Vol 18 No 5, May 2009
1674-1056/2009/18(05)/1814-07

Chinese Physics B

© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

Knot solitons in the AFZ model*

Ren Ji-Rong(任继荣)†, Mo Shu-Fan(墨淑凡)‡, and Zhu Tao(朱 涛)§

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, *China*

(Received 2 September 2008; revised manuscript received 10 October 2008)

This paper studies the topological properties of knotted solitons in the $(3+1)$-dimensional Aratyn–Ferreira–Zimerman (AFZ) model. Topologically, these solitons are characterized by the Hopf invariant I, which is an integral class in the homotopy group $\pi_3(S^3) = Z$. By making use of the decomposition of $U(1)$ gauge potential theory and Duan's topological current theory, it is shown that the invariant is just the total sum of all the self-linking and linking numbers of the knot family while only linking numbers are considered in other papers. Furthermore, it is pointed out that this invariant is preserved in the branch processes (splitting, merging and intersection) of these knot vortex lines.

Keywords: knot theory, solitons, topology, AFZ model
PACC: 1110, 0240, 0420M, 0420J

1. Introduction

Knotted solitons, i.e. topological solutions with a non-trivial value of the Hopf invariant, play a prominent role in modern physics, chemistry and biology scenarios: including the structure of elementary particles,[1] early universe cosmology,[2] the Bose–Einstein condensate,[3] polymer folding,[4] and DNA replication, transcription and recombination.[5] In 1975, Faddeev firstly proposed the possibility of knot-like solitons in nonlinear field theory.[6] Since then, a wide class of topological knot solitons have been proposed in various physical models of gauge field theory and condensed matter physics: in the effective theory of quantum chromodynamics (QCD),[7] in two-gap superconductivity models,[8] in two-component Bose–Einstein condensate theory,[9] in high energy physics, namely, in the Weinberg–Salam model,[10] in classical field theory[11] and in other systems.

Recently, a great deal of physical interest has been focused on the field theoretical model to which it is possible to write down in a closed form explicit knot solitons with nonzero Hopf index. This model was first proposed by Aratyn–Ferreira–Zimerman (AFZ)[12] in (3+1)-dimensional integrable theory and called the AFZ model. To fully establish a connection between integrability and soliton solutions, it would require finding topological solitons with arbitrary topological charges.[13] In the AFZ model, the connection has been investigated and the knot soliton has been studied in terms of its topological invariant, namely, the Hopf invariant $Q_{\rm H}$. Moreover, $Q_{\rm H} = -nm$ bas been obtained, where n and m show how many times one knot winds the other, respectively. This just responds to the linking numbers, but the self-linking numbers have not been considered in those papers.[5,13] This will be completed in the following.

In this paper, we mainly focus on the knot solitons in the AFZ model. Since the topological properties of knot solitons have played an important role in this model, we shall be particularly interested in those aspects of the AFZ model. Using $U(1)$ gauge potential theory and Duan's topological current theory, we study the topological properties of knot solitons in the AFZ model in detail and naturally obtain the inner structure of the nontrivial Hopf invariant. Furthermore, by comparing with others, this paper considers not only linking numbers but also self-linking numbers in detail.

The structure of this paper is as follows. In Section 2, using the decomposition of $U(1)$ gauge potential theory[14] and Duan's topological current theory,[15] we reveal that there are knot solitons in the AFZ model. In Section 3, we mainly study the knot

* Project supported by the National Natural Science Foundation of China (Grant No 10275030) and the Cuiying Program of Lanzhou University, China (Grant No 22500-582404).
†E-mail: renjr@lzu.edu.cn
‡E-mail: meshf07@lzu.cn
§Corresponding author. E-mail: zhut05@lzu.cn

configurations in these string structures. It shows that the Hopf invariant of knot solitons is just the total sum of all the self-linking and linking numbers of the knot family. In Section 4, we can see that the knot topological invariant is preserved in the branch processes (splitting, merging and intersection) of the knot solitons.

2. Topological solitons in the AFZ model

The AFZ model under consideration is described by the Lagrangian density

$$L = -\eta_0 \left(H_{\mu\nu}^2\right)^{\frac{3}{4}}, \tag{1}$$

where $\eta_0 = \pm 1$ determines the choice of the signature of the Minkowski metric $g_{\mu\nu} = \eta_0 \mathrm{diag}(1, -1, -1, -1)$ and the field tensor $H_{\mu\nu}$ is defined in terms of the three components, unit vector field $\boldsymbol{n} \in S^2$ is the pull-back of the area form on S^2

$$H_{\mu\nu} = \boldsymbol{n} \cdot (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}). \tag{2}$$

It is easy to prove that $H_{\mu\nu}$ can be reexpressed in an Abelian field tensor form $\boldsymbol{n} \cdot (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) = \partial_\mu b_\nu - \partial_\nu b_\mu$. Here b_μ is the Wu–Yang potential[16]

$$b_\mu = \boldsymbol{e}_1 \cdot \partial_\mu \boldsymbol{e}_2, \tag{3}$$

in which $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are two perpendicular unit vectors normal to $\boldsymbol{n}$, and ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{n}$) forms an orthogonal frame:

$$\boldsymbol{n} = \boldsymbol{e}_1 \times \boldsymbol{e}_2, \quad \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0. \tag{4}$$

Now, we consider a two-component vector field $\boldsymbol{\phi} = (\phi^1, \phi^2)$ residing in the plane forms by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$:

$$\begin{aligned} &e_1^a = \frac{\phi^a}{\| \phi \|}, \quad e_2^a = \varepsilon_{ab} \frac{\phi^b}{\| \phi \|}, \\ &(\phi^2 = \phi^a \phi^a; \quad a, b = 1, 2). \end{aligned} \tag{5}$$

It can be proven that the expression Eq.(5) for $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ satisfies the restriction Eq.(4). Obviously, the zero points of $\boldsymbol{\phi}$ are just the singular points of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. We will find that the topological defects just originate from these points. Using the $\boldsymbol{\phi}$ field, we can write the Wu–Yang potential Eq.(3) as

$$b_\mu = \varepsilon_{ab} \frac{\phi^a}{\| \phi \|} \partial_\mu \frac{\phi^b}{\| \phi \|}. \tag{6}$$

Comparing with the expression of the $U(1)$ gauge potential decomposition, we learn that b_μ satisfies the $U(1)$ gauge transformation.

Since the tensor $H_{\mu\nu}$ in Eq.(1) plays an important role in the topological feature of the AFZ model, we will make use of Duan's topological current theory to research into the properties hidden in the tensor. To do so, we introduce a topological tensor current $K^{\mu\nu}$,[17]

$$\begin{aligned} K^{\mu\nu} &= \frac{1}{8\pi} \varepsilon^{\mu\nu\lambda\rho} H_{\lambda\rho} \\ &= \frac{1}{4\pi} \varepsilon^{\mu\nu\lambda\rho} \varepsilon_{ab} \partial_\lambda \frac{\phi^a}{\| \phi \|} \partial_\rho \frac{\phi^b}{\| \phi \|}. \end{aligned} \tag{7}$$

Using $\partial_\mu \left(\dfrac{\phi^a}{\| \phi \|} \right) = \dfrac{\partial_\mu \phi^a}{\| \phi \|} + \phi^a \partial_\mu \left(\dfrac{1}{\| \phi \|} \right)$ and the Green's function relation in ϕ space, $\partial_a \partial_a \mathrm{In} \parallel \phi \parallel = 2\pi \delta^2 (\boldsymbol{\phi}) \left(\partial_a = \dfrac{\partial}{\partial \phi^a} \right)$, one can prove that

$$K^{\mu\nu} = \delta^2(\boldsymbol{\phi}) D^{\mu\nu} \left(\frac{\phi}{x} \right), \tag{8}$$

where $D^{\mu\nu} \left(\dfrac{\phi}{x} \right) = \dfrac{1}{2} \varepsilon^{\mu\nu\lambda\rho} \varepsilon_{ab} \partial_\lambda \phi^a \partial_\rho \phi^b$. Defining the spatial component of $K^{\mu\nu}$ as

$$j^i = K^{0i} = \frac{1}{8\pi} \varepsilon^{ijk} H_{jk} \quad (i, j, k = 1, 2, 3), \tag{9}$$

we have

$$j^i = \delta^2(\boldsymbol{\phi}) D^i \left(\frac{\phi}{x} \right), \tag{10}$$

where $D^i \left(\dfrac{\phi}{x} \right) = \dfrac{1}{2} \varepsilon^{ijk} \varepsilon_{ab} \partial_j \phi^a \partial_k \phi^b$ is the Jacobian vector.

The expression Eq.(10) provides an important conclusion: $j^i = 0$, if and only if $\boldsymbol{\phi} \neq 0$; $j^i \neq 0$, if and only if $\boldsymbol{\phi} = 0$. Therefore, it is necessary to study the zero points of $\boldsymbol{\phi}$ to determine the nonzero solutions of j^i. The implicit function theory[18] shows that under the regular condition

$$D^i \left(\frac{\phi}{x} \right) \neq 0. \tag{11}$$

The general solutions of

$$\phi^1(t, x^1, x^2, x^3) = 0, \quad \phi^2(t, x^1, x^2, x^3) = 0 \tag{12}$$

can be expressed as

$$x^1 = x_k^1(s, t), \quad x^2 = x_k^2(s, t), \quad x^3 = x_k^3(s, t), \tag{13}$$

which represent the world surfaces of N moving isolated singular strings L_k with string parameter $s (k = 1, 2, \ldots, N)$. This indicates that in the AFZ model,

there are vortex lines of solitons located at the zero points of the $\boldsymbol{\phi}$ field.

Now, we will investigate the topological charges of the vortex lines and their quantization. In δ-function theory,[19] one can prove that in three-dimensional space

$$\delta^2(\boldsymbol{\phi}) = \sum_{k=1}^{N} \beta_k \int_{L_k} \frac{\delta^3(\boldsymbol{x} - \boldsymbol{x}_k(s))}{\left| D\left(\frac{\phi}{u}\right) \right|_{\Sigma_k}} \mathrm{d}s, \tag{14}$$

where $D\left(\frac{\phi}{u}\right)_{\Sigma_k} = \frac{1}{2}\varepsilon^{\mu\nu}\varepsilon_{mn}\left(\frac{\partial\phi^m}{\partial u^\mu}\right)\left(\frac{\partial\phi^n}{\partial u^\nu}\right)$ and Σ_k is the kth planar element transverse to L_k with local coordinates (u^1, u^2). The positive integer β_k is the Hopf index of ϕ-mapping, which means that when $\boldsymbol{x}$ covers the neighbourhood of the zero point $\boldsymbol{x}_k(s)$ once, the vector field $\boldsymbol{\phi}$ covers the corresponding region in ϕ space β_k times. Meanwhile the direction vector of L_k is given by

$$\frac{\mathrm{d}x^i}{\mathrm{d}s}\Big|_{\boldsymbol{x}_k} = \frac{D^i\left(\frac{\phi}{x}\right)}{D\left(\frac{\phi}{u}\right)}\Big|_{\boldsymbol{x}_k}. \tag{15}$$

Then from Eqs.(14) and (15), we obtain the inner structure of j^i:

$$\begin{aligned} j^i &= \delta^2(\boldsymbol{\phi}) D^i\left(\frac{\phi}{x}\right) \\ &= \sum_{k=1}^{N} W_k \int_{L_k} \frac{\mathrm{d}x^i}{\mathrm{d}s} \delta^3(\boldsymbol{x} - \boldsymbol{x}_k(s))\, \mathrm{d}s, \end{aligned} \tag{16}$$

where $W_k = \beta_k \eta_k$ is the winding number of $\boldsymbol{\phi}$ around L_k, with $\eta_k = \mathrm{sgn} D\left(\frac{\phi}{u}\right)_{\boldsymbol{x}_j} = \pm 1$ being the Brouwer degree of ϕ-mapping. Hence the topological charge of the vortex line L_k is

$$Q_k = \int_{\Sigma_k} j^i \mathrm{d}\sigma_i = W_k. \tag{17}$$

Then we come to the conclusion: under the regular condition $D^i\left(\frac{\phi}{x}\right) \neq 0$ there exist vortex lines of solitons in the AFZ model, whose topological charges are just the winding numbers of ϕ-mapping.

3. Topological invariant of knotted solitons

In this section, we research into stationary knots corresponding to the vortex lines of solitons in the AFZ model. When we are going to a constant vector $\boldsymbol{n}(x) \to \boldsymbol{n}_0$, $\boldsymbol{n}(x)$ naturally defines a mapping $R_3 \sim S^3 \to S^2$. Such mapping falls into an integral class in the homotopy group $\pi_3(S^3) = Z$ and can be characterized by a topological invariant called the Hoft invariant,[20] an integral formula which counts the number of times the two-sphere S^2 is covered by the three-sphere S^3 where the spatial infinity is identified with the north pole. From the above discussion, the Hoft invariant can be given in terms of b_μ and $H_{\mu\nu}$ by

$$I = \frac{1}{16\pi^2} \int \varepsilon^{ijk} b_i H_{jk} \mathrm{d}^3x, \tag{18}$$

where b_i and H_{jk} are the spatial components of b_μ and $H_{\mu\nu}$, respectively. It can be seen that when the N vortex lines of solitons are closed curves, i.e., a family of knots γ_k $(k = 1, \ldots, N)$, considering Eq.(7) and substituting Eq.(16) into Eq.(18), we obtain

$$I = \int_M b_i j^i \mathrm{d}^3x = \sum_{k=1}^{N} W_k \int_{L_k} b_i \mathrm{d}x^i. \tag{19}$$

This is a very important expression. Consider the $U(1)$ gauge transformation of b_i

$$b_i' = b_i + \partial_i \alpha, \tag{20}$$

where $\alpha \in R$ is a phase factor denoting the $U(1)$ transformation. The $(\partial_i \alpha)$ term in Eq.(20) contributes nothing to the integral I; hence Eq.(20) is invariant under the gauge transformation. Meanwhile we know that I is independent of the metric $g_{\mu\nu}$. Therefore, one can conclude that I is a topological invariant for knotted vortex lines of solitons in the AFZ model.

It is well known that many important topological numbers are related to a knot family such as the self-linking number and Gauss linking number. Next we will research into the relationship between the action Eq.(19) and the self-linking and Gauss linking number of the knot family. First we should express b_i in terms of the vector field which carries the geometric information of the knot family. Define the Gauss mapping $\boldsymbol{n}$:

$$\boldsymbol{n} : S^1 \times S^1 \to S^2, \tag{21}$$

in which $\boldsymbol{n}$ is an unit vector

$$\boldsymbol{n}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{y} - \boldsymbol{x}}{\| \boldsymbol{y} - \boldsymbol{x} \|}, \tag{22}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are two points, respectively, on the knots γ_k and γ_l.[21] Let a two-dimensional unit vector $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ satisfy

$$u^a u^a = 1, \quad (a = 1, 2; \boldsymbol{u} \perp \boldsymbol{n}). \tag{23}$$

Then b_i can be decomposed in terms of this two-dimensional vector $u^a : b_i = \varepsilon_{ab}u^a\partial_i u^b - \partial_i\alpha$, in which α is a phase factor. It is due to the term $\partial_i\alpha$ which contributes nothing to the integral Eq.(19) that b_i can be written as

$$b_i = \varepsilon_{ab}u^a\partial_i u^b. \tag{24}$$

Substituting Eq.(24) into Eq.(19), we have

$$I = \sum_{k=1}^{N} W_k \oint_{\gamma_k} \varepsilon_{ab}u^a(\boldsymbol{x},\boldsymbol{y})\partial_i u^b(\boldsymbol{x},\boldsymbol{y})\mathrm{d}x^i. \tag{25}$$

Noticing the symmetry between the point $\boldsymbol{x}$ and $\boldsymbol{y}$ in Eq.(22), we should reexpress Eq.(25) as

$$I = \sum_{k,l=1}^{N} W_k W_l \oint_{\gamma_k}\oint_{\gamma_l} \varepsilon_{ab}\partial_i u^a(\boldsymbol{x},\boldsymbol{y})\partial_j u^b(\boldsymbol{x},\boldsymbol{y})\mathrm{d}x^i\wedge\mathrm{d}y^j. \tag{26}$$

There are three cases: (1) γ_k and γ_l are two different knots i.e., $k\neq l$; (2) γ_k and γ_l are the same knots but $\boldsymbol{x}$ and $\boldsymbol{y}$ are two different points i.e., $k=l$, $\boldsymbol{x}\neq\boldsymbol{y}$; (3) γ_k and γ_l are the same knots and $\boldsymbol{x}$ and $\boldsymbol{y}$ are the same points i.e., $k=l, \boldsymbol{x}=\boldsymbol{y}$. Therefore, Eq.(26) can be written as three terms:

$$\begin{aligned} I = &\sum_{k=1(k=l,\boldsymbol{x}\neq\boldsymbol{y})}^{N} W_k^2 \oint_{\gamma_k}\oint_{\gamma_k} \varepsilon_{ab}\partial_i u^a\partial_j u^b\mathrm{d}x^i\wedge\mathrm{d}x^j + \sum_{k=1}^{N} W_k^2 \oint_{\gamma_k} \varepsilon_{ab}u^a\partial_i u^b\mathrm{d}x^i \\ &+ \sum_{k,l=1(k\neq l)}^{N} W_k W_l \oint_{\gamma_k}\oint_{\gamma_l} \varepsilon_{ab}\partial_i u^a\partial_j u^b\mathrm{d}x^i\wedge\mathrm{d}x^j. \end{aligned} \tag{27}$$

Using the relation $\varepsilon_{ab}\partial_i u^a\partial_j u^b = \dfrac{1}{2}\boldsymbol{n}\cdot(\partial_i\boldsymbol{n}\times\partial_j\boldsymbol{n})$, from Eq.(27) we just obtain

$$I = \sum_{k=1(k=l,\boldsymbol{x}\neq\boldsymbol{y})}^{N} \frac{1}{2}W_k^2 \oint_{\gamma_k}\oint_{\gamma_k} \boldsymbol{n}^*\mathrm{d}s + \sum_{k=1}^{N} W_k^2 \oint_{\gamma_k} \varepsilon_{ab}u^a\partial_i u^b\mathrm{d}x^i + \frac{1}{2}\sum_{k,l=1(k\neq l)}^{N} W_k W_l \oint_{\gamma_k}\oint_{\gamma_l} \boldsymbol{n}^*\mathrm{d}s, \tag{28}$$

where $\boldsymbol{n}^*\mathrm{d}s = \boldsymbol{n}\cdot(\partial_i\boldsymbol{n}\times\partial_j\boldsymbol{n})\,\mathrm{d}x^i\wedge\mathrm{d}x^j \quad (\boldsymbol{x}\neq\boldsymbol{y})$ denotes the pullback of the S^2 surface element.

Let us discuss the three terms in detail. The first term of Eq.(28) is just related to the writhing number $\mathcal{W}r(\gamma_k)$ of γ_k:[22,23]

$$\mathcal{W}r(\gamma_k) = \frac{1}{4\pi}\oint_{\gamma_k}\oint_{\gamma_k} \boldsymbol{n}^*\mathrm{d}s. \tag{29}$$

Because of the $\boldsymbol{x}=\boldsymbol{y}$ term, the second term of Eq.(28) is related to the twisting number $\mathcal{T}w(\gamma_k)$ of γ_k:[22,23]

$$\begin{aligned} \frac{1}{2\pi}\oint_{\gamma_k} \varepsilon_{ab}u^a\partial_i u^b\mathrm{d}x^i &= \frac{1}{2\pi}\oint_{\gamma_k} (\boldsymbol{T}\times\boldsymbol{V})\cdot\mathrm{d}\boldsymbol{V} \\ &= \mathcal{T}w(\gamma_k), \end{aligned} \tag{30}$$

where $\boldsymbol{T}$ is the unit tangent vector of knot γ_k at $\boldsymbol{x}$ ($\boldsymbol{n}=\boldsymbol{T}$ when $\boldsymbol{x}=\boldsymbol{y}$), and $\boldsymbol{V}$ is defined as $u^a = \varepsilon_{ab}V^b \quad (a,b=1,2; \quad \boldsymbol{V}\perp\boldsymbol{T}, \quad \boldsymbol{u}=\boldsymbol{T}\times\boldsymbol{V})$. From the White formula

$$S(\gamma_k) = \mathcal{W}r(\gamma_k) + \mathcal{T}w(\gamma_k), \tag{31}$$

[where $S(\gamma_k)$ is the self-linking number of γ_k], we see that the first and second terms of Eq.(28) just compose the self-linking number of the N knots.

The third term of Eq.(28) can be reexpressed as

$$\begin{aligned} &\frac{1}{4\pi}\oint_{\gamma_k}\oint_{\gamma_l} \boldsymbol{n}^*\mathrm{d}s \\ &= \frac{1}{4\pi}\varepsilon^{ijk}\oint_{\gamma_k}\mathrm{d}x^i\oint_{\gamma_l}\mathrm{d}y^j\frac{(x^k-y^k)}{\parallel\boldsymbol{x}-\boldsymbol{y}\parallel^3} \\ &= \mathcal{L}(\gamma_k,\gamma_l), \quad (k\neq l), \end{aligned} \tag{32}$$

where $\mathcal{L}(\gamma_k,\gamma_l)$ is the Gauss linking number between γ_k and γ_l. Therefore, the third term is just related to the linking number between the N knots.

From Eqs.(29), (30), (31) and (32), we arrive at the important result[24]

$$I = 2\pi\left[\sum_{k=1}^{N} W_k^2 S(\gamma_k) + \sum_{k,l=1(k\neq l)}^{N} W_k W_l \mathcal{L}(\gamma_k,\gamma_l)\right]. \tag{33}$$

This precise expression just reveals the relationship between the action Eq.(19) and the self-linking and linking numbers of the knot family.[25] Since the self-linking and linking numbers are both invariant characteristic numbers of the knot closed curves in topology, Eq.(33) directly relates I to the topology of the knot family, and so I is an important invariant required to describe the knot vortex lines of solitons in the AFZ model. This is just the significance of the introduction and research of the topological invariant I.

Furthermore, comparing with others, we find that this paper considers not only linking numbers but also self-linking numbers in detail. We can see that, when we only consider the linking numbers, we also obtain the same result as other papers. Therefore, this paper has a deeper insight on knot solitons in the AFZ model.

4. The conversion of I in the branch processes of knot solitons

In Section 2, we have used the regular condition $D^i\left(\frac{\phi}{x}\right) \neq 0$; when this condition fails, branch processes will occur. In this section, we discuss the conversion of topological invariant I in the branch processes (splitting, merging and intersection) of knot solitons.

Generally speaking, the evolution of a vortex line L of a soliton can be discussed from Eq.(8). Here we fix the $x^3 = z$ coordinate for simplicity and take the XOY plane as the cross section, so the intersection line between the L's evolution surface and the cross section is just the motion curve of L.[26] In this case the two-dimensional topological current is defined as

$$j^3 = K^{03} = \delta^2(\boldsymbol{\phi}) D^0\left(\frac{\phi}{x}\right) \tag{34}$$

and

$$K^i = K^{i3} = \delta^2(\boldsymbol{\phi}) D^i\left(\frac{\phi}{x}\right), \quad (i = 1, 2), \tag{35}$$

they satisfy the continuity equation

$$\partial_t j^3 + \partial_i K^i = 0. \tag{36}$$

The velocity of the intersection point between L and the cross section is given by

$$v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} = \left.\frac{D^i\left(\frac{\phi}{x}\right)}{D^0\left(\frac{\phi}{x}\right)}\right|_{\boldsymbol{x}}, \quad (i = 1, 2), \tag{37}$$

where $D^0\left(\frac{\phi}{x}\right) = \varepsilon_{ab}\partial_1\phi^a\partial_2\phi^b$, $D^1\left(\frac{\phi}{x}\right) = \varepsilon_{ab}\partial_2\phi^a\partial_0\phi^b$ and $D^2\left(\frac{\phi}{x}\right) = \varepsilon_{ab}\partial_0\phi^a\partial_1\phi^b$. From Eq.(37) it is obvious that when

$$D^0\left(\frac{\phi}{x}\right) = 0, \tag{38}$$

at the very point $(t^*, \boldsymbol{x}^*)$, the velocity

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = \left.\frac{D^1\left(\frac{\phi}{x}\right)}{D^0\left(\frac{\phi}{x}\right)}\right|_{(t^*,\boldsymbol{x}^*)}, \quad \frac{\mathrm{d}x^2}{\mathrm{d}t} = \left.\frac{D^2\left(\frac{\phi}{x}\right)}{D^0\left(\frac{\phi}{x}\right)}\right|_{(t^*,\boldsymbol{x}^*)} \tag{39}$$

is not unique. This very point $(t^*, \boldsymbol{x}^*)$ is called the bifurcation point. Without loss of generality, we discuss only the branch of the velocity component $\frac{\mathrm{d}x^1}{\mathrm{d}t}$ at $(t^*, \boldsymbol{x}^*)$. It is known[27] that the Taylor expansion of the solution of Eq.(12) in the neighbourhood of $(t^*, \boldsymbol{x}^*)$ can generally be expressed as $A(x^1 - x^{1*})^2 + 2B(x^1 - x^{1*})(t - t^*) + C(t - t^*)^2 + \ldots = 0$, where A, B and C are three constants. (In this paper, we do not consider the intersection of three knots, so we assume $A \neq 0$ and the one-split-into-three or the three-merge-into-one branch cases are not considered.) Then the above Taylor expansion leads to

$$A\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + 2B\frac{\mathrm{d}x^1}{\mathrm{d}t} + c = 0, \quad (A \neq 0). \tag{40}$$

The solutions of Eq.(40) give different motion directions of the zero point on the cross section at the bifurcation point. There are two possible cases.

Case 1 For $\Delta = 4(B^2 - AC) = 0$, we obtain only one motion direction of the zero point on the cross section: $\left.\frac{\mathrm{d}x^1}{\mathrm{d}t}\right|_{1,2} = -\frac{B}{A}$, which includes three cases: (1) a vortex line splits into two vortex lines; (2) two vortex lines merge into one vortex line; (3) two vortex lines tangentially intersect at the bifurcation point.

Case 2 For $\Delta = 4(B^2 - AC) > 0$, we obtain two different motion directions of the zero point on the cross section: $\left.\frac{\mathrm{d}x^1}{\mathrm{d}t}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$. This is the intersection of two vortex lines, which means that

two vortex lines meet and then depart at the bifurcation point.

In both cases 1 and 2, from the continuity equation Eq.(36) we know that the sum of the topological charges of final vortex line(s) must be equal to that of the initial vortex line(s) at the bifurcation point for fixed k,

$$\sum_f W_{k_f} = \sum_i W_{k_i}; \tag{41}$$

namely, (1) for the case that one line L splits into two lines L_1 and L_2, we have $W_L = W_{L_1} + W_{L_2}$; (2) for the case that two lines L_1 and L_2 merge into one line L, $W_{L_1} + W_{L_2} = W_L$; (3) for the case that two lines L_1 and L_2 meet and depart as two other lines L_3 and L_4, $W_{L_1} + W_{L_2} = W_{L_3} + W_{L_4}$.

In the following we will show that in the branch processes of knot solitons, the topological invariant I is preserved.

(a) The splitting case. We consider that one knot γ splits into two knots γ_1 and γ_2 which are of the same self-linking numbers as $\gamma\,(S(\gamma) = S(\gamma_1) = S(\gamma_2))$, and then we will compare the two numbers I_γ and $I_{\gamma_1+\gamma_2}$ (where I_γ is the contribution of γ to I before splitting, and $I_{\gamma_1+\gamma_2}$ is the total contribution of γ_1 and γ_2 to I after splitting). First, from the above text we have $W_\gamma = W_{\gamma_1} + W_{\gamma_2}$ in the splitting process. Second, on the one hand, we note that in the neighbourhood of the bifurcation point $(\boldsymbol{x}^*, t^*)$, γ_1 and γ_2 are infinitesimally displaced from each other; on the other hand, for a knot γ its self-linking number $S(\gamma)$ is defined as

$$S(\gamma) = \mathcal{L}(\gamma, \gamma_V), \tag{42}$$

where γ_V is another knot obtained by infinitesimally displacing γ in the normal direction $\boldsymbol{V}$. Therefore

$$S(\gamma) = S(\gamma_1) = S(\gamma_2) = \mathcal{L}(\gamma_1, \gamma_2) = \mathcal{L}(\gamma_2, \gamma_1) \tag{43}$$

and

$$\mathcal{L}(\gamma, \gamma_k') = \mathcal{L}(\gamma_1, \gamma_k') = \mathcal{L}(\gamma_2, \gamma_k'), \tag{44}$$

where γ_k' denotes another arbitrary knot in the family $(\gamma_k' \neq \gamma, \gamma_k' \neq \gamma_{1,2})$. Then, third, we can compare I_γ and $I_{\gamma_1+\gamma_2}$ as follows: before splitting, we have

$$I_\gamma = 2\pi \left[W_\gamma^2 S(\gamma) + \sum_{k=1(\gamma_k' \neq \gamma)}^{N} 2W_\gamma W_{\gamma_k'} \mathcal{L}(\gamma, \gamma_k') \right], \tag{45}$$

where $\mathcal{L}(\gamma, \gamma_k') = \mathcal{L}(\gamma_k', \gamma)$; after splitting,

$$\begin{aligned} & I_{\gamma_1+\gamma_2} \\ &= 2\pi \Bigg[W_{\gamma_1}^2 S(\gamma_1) + W_{\gamma_2}^2 S(\gamma_2) + 2W_{\gamma_1} W_{\gamma_2} \mathcal{L}(\gamma_1, \gamma_2) \\ &\quad + \sum_{k=1(\gamma_k' \neq \gamma_{1,2})}^{N} 2W_{\gamma_1} W_{\gamma_k'} \mathcal{L}(\gamma_1, \gamma_k') \\ &\quad + \sum_{k=1(\gamma_k' \neq \gamma_{1,2})}^{N} 2W_{\gamma_2} W_{\gamma_k'} \mathcal{L}(\gamma_2, \gamma_k') \Bigg]. \end{aligned} \tag{46}$$

Comparing Eq.(45) with Eq.(46), we just have

$$I_\gamma = I_{\gamma_1+\gamma_2}. \tag{47}$$

This means that in the splitting process I is conserved.

(b) The merging case. We consider two knots γ_1 and γ_2, which are of the same self-linking numbers, merging into one knot γ which is of the same self-linking numbers as γ_1 and γ_2. This is obviously the inverse process of the above splitting case, therefore we have

$$I_{\gamma_1+\gamma_2} = I_\gamma. \tag{48}$$

(c) The intersection case. This case is related to the collision of two knots. We consider that two knots γ_1 and γ_2, which are of the same self-linking numbers, meet and then depart as other two knots γ_3 and γ_4. This process can be identified as two subprocesses: γ_1 and γ_2 merge into one knot γ, and then γ splits into γ_3 and γ_4. Thus from the above two cases (a) and (b) we have

$$I_{\gamma_1+\gamma_2} = I_{\gamma_3+\gamma_4}. \tag{49}$$

Therefore, we obtain the result that the topological invariant I is preserved in the branch processes during the evolution of knot solitons.

5. Conclusion

In conclusion, based on the decomposition of $U(1)$ gauge potential theory and Duan's topological current theory, we point out that there are allowed vortex lines and knot solitons realized in the AFZ model. In Section 2, we reveal that, under the regular condition $D^i\left(\dfrac{\phi}{x}\right) \neq 0$, vortex lines are originated from the zero points of the $\boldsymbol{\phi}$ field and the topological charges can be expressed in terms of the Hopf indices and the Brouwer degrees of $\boldsymbol{\phi}$-mapping. In Section 3, we mainly study the knot configurations in these string

structures. Based on the integral theory we construct a topological invariant for the vortex lines. It is shown that this invariant is just the total sum of all the self-linking and linking numbers of the knot family. In Section 4, moreover, we show that the knot topological invariant is preserved during the evolution of the knot vortex lines.

References

[1] Faddeev L D and Niemi A J 1999 *Phys. Rev. Lett.* **82** 1624
Faddeev L D and Niemi A J 2002 *Phys. Lett.* B **525** 195
Niemi A J 2000 *Phys. Rev.* D **61** 125006

[2] Duan Y S and Liu X 2004 *JHEP* **02** 028
Dilao R and Schiappa R 1997 *Phys. Lett.* B **404** 57
Dilao R and Schiappa R 1998 *Phys. Lett.* B **427** 26

[3] Babaev E, Faddeev L D and Niemi A J 2002 *Phys. Rev.* B **65** 100512
Babaev E 2002 *Phys. Rev. Lett.* **88** 177002
Xue J K and Peng P 2006 *Chin. Phys.* **15** 1149

[4] Saitta A M, Soper P D, Wasserman E and Klein M L 1999 *Natrue* **399** 46

[5] Sumners D W 1995 *Notices A. M. S.* **42** 528

[6] Faddeev L D 1975 Preprint Repont No.IAS-75-QS70

[7] Faddeev L D and Niemi A J 1997 *Nature* **387** 58
Battye R and Sutcliffe P 1998 *Phys. Rev. Lett.* **81** 4798

[8] Duan Y S, Zhao L and Zhang X H 2006 *Phys. Lett.* A **360** 183

[9] Jiang Y 2004 *Phys. Rev.* B **70** 012501
Duan Y S, Zhang X H, Liu Y X and Zhao L 2006 *Phys. Rev.* B **74** 144508

[10] Fayzullaev B A, Musakhanov M M, Pakc D G and Siddikov M 2005 *Phys. Lett.* B **609** 442

[11] Faddeev L D and Niemi A J 1997 *Natrue* **387** 58

[12] Aratyn H, Ferreira L A and Zimerman A H 1999 *Phys. Lett.* B **456** 162

[13] Aratyn H, Ferreira L A and Zimerman A H 1999 *Phys. Rev. Lett.* **83** 9
Ying H P and Zhang J B 2003 *Chin. Phys.* **12** 1374

[14] Duan Y S, Liu X and Zhang P M 2002 *J. Phys.* **14** 7941
Duan Y S, Zhang S L and Feng S S 1994 *J. Math. Phys.* **35** 4463

[15] Mo S F, Ren J R and Zhu T 2008 *J. Phys.* A **41** 315412
Mo S F, Ren J R and Zhu T 2008 *Int. J. Theor. Phys.* preprint http://dx.doi.org/10.1007/s10773-008-9838-3
Duan Y S, Zhao L, Liu Y X and Ren J R 2007 *Chin. Phys.* **16** 0881

[16] Wu T T and Yang C N 1975 *Phys. Rev.* D **12** 3845
Wu T T and Yang C N 1975 *Phys. Rev.* D **14** 437

[17] Duan Y S, Liu X and Fu L B 2003 *Phys. Rev.* D **67** 085022
Duan Y S, Liu X and Zhang P M 2003 *J. Phys.* A **36** 563
Duan Y S, Zhang H and Li S 1998 *Phys. Rev.* B **58** 125

[18] Goursat E 1904 *A Course in Mathematical Analysis* Vol. I (translated by Hedrick E R) (New York: Dover)

[19] Schouten J A 1951 *Tensor Analysis for Physicists* (Oxford: Clarendon)

[20] Duan Y S, Liu X and Fu L B 2003 *Commun. Theor. Phys.* **40** 447
Ren J R, Li R and Duan Y S 2007 *J. Math. Phys.* **48** 073502
Ren J R, Zhu T and Duan Y S 2008 *Chin. Phys. Lett.* **25** 353

[21] Polyakov A M 1988 *Mod. Phys. Lett.* A **3** 325
Duan Y S, Zhang X H and Zhao L 2007 *Mod. Phys. Lett.* A **22** 2201

[22] Rolfsen D 1976 *Knots and Links* (Berkeley, CA: Publish or Perish)

[23] Pohl W 1968 *J. Math. Mech.* **17** 975

[24] Ren J R, Zhu T and Duan Y S 2008 *Chin. Phys. Lett.* **25** 367
Ren J R, Zhu T and Duan Y S 2008 *Commun. Theor. Phys.* **50** 345

[25] Written E 1989 *Commun. Math. Phys.* **121** 351
Polyakov A M 1988 *Mod. Phys. Lett.* A **3** 325

[26] Duan Y S and Zhang H 1999 *Phys. Rev.* E **60** 2568
Fu L B, Duan Y S and Zhang H 2000 *Phys. Rev.* D **61** 045004

[27] Yang G H and Duan Y S 1998 *Int. J. Theor. Phys.* **37** 2435