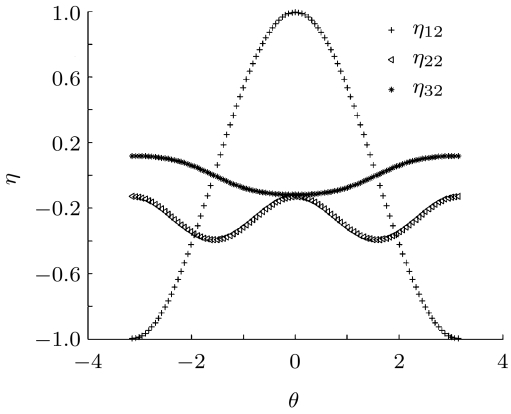


Fig.3. Displacements of the lower interfacial wave.

5. Conclusions

Interfacial internal waves in a three-layer density-stratified fluid have been investigated using a singular perturbation method. The third-order asymptotic solutions of the velocity potentials and the third-order Stokes wave solutions of the associated elevations of the interfacial waves are presented based on the small amplitude wave theory. The results obtained show that the third-order solutions can describe the third-order nonlinear modification and the third-order nonlinear interactions between interfacial waves, and the nonlinear influence appears not only in the $n(n \geq 2)$ times high-frequency components of waves, but also in the frequency of waves or the stationary low-frequency components. When approximating to third-order, the wave velocity depends not only on the wave number and the depth of each layer but also on the wave amplitude.

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