Vol 18 No 5, May 2009
1674-1056/2009/18(05)/1821-07

Chinese Physics B

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Infinite series symmetry reduction solutions to the modified KdV–Burgers equation*

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(Received 3 July 2008; revised manuscript received 5 August 2008)

From the point of view of approximate symmetry, the modified Korteweg–de Vries–Burgers (mKdV–Burgers) equation with weak dissipation is investigated. The symmetry of a system of the corresponding partial differential equations which approximate the perturbed mKdV–Burgers equation is constructed and the corresponding general approximate symmetry reduction is derived; thereby infinite series solutions and general formulae can be obtained. The obtained result shows that the zero-order similarity solution to the mKdV–Burgers equation satisfies the Painlevé II equation. Also, at the level of travelling wave reduction, the general solution formulae are given for any travelling wave solution of an unperturbed mKdV equation. As an illustrative example, when the zero-order tanh profile solution is chosen as an initial approximate solution, physically approximate similarity solutions are obtained recursively under the appropriate choice of parameters occurring during computation.

Keywords: modified Korteweg–de Vries–Burgers (mKdV–Burgers) equation, approximate symmetry reduction, series reduction solution

PACC: 1130, 0420J, 0220, 0340K

1. Introduction

In the last few decades, much active research effort has been focused on nonlinear dynamical systems that have emerged in various fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid-state physics and optical fibres. The nonlinear phenomena occurring in such fields are often related to some nonlinear wave equations, which are, in general, chosen as a starting point of exactly solving the relevant unperturbed partial differential equation systems. However, many nonlinear partial differential equations (PDEs) are, of course, approximate in their derivation, i.e. higher-order terms and dissipation have been neglected. In order to better understand such phenomena as well as to further use them in practical scientific research, it is important to study the properties of the perturbed PDEs. The approximate symmetry perturbation approach[1] is a powerful method and has been applied to many classical PDEs.[2−7] Although in theory it is not difficult, concrete study shows that it is indeed not trivial work to construct different order perturbation solutions and even give general formulae. One must appropriately choose not only the types of the solution functions but also the parameters occurring during computation, such as some integral constants, to render both the solutions physically meaningful and the asymptotic forms correct. Next, we take the well-known modified Korteweg–de Vries–Burgers (mKdV–Burgers) equation as an example, which reads

$$u_t + u_{xxx} + 3\alpha u^2 u_x + \varepsilon u_{xx} = 0, \tag{1}$$

where α and ε are constant coefficients, and they incorporate the effects of nonlinearity ($\alpha u^2 u_x$) and dissipation (εu_{xx}) into the equation; ε is the coefficient of the kinematic viscosity of a fluid ($\varepsilon < 0$). When the dispersion term u_{xxx} is neglected, $u_{xxx} = 0$. Equation (1) was formulated from the modified Burgers equation.[8] When $\varepsilon = 0$, Eq.(1) is just the so-called mKdV equation, which originates from nonlinear optics[9] and the propagation of long internal waves

*Project supported by the National Natural Science Foundations of China (Grant Nos 10735030, 10475055, and 90503006), the National Basic Research Program of China (Grant No 2007CB814800), the Science Foundation for Post Doctorate Research from the Ministry of Science and Technology of China (Grant No 20070410727), and the Natural Science Basic Research Plan in Shaanxi Province, China (Grant No SJ08A09).

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in a fluid when the coefficient of the ordinary nonlinear term in the KdV equation, uu_x, is zero and the higher order nonlinear term u^2u_x dominates over higher order dispersive terms.[10] Moreover, in Ref.[11], Eq.(1) has been studied as a special case of the compound mKdV–Burgers equation.

Equation (1) has recently become a popular model for describing internal solitary waves in shallow seas. In the present paper, we take account of the special case of the mKdV–Burgers equation with weak dissipation

$$u_t + u_{xxx} + 3\alpha u^2 u_x + \varepsilon u_{xx} = 0, \ \ |\varepsilon| \ll 1 \qquad (2)$$

which belongs to the perturbed mKdV equation. Dodd[2] inspected the solitary wave solutions to Eq.(2) by an inverse scattering transformation related perturbation method.

Equation (2) can be treated through the approximate symmetry perturbation approach which is an integration of the perturbation method and the symmetry reduction method first proposed by Fushchich[1] and called approximate symmetry. The crucial point is that a physically small parameter appears when the equations are cast into a dimensionless form and a perturbation solution is possible in terms of this small parameter. The remainder of the present paper is organized as follows. Section 2 is devoted to the application of the approximate symmetry perturbation approach to Eq.(2). In Subsection 2.1 approximate reduction solutions for the Painlevé II reduction are presented. The general travelling wave reduction for the solitary wave solutions of Eq.(1) via the approximate symmetry perturbation approach and relevant discussion are given in detail in Subsection 2.2. Finally, the conclusions drawn from the present study and a discussion of the results are presented in Section 3.

2. Approximate symmetry reduction approach to Eq.(2)

According to perturbation theory, in the actual case, solutions of perturbed partial differential equations can be expressed in a form with a finite portion of the series up to some order of a small parameter. Obviously, one can consider higher orders of approximation of u in ε, i.e. $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$, and can study the symmetry of the corresponding approximate system of the PDE for functions $u_j, \ j = 0, 1, 2, \ldots$. Now, we start directly from the following infinity case. For Eq.(2), the solution can be assumed to be in the general form

$$u = \sum_{j=0}^{\infty} \varepsilon^j u_j, \qquad (3)$$

where u_j is a function of x and t.

Substituting expression (3) into Eq.(2) and vanishing the coefficients of the various powers of ε yields a system of partial differential equations denoted as $\mathbb{F}$

$$\begin{aligned}
&\mathrm{O}(\varepsilon^0): u_{0t} + u_{0xxx} + 3\alpha u_0^2 u_{0x} = 0,\\
&\mathrm{O}(\varepsilon^1): u_{1t} + u_{1xxx} + u_{0xx} + 3\alpha(u_0^2 u_1)_x = 0,\\
&\mathrm{O}(\varepsilon^2): u_{2t} + u_{2xxx} + u_{1xx} + 3\alpha(u_0^2 u_2 + u_1^2 u_0)_x = 0,\\
&\mathrm{O}(\varepsilon^3): u_{3t} + u_{3xxx} + u_{2xx}\\
&\qquad + 3\alpha\left(u_0^2 u_3 + u_0 u_1 u_2 + \frac{1}{3}u_1^3\right)_x = 0,\\
&\qquad \ldots,\\
&\mathrm{O}(\varepsilon^j): u_{jt} + u_{jxxx} + u_{j-1,xx}\\
&\qquad + \alpha \sum_{k=0}^{j}\sum_{l=0}^{k} (u_l u_{k-l} u_{j-k})_x = 0,\\
&\qquad \ldots .
\end{aligned} \qquad (4)$$

To find exact solutions of Eqs.(4), we first construct their Lie point symmetry and then make the corresponding symmetry reduction. Supposing that the Lie point symmetry of Eqs.(4) is in the form

$$\sigma_j = X\frac{\partial}{\partial x}u_j + T\frac{\partial}{\partial t}u_j - U_j \ \ (j = 0,\ 1,\ 2, \ldots), \qquad (5)$$

where X, T and U_j $(j = 0, 1, \ldots)$ are functions of x, t and u_j $(j = 0, 1, \ldots)$, which means that the system of Eq.(1) is invariant under the following transformation:

$$x \to x, \ t \to t, \ \ \text{and} \ \ u_j \to u_j + \varepsilon' \sigma_j + \mathrm{O}(\varepsilon'^2) \qquad (6)$$

with a small parameter ε'. Obviously, σ_j $(j = 0, 1, \ldots)$ satisfies

$$\frac{\partial}{\partial \varepsilon'}\mathbb{F}(u_j + \varepsilon' \sigma_j)\,|_{\varepsilon'=0} = 0. \qquad (7)$$

Then from condition (7) and Eq.(4) we obtain the lin-

earized equations for Eq.(4) listed below:

$$\begin{aligned}
&\sigma_{0t}+\sigma_{0xxx}+3\alpha(u_0^2\sigma_0)_x=0,\\
&\sigma_{1t}+\sigma_{1xxx}+\sigma_{0xx}+3\alpha(u_0^2\sigma_1+2u_0u_1\sigma_0)_x=0,\\
&\sigma_{2t}+\sigma_{2xxx}+\sigma_{1xx}+3\alpha(u_0^2\sigma_2+u_1^2\sigma_0\\
&+2u_0u_1\sigma_1+2u_0u_2\sigma_0)_x=0,\\
&\dots,\\
&\sigma_{jt}+\sigma_{jxxx}+\sigma_{j-1,xx}+\alpha\sum_{k=0}^{j}\sum_{l=0}^{k}(u_lu_{k-l}\sigma_{j-k}\\
&+u_l\sigma_{k-l}u_{j-k}+\sigma_lu_{k-l}u_{j-k})_x=0,\\
&\dots.
\end{aligned}\tag{8}$$

Substituting Eq.(5) into Eqs.(8) and eliminating u_{jt} $(j=0,1,\dots)$ in terms of Eq.(4) leads to the determining equations by vanishing all coefficients of the different partial derivatives of u_j for the unknown functions X, T and U_j $(j=0,1,\dots)$, which are overdetermined and have the solution

$$\begin{aligned}
&X=\frac{1}{3}cx+b,\quad T=ct+t_0,\\
&U_0=-\frac{1}{3}cu_0,\quad U_1=0,\quad U_2=\frac{1}{3}cu_2,\\
&U_3=\frac{2}{3}cu_3,\quad U_4=cu_4,\dots,\\
&U_j=\frac{1}{3}(j-1)cu_j,\dots,
\end{aligned}\tag{9}$$

where b, c and t_0 are arbitrary constants. Subsequently, solving the characteristic equations

$$\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}t}{T}=\frac{\mathrm{d}u_0}{U_0}=\frac{\mathrm{d}u_1}{U_1}=\dots=\frac{\mathrm{d}u_j}{U_j}=\dots\tag{10}$$

leads to the similarity solutions of Eq.(4), which we will discuss in detail in the following two subcases.

2.1. Symmetry perturbation of the Painlevé II solution

When $c\neq 0$, from Eq.(10), we can choose the similarity variable as

$$\xi=(cx+3b)/c(ct+t_0)^{\frac{1}{3}}.\tag{11}$$

Then the similarity solutions for the fields u_j are

$$\begin{aligned}
&u_0=(ct+t_0)^{-\frac{1}{3}}V_0(\xi),\quad u_1=V_1(\xi),\\
&u_2=(ct+t_0)^{\frac{1}{3}}V_2(\xi),\quad u_3=(ct+t_0)^{\frac{2}{3}}V_3(\xi),\dots,\\
&u_j=(ct+t_0)^{\frac{1}{3}(j-1)}V_j(\xi),\dots.
\end{aligned}\tag{12}$$

Accordingly, the perturbation series solution of Eq.(2) is in the form

$$u=\sum_{j=0}^{\infty}\varepsilon^j(ct+t_0)^{\frac{1}{3}(j-1)}V_j(\xi)\tag{13}$$

and the similarity reduction equations related to the similarity solutions Eqs.(12) are

$$\mathrm{O}(\varepsilon^0):V_{0\xi\xi\xi}=-3\alpha V_0^2V_{0\xi}+\frac{1}{3}\xi cV_{0\xi}+\frac{1}{3}cV_0,\tag{14}$$

$$\mathrm{O}(\varepsilon^1):V_{1\xi\xi\xi}=-3\alpha V_0(2V_1V_{0\xi}+V_0V_{1\xi})+\frac{1}{3}\xi cV_{1\xi}-V_{0\xi\xi},\tag{15}$$

$$\mathrm{O}(\varepsilon^2):V_{2\xi\xi\xi}=-3\alpha(V_0V_1^2+V_0^2V_2)_\xi+\frac{1}{3}\xi cV_{2\xi}-\frac{1}{3}cV_2-V_{1\xi\xi},\tag{16}$$

$$\mathrm{O}(\varepsilon^3):V_{3\xi\xi\xi}=-3\alpha\left(V_3V_0^2+2V_0V_1V_3+\frac{1}{3}V_1^3\right)_\xi+\frac{1}{3}\xi cV_{3\xi}-\frac{2}{3}cV_3-\frac{1}{3}V_{2\xi\xi},\tag{17}$$

$$\dots,$$

$$\mathrm{O}(\varepsilon^j):V_{j\xi\xi\xi}=-3\alpha\sum_{k=0}^{j}\sum_{l=0}^{k}(V_lV_{k-l}V_{j-k})_\xi+\frac{1}{3}\xi cV_{j\xi}-\frac{1}{3}jcV_j+\frac{1}{3}cV_j-V_{j-1,\xi\xi},\tag{18}$$

$$\dots,\tag{19}$$

with $V_{-1}=0$. Equation (14) will be equivalent to the Painlevé II type equation if it is integrated once, and obviously, for a sufficiently long time t, the general term may become infinite. Hence the infinite series of Eqs.(14–19) possesses inferior convergence.

2.2. Symmetry perturbation of travelling wave solutions

When $c=0$, the similarity solutions are

$$u_0=V_0(\xi),\ u_1=V_1(\xi),\ \dots,u_j=V_j(\xi),\ \dots\tag{20}$$

with the similarity variable $\xi = x + \bar{c}t$ and $\bar{c}$ being a real constant.

From expression (3), the perturbation series solution to Eq.(2) is

$$u = \sum_{j=0}^{\infty} \varepsilon^j V_j(\xi). \tag{21}$$

Then the relevant similarity reduction equations are

$$\begin{aligned}
&\mathrm{O}(\varepsilon^0): V_{0\xi\xi\xi} = \bar{c}V_{0\xi} - 3\alpha V_0^2 V_{0\xi}, && (22)\\
&\mathrm{O}(\varepsilon^1): V_{1\xi\xi\xi} = -V_{0\xi\xi} + \bar{c}V_{1\xi} - 3\alpha(V_0^2 V_1)_\xi, && (23)\\
&\mathrm{O}(\varepsilon^2): V_{2\xi\xi\xi} = -V_{1\xi\xi} + \bar{c}V_{2\xi} - 3\alpha(V_0 V_1^2 + V_0^2 V_2)_\xi,\\
&\qquad \dots,\\
&\mathrm{O}(\varepsilon^j): V_{j\xi\xi\xi} = -V_{j-1,\xi\xi} + \bar{c}V_{j\xi}\\
&\qquad\qquad - \alpha \sum_{k=0}^{j}\sum_{l=0}^{k} (V_l V_{k-l} V_{j-k})_\xi,\\
&\qquad \dots && (24)
\end{aligned}$$

with $V_{-1} = 0$.

Equation (22) that actually comes from the unperturbed equation, namely the mKdV equation, possesses a general solution which can be expressed, by means of the following elliptic integral, as

$$\int^{V_0} \frac{\mathrm{d}Z}{\sqrt{\bar{c}Z^2 - \frac{1}{2}\alpha Z^4 - 2A_0 Z - B_0}} = \pm(\xi - \xi_0). \tag{25}$$

Equation (25) can be explicitly written, by means of Jacobi elliptic functions, say, as

$$V_0 = \frac{a_0 + a_1 \mathrm{sn}(k\xi - k\xi_0, m)}{b_0 + b_1 \mathrm{sn}(k\xi - k\xi_0, m)} \tag{26}$$

with

$$a_0 = \frac{a_1 b_1 (2b_1^2 - b_0^2 - m^2 b_0^2)}{b_0(b_1^2 - m^2 b_1^2 - 2m^2 b_0^2)},$$

$$a_1^2 = \frac{k^2 b_0^2 (b_1^2 - m^2 b_1^2 - 2m^2 b_0^2)^2}{2\alpha(b_1^2 - b_0^2)(m^2 b_0^2 - b_1^2)},$$

$$\bar{c} = \frac{k^2[2(1+m^2)(b_1^4 + m^2 b_0^4) + b_0^2 b_1^2(1 + m^4 - 10m^2)]}{2(b_1^2 - b_0^2)(m^2 b_0^2 - b_1^2)},$$

where k, m, b_0, b_1 and ξ_0 are arbitrary constants and the constants A_0 and B_0 should be approximately related to other parameters omitted here because they are just the re-denoting of the arbitrary constants. When $m = 1$, solution (26) becomes the grey soliton of the mKdV equation

$$V_0 = \pm k\sqrt{\frac{2}{\alpha}} \frac{b_1 + b_0 \tanh(k\xi - k\xi_0)}{b_0 + b_1 \tanh(k\xi - k\xi_0)}. \tag{27}$$

with $c_1 = -2k^2$.

Furthermore, when $b_1 = 0$ and $\alpha < 0$, the grey soliton (27) becomes a tanh profile dark soliton solution

$$V_0 = \pm\sqrt{-\frac{2}{\alpha}} k \tanh[k(x + 2k^2 t)]. \tag{28}$$

For the case of $\alpha > 0$, it is convenient to write the elliptic integral solution (26) in the form

$$V_0 = \frac{a_0 + a_1 \mathrm{cn}(k\xi - k\xi_0, m)}{b_0 + b_1 \mathrm{cn}(k\xi - k\xi_0, m)} \tag{29}$$

with

$$a_0 = \frac{a_1 b_1 [2(b_1^2 - b_0^2)m^2 - 2b_1^2 + b_0^2)]}{b_0[2m^2(b_1^2 - b_0^2) - b_1^2]},$$

$$a_1^2 = \frac{k^2 b_0^2 [2m^2(b_1^2 - b_0^2) - b_1^2]^2}{2\alpha(b_1^2 - b_0^2)[m^2(b_1^2 - b_0^2) - b_1^2]},$$

$$\bar{c} = \frac{k^2\{2(2m^2 - 1)[m^2(b_1^4 + b_0^4) - b_1^4] - b_0^2 b_1^2(8m^4 - 8m^2 - 1)\}}{2(b_1^2 - b_0^2)[m^2(b_1^2 - b_0^2) - b_1^2]}.$$

When $m = 1$, the periodic solution becomes that of a bright or grey soliton for both cases $\{\alpha > 0,\ b_0^2 > b_1^2\}$ and $\{\alpha < 0,\ b_0^2 < b_1^2\}$. In particular, when $b_1 = 0$ and $\alpha > 0$, we can have only a sech profile bright soliton solution

$$V_0 = \pm k\sqrt{\frac{2}{\alpha}} k \mathrm{sech}[k(x - k^2 t)]. \tag{30}$$

It is interesting and encouraging that the general symmetry reduction equations (22)–(24) can be solved recursively by means of any zero order solution of Eq.(42). The final result reads

$$V_j = V_{0\xi}\left[\int V_{0\xi}^{-2}\left(\int V_{0\xi'}F_j(\xi')\,\mathrm{d}\xi' + A_j\right)\,\mathrm{d}\xi + B_j\right], \tag{31}$$

where

$$\begin{aligned} F_j(\xi) =& -V_{j-1,\xi\xi} - \alpha\sum_{k=1}^{j-1}\sum_{l=0}^{k} V_l V_{k-l} V_{j-k} \\ & - 2\alpha V_0\sum_{k=1}^{j-1} V_k V_{j-k} - C_j \end{aligned} \tag{32}$$

with A_j, B_j and C_j being arbitrary constants.

Next we discuss the similarity solutions in the following two special cases for V_0 given by expressions (28) and (30) respectively.

2.3. Case 1 Symmetry perturbation reduction for dark soliton solution (28)

In this case, we construct the similarity solutions of various orders using solution (28) as the initial approximation of the perturbed equation (1). Substituting expression (28) into Eq.(23) and then eliminating the nonphysical secular terms by appropriately selecting the integral constants gives a constant solution of Eq.(23)

$$V_0 = \pm\frac{1}{6}\sqrt{-\frac{2}{\alpha}}. \tag{33}$$

This shows that the addition of a small amount of the first order approximate solution to u_0 can merely be thought of as a small background shift of the solitary wave as shown in Fig.1.

After detailed calculations, one can find that V_j should have the form $V_j(\xi) = V_j\ (\tanh(k\xi))$. Then in the same way, we obtain the corresponding solutions as follows:

$$\begin{aligned} V_2 =& \frac{1}{4608}\frac{\operatorname{sech}(k\xi)^2(-96A(\ln(1+\tanh(k\xi)) - \ln(1-\tanh(k\xi)))k^2 + 162A\tanh(k\xi)k^2 - C)}{k^4} \\ & + \frac{1}{768}\frac{A\tanh(k\xi)(27\tanh(k\xi)^2 - 59)}{k^2}, \end{aligned} \tag{34}$$

$$V_3 = B\operatorname{sech}(k\xi)^2, \tag{35}$$

$$\begin{aligned} V_4 =& \operatorname{sech}(k\xi)^2\left(-\frac{1}{2304}\frac{A\tanh(k\xi)(\ln(1+\tanh(k\xi)))^2}{k^4} - \frac{1}{2304}\frac{A\tanh(k\xi)(\ln(1-\tanh(k\xi)))^2}{k^4}\right. \\ & + \ln(\tanh(k\xi)+1)\left(\frac{1}{1152}\frac{A\tanh(k\xi)\ln(1-\tanh(k\xi))}{k^4} - \frac{1}{110592}\frac{C\tanh(k\xi) - 48Ak^2}{k^6}\right) \\ & \left. + \frac{1}{110592}\ln(1-\tanh(k\xi))\frac{C(\tanh(k\xi) - 48Ak^2)}{k^6} + \frac{1}{42467328}\frac{\alpha AC^2\tanh(k\xi)}{k^{10}} - D\right) \\ & - \frac{1}{1152}A\frac{\tanh(k\xi)}{k^4}, \end{aligned} \tag{36}$$

where $\xi = x + 2k^2t$, $A = \sqrt{-\frac{2}{\alpha}}$ ($\alpha<0$), B, C and D are arbitrary constants. It is worth mentioning again that the parameters occurring during computation were all selected technically to vanish the divergence terms and to render the solutions physically meaningful.

To identify the evolution of the perturbation series solution for Eq.(1), we depict its picture. For simplicity, we give the perturbation series solution up to order 3, that is,

$$u = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \varepsilon^3 V_3 + \mathrm{O}(\varepsilon^4), \tag{37}$$

where V_i ($i = 0, 1, 2, 3$) are in the forms of expressions (28), (33), (34) and (35) respectively. Then with the choices of $A = 1$, $k = 1$, $C = 0$, $B = 1$, $\varepsilon = 0.8$, we depict in Fig.1 the following three curves: the zero-order (the upper line for $\eta > 0$), first-order (the middle line

for $\eta > 0$) and second-order (the lower line for $\eta > 0$) perturbation series solutions. By selecting the model parameter ε to be a smaller value, say, $\varepsilon = 0.1$, the two lines for the first and second order approximations have almost completely identical profiles.

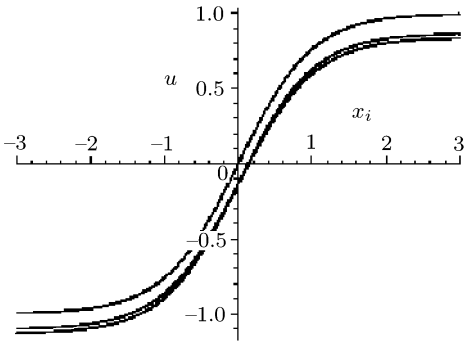


Fig.1. Curves for mKdV–Burgers equation perturbation series solutions.

2.4. Case 2 Symmetry perturbation reduction for bright soliton solution (30)

Substituting expression (30) into Eq.(23) gives

$$\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}V_1(\xi) + 6\left(\frac{k^2}{\cosh(k\xi)^2} - k^2\right)\frac{\mathrm{d}}{\mathrm{d}\xi}V_1(\xi) - 12\frac{k^3\sinh(k\xi)}{\cosh(k\xi)^3}V_1(\xi) + \frac{a_2\,k^2}{\cosh(k\xi)} - 2\frac{a_2\,k^2}{\cosh(k\xi)^3} = 0, \tag{38}$$

which is difficult to solve directly. At this point, it is helpful to introduce a transformation defined as

$$V_1(\xi) = \mathrm{sech}(k\xi)U_1(\tanh(k\xi)). \tag{39}$$

Then substituting expression (39) along with

$$\tanh(k\xi) = \varphi, \quad \sinh(k\xi) = \varphi\cosh(k\xi), \quad 1/\cosh(k\xi)^2 = 1 - \varphi^2$$

into Eq.(38) gives

$$\begin{aligned}&-12k^3\varphi U_1(\varphi) + 12k^3\varphi^3U_1(\varphi) + 12k^3\varphi^2\frac{\mathrm{d}}{\mathrm{d}\varphi}U_1(\varphi) - 12k^3\varphi^4\frac{\mathrm{d}}{\mathrm{d}\varphi}U_1(\varphi) - 9k^3\varphi\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}U_1(\varphi)\\ &+ 18k^3\varphi^3\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}U_1(\varphi) - 9k^3\varphi^5\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}U_1(\varphi) + k^3\frac{\mathrm{d}^3}{\mathrm{d}\varphi^3}U_1(\varphi)\frac{\mathrm{d}^3}{\mathrm{d}\varphi^3}U_1(\varphi) - 3k^3\varphi^2\frac{\mathrm{d}^3}{\mathrm{d}\varphi^3}U_1(\varphi) + 3k^3\varphi^4\frac{\mathrm{d}^3}{\mathrm{d}\varphi^3}U_1(\varphi)\\ &- k^3\varphi^6\frac{\mathrm{d}^3}{\mathrm{d}\varphi^3}U_1(\varphi) - Ak^2 + 2Ak^2\varphi^2 = 0,\end{aligned} \tag{40}$$

which gives

$$\begin{aligned}U_1(\varphi) = {}&\frac{3}{4}\frac{\varphi\,C_1\ln(\varphi+1)}{k^3} - \frac{3}{4}\frac{\varphi C_1\ln(1-\varphi)}{k^3} - \frac{1}{2}\frac{C_1(3\,\varphi^2-2)}{(\varphi+1)(\varphi-1)k^3}\\ &+ \frac{C_2(\varphi^2-1)}{\sqrt{(1-\varphi)(1+\varphi)}} - \frac{1}{6}\frac{A\varphi(2\,\varphi^2-1)}{k(\varphi+1)(\varphi-1)} + C_3\,\varphi,\end{aligned} \tag{41}$$

where $A = \sqrt{\dfrac{2}{\alpha}}k$ $(\alpha > 0)$, and C_i $(i = 1,\ 2,\ 3)$ are integral constants.

Thus, from expression (39) we obtain the first-order approximate solution

$$\begin{aligned}V_1 = {}&\frac{3}{4}\frac{C_1\sinh(k\xi)\ln(1+\tanh(k\xi))}{\cosh(k\xi)^2k^3} - \frac{3}{4}\frac{C_1\sinh(k\xi)\ \ln(1-\tanh(k\xi))}{\cosh(k\xi)^2k^3} + \frac{C_2\left(\cosh(k\xi)^2-2\right)}{\cosh(k\xi)^2}\\ &+ \frac{C_3\sinh(k\xi)}{\cosh(k\xi)^2} + \frac{1}{2}\frac{C_1\left(\cosh(k\xi)^2-3\right)}{\cosh(k\xi)k^3} + \frac{1}{6}\frac{A\sinh(k\xi)\left(\cosh(k\xi)^2-3\right)2}{\cosh(k\xi)k^3}.\end{aligned} \tag{42}$$

In solution (42), we notice that, for arbitrary ξ, all terms except the following two:

$$\frac{1}{2}\frac{C_1\cosh(k\xi)}{k^3} \quad \text{and} \quad \frac{1}{6}\frac{A\sinh(k\xi)}{k}, \tag{43}$$

are convergent. Naturally, we attempt to remove the terms given in expression (43) by appropriately choosing the parameters. However, the rewriting of terms (43) as the exponential function

$$\frac{1}{12}\frac{(3C_1+Ak^2)\mathrm{e}^{k\xi}}{k^3}-\frac{1}{12}\frac{(-3C_1+Ak^2)\mathrm{e}^{-k\xi}}{k^3}$$

shows that this is impossible. In other words, solution (42) will blow up soon and becomes physically meaningless, which means that for the bright solitary waves of the mKdV system, it is impossible to maintain a travelling wave form no matter how small the dissipation is. Hence, it is not necessary to investigate this case further.

3. Conclusions and discussion

The perturbed mKdV–Burgers equation is studied by applying the approximate symmetry perturbation approach. The symmetries of a system of the corresponding PDEs which approximate the perturbed mKdV–Burgers equation are constructed, and the general approximate symmetry reductions and the general form of the infinite approximate series solutions are obtained separately. The obtained result shows that the similarity equation of zero-order is equivalent to the Painlevé II equation. Also, when the similarity solutions of zero-order are chosen as the initial approximate solutions, physically approximate similarity solutions can be constructed one by one under the appropriate choice of parameters occurring during computation. More importantly, choosing any solution of the related unperturbed equation, we could give the general similarity solution formula for the infinite series symmetry reduction equations at the level of travelling wave reduction.

Applying the method to the special dark and bright solitary wave solutions, we find that the dark solitary wave may maintain its travelling wave form even for a quite large dissipation (say, $\varepsilon = 0.8$ as shown in Fig.1), while the bright solitary wave cannot preserve its travelling wave form no matter how tiny the dissipation is.

Moreover, as mentioned in Ref.[12], the approximate symmetry reduction approach can also be used to search for more results by expanding in the ε-series not only dependent variables, but also independent ones, e.g. $t = t_0 + \varepsilon t_1 + \varepsilon^2\ t_2 + \ldots$, and to construct in this way a corresponding approximate system and even obtain the similarity solutions.

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