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N-soliton solution of a coupled integrable dispersionless equation*

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A new coupled integrable dispersionless equation is presented by considering a spectral problem. A Darboux transformation for the resulting coupled integrable dispersionless equation is constructed with the help of spectral problems. As an application, the N-soliton solution of the coupled integrable dispersionless equation is explicitly given.

Keywords: soliton solution, Darboux transformation, coupled integrable dispersionless equation
PACC: 0545, 0420J

1. Introduction

The interaction between solitons plays an important role in nonlinear science. Many methods have been used to construct soliton solutions of nonlinear evolution equations, such as the inverse scattering transformation (IST), Hirota's bilinear method, the Bäcklund and Darboux transformation(DT) and so on.[1−13] Among them, the DT method based on Lax pairs has been proven to be one of the most fruitful algorithmic procedures to obtain soliton solutions of nonlinear evolution equations.

The study of dispersionless systems is of great importance, for they arise from the analysis of various problems, such as physics and mathematics.[14−18] In the present paper, we consider a coupled dispersionless equation

$$\begin{aligned} &q_{xt}+\beta\left(q+\frac{\beta}{\alpha}r\right)r_x=0,\\ &r_{xt}-\alpha\left(\frac{\alpha}{\beta}q+r\right)q_x=0, \end{aligned} \tag{1}$$

where α and β are arbitrary non-zero constants, and q and r are both functions of x and t. By applying the Painlevé analysis[19−21] to coupled dispersionless equation (1), resonances are found to be -1, 1, 2 and 4. The compatibility condition for each positive resonance is fully satisfied. Hence we ascertain that Eq.(1) passes the Painlevé test and it is called a coupled Painlevé integrable dispersionless equation.

We construct a new DT for the coupled integrable dispersionless equation (1) with the help of its spectral problem. Then we obtain a one-soliton solution, a two-soliton solution and an $N-$soliton solution of the coupled integrable dispersionless equation (1) by means of the DT.[22,23]

2. Darboux transformation

In this section, we construct a new DT for Eq.(1). Consider the spectral problem

$$\phi_x=U\phi,\quad U=\begin{pmatrix}\alpha q_x\lambda & \beta r_x\lambda\\ \beta r_x\lambda & -\alpha q_x\lambda\end{pmatrix}, \tag{2}$$

and the auxiliary problem

$$\begin{aligned} &\phi_t=V\phi,\\ &V=\begin{pmatrix}\dfrac{1}{4\lambda} & -\dfrac{1}{4\lambda}-\dfrac{1}{2}(\alpha q+\beta r)\\ -\dfrac{1}{4\lambda}+\dfrac{1}{2}(\alpha q+\beta r) & -\dfrac{1}{4\lambda}\end{pmatrix}. \end{aligned} \tag{3}$$

The compatible condition of expressions (2) and (3) is the zero curvature equation

$$U_t-V_x+[U,V]=0. \tag{4}$$

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Through the direct calculation of Eq.(4), we can yield Eq.(1).

The DT is actually a gauge transformation

$$\bar{\boldsymbol{\phi}} = \boldsymbol{T}\boldsymbol{\phi} \tag{5}$$

of spectral problems (2) and (3). It is required that $\bar{\phi}$ also satisfies the spectral problems

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \quad \bar{U} = \begin{pmatrix} \alpha\bar{q}_x\lambda & \beta\bar{r}_x\lambda \\ \beta\bar{r}_x\lambda & -\alpha\bar{q}_x\lambda \end{pmatrix}, \tag{6}$$

$$\bar{\phi}_t = \bar{V}\bar{\phi},$$
$$\bar{\boldsymbol{V}} = \begin{pmatrix} \dfrac{1}{4\lambda} & -\dfrac{1}{4\lambda} - \dfrac{1}{2}(\alpha\bar{q} + \beta\bar{r}) \\ -\dfrac{1}{4\lambda} + \dfrac{1}{2}(\alpha\bar{q} + \beta\bar{r}) & -\dfrac{1}{4\lambda} \end{pmatrix}. \tag{7}$$

This means that we have to find a matrix $\boldsymbol{T}$ such that the old potentials q, r in U and V are replaced with new ones $\bar{q}$, $\bar{r}$.

Now we come to construct a concrete transformation. Let $\lambda = \lambda_k$, the solution of spectral problems (2) and (3) is $h^{(k)} = \left(h_1^{(k)}, h_2^{(k)}\right)^{\mathrm{T}}$. It is easy to see that $h^{(k)-} = \left(h_2^{(k)}, -h_1^{(k)}\right)^{\mathrm{T}}$ is the solution of $\lambda = -\lambda_k$. Suppose that

$$H = \begin{pmatrix} h_1^{(k)} & h_2^{(k)} \\ h_2^{(k)} & -h_1^{(k)} \end{pmatrix}, \quad \text{and} \quad \Lambda = \begin{pmatrix} \dfrac{1}{\lambda_k} & 0 \\ 0 & -\dfrac{1}{\lambda_k} \end{pmatrix}, \tag{8}$$

let $\sigma = h_2^{(k)}/h_1^{(k)}$,

$$\begin{aligned} S^{(k)} &= H\Lambda H^{-1} \\ &= \frac{1}{\lambda_k(1+\sigma^2)} \begin{pmatrix} 1-\sigma^2 & 2\sigma \\ 2\sigma & \sigma^2 - 1 \end{pmatrix}, \end{aligned} \tag{9}$$

and for simplicity, further let $\tan\dfrac{\theta}{2} = \sigma$, then Eq.(9) will become

$$S^{(k)} = \frac{1}{\lambda_k} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \triangleq \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \tag{10}$$

With the help of

$$\sigma_x = \beta r_x \lambda_k (1-\sigma^2) - 2\alpha q_x \lambda_k \sigma, \tag{11}$$

we obtain

$$\theta_x = 2\beta r_x \lambda_k \cos\theta - 2\alpha q_x \lambda_k \sin\theta. \tag{12}$$

Taking

$$\begin{aligned} T &= \frac{1}{\lambda} I - S^{(k)} \\ &= \begin{pmatrix} \dfrac{1}{\lambda} - \dfrac{1}{\lambda_k}\cos\theta & -\dfrac{1}{\lambda_k}\sin\theta \\ -\dfrac{1}{\lambda_k}\sin\theta & \dfrac{1}{\lambda} + \dfrac{1}{\lambda_k}\cos\theta \end{pmatrix}, \end{aligned} \tag{13}$$

we have

$$\bar{U} = (T_x + TU)T^{-1} = \begin{pmatrix} \alpha\left(q_x\cos(2\theta) + \dfrac{\beta}{\alpha}r_x\sin(2\theta)\right)\lambda & \beta\left(\dfrac{\alpha}{\beta}q_x\sin(2\theta) - r_x\cos(2\theta)\right)\lambda \\ \beta\left(\dfrac{\alpha}{\beta}q_x\sin(2\theta) - r_x\cos(2\theta)\right)\lambda & -\alpha\left(q_x\cos(2\theta) + \dfrac{\beta}{\alpha}r_x\sin(2\theta)\right)\lambda \end{pmatrix}. \tag{14}$$

Using expressions (6) and (14) yields

$$\bar{q}_x = q_x\cos(2\theta) + \frac{\beta}{\alpha}r_x\sin(2\theta) = \left(q - \frac{1}{\alpha\lambda_k}\cos\theta\right)_x, \quad \bar{r}_x = \frac{\alpha}{\beta}q_x\sin(2\theta) - r_x\cos(2\theta) = \left(r - \frac{1}{\beta\lambda_k}\sin\theta\right)_x, \tag{15}$$

i.e.

$$\bar{q} = q - \frac{1}{\alpha\lambda_k}\cos\theta = q - \frac{1}{\alpha}A, \quad \text{and} \quad \bar{r} = r - \frac{1}{\beta\lambda_k}\sin\theta = r - \frac{1}{\beta}B. \tag{16}$$

According to the above computation, we have the following propositions.

Proposition 1 When q and r are given, let $\left(h_1^{(k)}, h_2^{(k)}\right)^{\mathrm{T}}$ be the solution of Eqs.(2) and (3) with $\lambda = \lambda_k$, then by using the Darboux matrix $\boldsymbol{T}$ and Eq.(16), we have

$$\bar{U} = (T_x + TU)T^{-1}. \tag{17}$$

We can obtain the same proposition about the auxiliary spectral problem.

Proposition 2 When q and r are given, let $\left(h_1^{(k)}, h_2^{(k)}\right)^{\mathrm{T}}$ be the solution of Eqs.(2) and (3) with $\lambda = \lambda_k$, then by using the Darboux matrix $\boldsymbol{T}$ and Eq.(16), we obtain

$$\bar{V} = (T_t + TV)T^{-1}. \tag{18}$$

Proof A direct calculation yields

$$\begin{aligned}\sigma_t =& \left(-\frac{1}{4\lambda_k} + \frac{1}{2}(\alpha q + \beta r)\right) - \frac{1}{2\lambda_k}\sigma \\ &+ \left(\frac{1}{4\lambda_k} + \frac{1}{2}(\alpha q + \beta r)\right)\sigma^2, \\ \theta_t =& (\alpha q + \beta r) - \frac{1}{2\lambda_k}(\cos\theta + \sin\theta).\end{aligned} \tag{19}$$

Substituting expressions (13), (16) and (19) into Eq.(18), we can obtain the conclusion above. Thus the proof is completed.

Propositions 1 and 2 show that the transformations (13) and (16) change the Lax pairs (2) and (3) into other Lax pairs (6) and (7), with U, V and $\bar{U}, \bar{V}$ being in identical forms respectively. Therefore both of the Lax pairs lead to the same Eq.(1), so expressions (13) and (16) are the DT of the coupled integrable dispersionless Eq.(1). From propositions 1 and 2 we have the following theorem.

Theorem 1 If (q, r) is a solution of the coupled integrable dispersionless equation (1) and $\lambda_k(\lambda_k \neq 0)$ is an arbitrary constant, when $\lambda = \lambda_k$, $\left(h_1^{(k)}, h_2^{(k)}\right)^{\mathrm{T}}$ is a solution of Lax pairs (2) and (3).

Let $\sigma = h_2^{(k)}/h_1^{(k)} = \tan\dfrac{\theta}{2}$, then

$$\begin{aligned}T =& \begin{pmatrix} \dfrac{1}{\lambda} - \dfrac{1}{\lambda_k}\cos\theta & -\dfrac{1}{\lambda_k}\sin\theta \\ -\dfrac{1}{\lambda_k}\sin\theta & \dfrac{1}{\lambda} + \dfrac{1}{\lambda_k}\cos\theta \end{pmatrix} \\ =& \begin{pmatrix} \dfrac{1}{\lambda} - \dfrac{1-\sigma^2}{\lambda_k(1+\sigma^2)} & -\dfrac{2\sigma}{\lambda_k(1+\sigma^2)} \\ -\dfrac{2\sigma}{\lambda_k(1+\sigma^2)} & \dfrac{1}{\lambda} + \dfrac{1-\sigma^2}{\lambda_k(1+\sigma^2)} \end{pmatrix}\end{aligned} \tag{20}$$

will be a Darboux matrix of the coupled integrable dispersionless equation (1). Mapping (q, r) into a new solution $(\bar{q}, \bar{r})$, the relations of $(\bar{q}, \bar{r})$ and (q, r) are determined by expression (16), i.e.

$$\bar{q} = q - \frac{1}{\alpha\lambda_k}\frac{1-\sigma^2}{1+\sigma^2} = q - \frac{1}{\alpha}A,$$

and

$$\bar{r} = r - \frac{1}{\beta\lambda_k}\frac{2\sigma}{1+\sigma^2} = r - \frac{1}{\beta}B. \tag{21}$$

3. The soliton solutions

We start from a particular solution of Eq.(1), which is $q = q_0$ and $r = r_0$, and we choose

$$h^{(k)} = \begin{pmatrix} h_1^{(k)} \\ h_2^{(k)} \end{pmatrix}, \quad 1 \leq k \leq N \tag{22}$$

as the solution of the Lax pairs (2) and (3) when $\lambda = \lambda_k$. Then we could construct a series of exact solutions of Eq.(1) as follows.

First, we construct

$$H^{(1)} = (h^{(1)}, h^{(1)-}), \quad \Lambda^{(1)} = \begin{pmatrix} \dfrac{1}{\lambda_1} & 0 \\ 0 & -\dfrac{1}{\lambda_1} \end{pmatrix}, \tag{23}$$

$$\begin{aligned}&S^{(1)} \\ &= H^{(1)}\Lambda^{(1)}(H^{(1)})^{-1} \\ &= \frac{1}{\lambda_1({h_1^{(1)}}^2 + {h_2^{(1)}}^2)} \begin{pmatrix} {h_1^{(1)}}^2 - {h_2^{(1)}}^2 & 2h_1^{(1)}h_2^{(1)} \\ 2h_1^{(1)}h_2^{(1)} & {h_2^{(1)}}^2 - {h_1^{(1)}}^2 \end{pmatrix} \\ &= \begin{pmatrix} A^{(1)} & B^{(1)} \\ B^{(1)} & -A^{(1)} \end{pmatrix}.\end{aligned} \tag{24}$$

Then by the use of theorem 1, we can obtain a new solution $q[1]$, $r[1]$ of Eq.(1) from the following equations:

$$q[1] = q_0 - \frac{1}{\alpha}A^{(1)}, \quad \text{and} \quad r[1] = r_0 - \frac{1}{\beta}B^{(1)}. \tag{25}$$

With the help of expressions (5) and (13) and after some calculations, we can acquire the solution of Lax pairs (2) and (3), when $q = q[1]$, $r = r[1]$ and $\lambda = \lambda_i$. the solution can be expressed as

$$\begin{aligned}\bar{h}_{[1]}^{(i)} =& \begin{pmatrix} \bar{h}_{1[1]}^{(i)} \\ \bar{h}_{2[1]}^{(i)} \end{pmatrix} \\ =& \left(\frac{1}{\lambda_i}I - S^{(1)}\right)\begin{pmatrix} h_1^{(i)} \\ h_2^{(i)} \end{pmatrix} \\ =& \frac{\dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_i}}{\left[h^{(1)}, h^{(1)}\right]} \begin{pmatrix} \begin{vmatrix} h_1^{(i)} & \left[h^{(i)}, h^{(1)}\right] \\ h_1^{(1)} & \left[h^{(1)}, h^{(1)}\right] \end{vmatrix} \\ \begin{vmatrix} h_2^{(i)} & \left[h^{(i)}, h^{(1)}\right] \\ h_2^{(1)} & \left[h^{(1)}, h^{(1)}\right] \end{vmatrix} \end{pmatrix},\end{aligned} \tag{26}$$

where $\left[h^{(i)}, h^{(j)}\right] = \dfrac{h_1^{(i)}h_1^{(j)} + h_2^{(i)}h_2^{(j)}}{(1/\lambda_i + 1/\lambda_j)}$. We construct

$$H^{(2)} = \left(\bar{h}^{(2)}_{[1]}, \bar{h}^{(2)-}_{[1]}\right), \quad \Lambda^{(2)} = \begin{pmatrix} \frac{1}{\lambda_2} & 0 \\ 0 & -\frac{1}{\lambda_2} \end{pmatrix},$$

$$\begin{aligned} S^{(2)} &= H^{(2)}\Lambda^{(2)}\left(H^{(2)}\right)^{-1} \\ &= \frac{1}{\lambda_2\left(\bar{h}^{(2)^2}_{1[1]} + \bar{h}^{(2)^2}_{2[1]}\right)} \begin{pmatrix} \bar{h}^{(2)^2}_{1[1]} - \bar{h}^{(2)^2}_{2[1]} & 2\bar{h}^{(2)}_{1[1]}\bar{h}^{(2)}_{2[1]} \\ 2\bar{h}^{(2)}_{1[1]}\bar{h}^{(2)}_{2[1]} & \bar{h}^{(2)^2}_{2[1]} - \bar{h}^{(2)^2}_{1[1]} \end{pmatrix} \\ &= \begin{pmatrix} A^{(2)} & B^{(2)} \\ B^{(2)} & -A^{(2)} \end{pmatrix}, \end{aligned}$$

where $h^{(2)}_{1[1]}$ and $h^{(2)}_{2[1]}$ are given by Eq.(26) when $i = 2$.

Then we can obtain a new solution $q[2], r[2]$ of Eq.(1) from the following equation:

$$q[2] = q[1] - \frac{1}{\alpha}A^{(2)}, \quad r[2] = r[1] - \frac{1}{\beta}B^{(2)}. \tag{27}$$

So with the help of Eq.(26) and $S^{(2)}$, after taking the DT once again, we can obtain the solution of Lax pairs (2) and (3) when $q = q[2], r = r[2]$, and $\lambda = \lambda_3$, then

$$\begin{aligned} \bar{h}^{(3)}_{[2]} &= \begin{pmatrix} \bar{h}^{(3)}_{1[2]} \\ \bar{h}^{(3)}_{2[2]} \end{pmatrix} \\ &= \left(\frac{1}{\lambda_3}I - S^{(2)}\right)\begin{pmatrix} \bar{h}^{(3)}_{1[1]} \\ \bar{h}^{(3)}_{2[1]} \end{pmatrix} \\ &= \frac{\left(\frac{1}{\lambda_3}+\frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_3}+\frac{1}{\lambda_1}\right)}{\left[h^{(1)},h^{(1)}\right]\left[h^{(2)},h^{(2)}\right] - \left[h^{(2)},h^{(1)}\right]^2} \\ &\quad \times \begin{pmatrix} \begin{vmatrix} h^{(3)}_1 & \left[h^{(3)},h^{(1)}\right] & \left[h^{(3)},h^{(2)}\right] \\ h^{(1)}_1 & \left[h^{(1)},h^{(1)}\right] & \left[h^{(1)},h^{(2)}\right] \\ h^{(2)}_1 & \left[h^{(2)},h^{(1)}\right] & \left[h^{(2)},h^{(2)}\right] \end{vmatrix} \\ \begin{vmatrix} h^{(3)}_2 & \left[h^{(3)},h^{(1)}\right] & \left[h^{(3)},h^{(2)}\right] \\ h^{(1)}_2 & \left[h^{(1)},h^{(1)}\right] & \left[h^{(1)},h^{(2)}\right] \\ h^{(2)}_2 & \left[h^{(2)},h^{(1)}\right] & \left[h^{(2)},h^{(2)}\right] \end{vmatrix} \end{pmatrix}. \end{aligned} \tag{28}$$

If we have carried out the DT N–1 times, we can obtain the solution of Eq.(1) as $q[N-1]$ and $r[N-1]$. Thus we can express the solution of Lax pairs (2) and (3) as follows:

$$\bar{h}^{(N)}_{[N-1]} = \begin{pmatrix} \bar{h}^{(N)}_{1[N-1]} \\ \bar{h}^{(N)}_{2[N-1]} \end{pmatrix} = \Delta_N \begin{pmatrix} \begin{vmatrix} h^{(N)}_1 & \left[h^{(N)},h^{(1)}\right] & \cdots & \left[h^{(N)},h^{(N-1)}\right] \\ h^{(1)}_1 & \left[h^{(1)},h^{(1)}\right] & \cdots & \left[h^{(1)},h^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ h^{(N-1)}_1 & \left[h^{(N-1)},h^{(1)}\right] & \cdots & \left[h^{(N-1)},h^{(N-1)}\right] \end{vmatrix} \\ \begin{vmatrix} h^{(N)}_2 & \left[h^{(N)},h^{(1)}\right] & \cdots & \left[h^{(N)},h^{(N-1)}\right] \\ h^{(1)}_2 & \left[h^{(1)},h^{(1)}\right] & \cdots & \left[h^{(1)},h^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ h^{(N-1)}_2 & \left[h^{(N-1)},h^{(1)}\right] & \cdots & \left[h^{(N-1)},h^{(N-1)}\right] \end{vmatrix} \end{pmatrix}, \tag{29}$$

where

$$\Delta_N = \frac{\left(\frac{1}{\lambda_N}+\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_N}+\frac{1}{\lambda_2}\right)\cdots\left(\frac{1}{\lambda_N}+\frac{1}{\lambda_{N-1}}\right)}{\begin{vmatrix} \left[h^{(1)},h^{(1)}\right] & \left[h^{(1)},h^{(2)}\right] & \cdots & \left[h^{(1)},h^{(N-1)}\right] \\ \left[h^{(2)},h^{(1)}\right] & \left[h^{(2)},h^{(2)}\right] & \cdots & \left[h^{(2)},h^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[h^{(N-1)},h^{(1)}\right] & \left[h^{(N-1)},h^{(2)}\right] & \cdots & \left[h^{(N-1)},h^{(N-1)}\right] \end{vmatrix}}.$$

We construct

$$H^{(N)}=\left(\bar{h}^{(N)}_{[N-1]},\bar{h}^{(N)-}_{[N-1]}\right),\quad \Lambda^{(N)}=\begin{pmatrix}\frac{1}{\lambda_N} & 0\\ 0 & -\frac{1}{\lambda_N}\end{pmatrix},$$

$$\begin{aligned}S^{(N)} &= H^{(N)}\Lambda^{(N)}\left(H^{(N)}\right)^{-1}\\ &= \frac{1}{\lambda_N\left(\bar{h}^{(N)^2}_{1[N-1]}+\bar{h}^{(N)^2}_{2[N-1]}\right)}\begin{pmatrix}\bar{h}^{(N)^2}_{1[N-1]}-\bar{h}^{(N)^2}_{2[N-1]} & 2\bar{h}^{(N)}_{1[N-1]}\bar{h}^{(N)}_{2[N-1]}\\ 2\bar{h}^{(N)}_{1[N-1]}\bar{h}^{(N)}_{2[N-1]} & \bar{h}^{(N)^2}_{2[N-1]}-\bar{h}^{(N)^2}_{1[N-1]}\end{pmatrix}\\ &= \begin{pmatrix}A^{(N)} & B^{(N)}\\ B^{(N)} & -A^{(N)}\end{pmatrix}.\end{aligned}$$

Then we can obtain a new solution $q[N]$ and $r[N]$ of Eq.(1) from the following equation:

$$q[N]=q[N-1]-\frac{1}{\alpha}A^{(N)},$$

and

$$r[N]=r[N-1]-\frac{1}{\beta}B^{(N)}. \tag{30}$$

This process can be iterated, thereby yielding a sequence of new solutions usually.

Finally, we give a simple example. We construct the soliton solution for Eq.(1). Substituting $q=\beta x$ and $r=-\alpha x$ into Lax pairs (2) and (3), and choosing basic solutions corresponding to $\lambda=\lambda_k$ as

$$h^{(k)}=\begin{pmatrix}h^{(k)}_1\\ h^{(k)}_2\end{pmatrix}=\begin{pmatrix}\cosh\xi_k\\ \cosh\xi_k-\sqrt{2}\sinh\xi_k\end{pmatrix},\quad 1\le k\le N, \tag{31}$$

where $\xi_k=\sqrt{2}\alpha\beta\lambda_k\left(x+\frac{t}{4\alpha\beta\lambda_k^2}\right)+\xi_k^{(0)}$, then we construct

$$H^{(1)}=\left(h^{(1)},h^{(1)-}\right)=\begin{pmatrix}\cosh\xi_1 & \cosh\xi_1-\sqrt{2}\sinh\xi_1\\ \cosh\xi_1-\sqrt{2}\sinh\xi_1 & -\cosh\xi_1\end{pmatrix}, \tag{32}$$

$$\Lambda^{(1)}=\begin{pmatrix}\frac{1}{\lambda_1} & 0\\ 0 & -\frac{1}{\lambda_1}\end{pmatrix},\quad S^{(1)}=H^{(1)}\Lambda^{(1)}(H^{(1)})^{-1}=\begin{pmatrix}A^{(1)} & B^{(1)}\\ B^{(1)} & -A^{(1)}\end{pmatrix}, \tag{33}$$

where

$$A^{(1)}=\frac{\sinh\xi_1(\sqrt{2}\cosh\xi_1-\sinh\xi_1)}{\lambda_1\cosh(2\xi_1)-\frac{\lambda_1}{\sqrt{2}}\sinh(2\xi_1)},\ \text{ and }\ B^{(1)}=\frac{\cosh\xi_1(-\sqrt{2}\sinh\xi_1+\cosh\xi_1)}{\lambda_1\cosh(2\xi_1)-\frac{\lambda_1}{\sqrt{2}}\sinh(2\xi_1)}.$$

Thus from expression (25), we can obtain one-soliton solution of Eq.(1)

$$\begin{aligned}q[1] &= \beta x-\frac{1}{\alpha}A^{(1)}=\beta x-\frac{1}{\alpha}\frac{\sinh\xi_1(\sqrt{2}\cosh\xi_1-\sinh\xi_1)}{\lambda_1\cosh(2\xi_1)-\frac{\lambda_1}{\sqrt{2}}\sinh(2\xi_1)},\\ r[1] &= -\alpha x-\frac{1}{\beta}B^{(1)}=-\alpha x-\frac{1}{\beta}\frac{\cosh\xi_1(-\sqrt{2}\sinh\xi_1+\cosh\xi_1)}{\lambda_1\cosh(2\xi_1)-\frac{\lambda_1}{\sqrt{2}}\sinh(2\xi_1)},\end{aligned} \tag{34}$$

where $\xi_1=\sqrt{2}\alpha\beta\lambda_1\left(x+\frac{t}{4\alpha\beta\lambda_1^2}\right)+\xi_1^{(0)}$. The one-soliton solution is plotted in Fig.1.

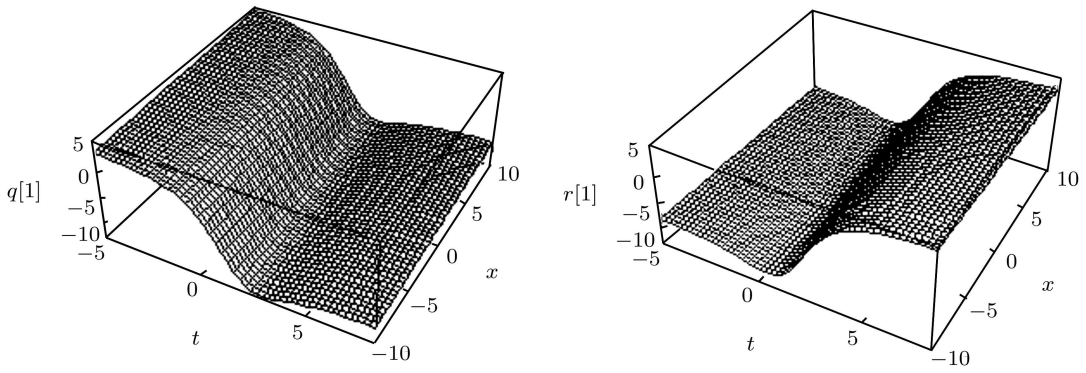


Fig.1. One-soliton solution (34) with $\alpha = \beta = 0.1, \lambda_1 = 1$, and $\xi_1^{(0)} = 0$.

If we choose another basic solution corresponding to $\lambda = \lambda_2$, given as follows:

$$h^{(2)} = \begin{pmatrix} h_1^{(2)} \\ h_2^{(2)} \end{pmatrix} = \begin{pmatrix} \cosh \xi_2 \\ \cosh \xi_2 - \sqrt{2} \sinh \xi_2 \end{pmatrix}, \tag{35}$$

where $\xi_2 = \sqrt{2}\alpha\beta\lambda_2 \left(x + \dfrac{t}{4\alpha\beta\lambda_2^2} \right) + \xi_2^{(0)}$, then we will have a two-soliton solution of Eq.(1) from expression (27)

$$\begin{aligned} q[2] &= \beta x - \frac{1}{\alpha} \frac{\sinh \xi_1 (\sqrt{2} \cosh \xi_1 - \sinh \xi_1)}{\lambda_1 \cosh(2\xi_1) - \dfrac{\lambda_1}{\sqrt{2}} \sinh(2\xi_1)} - \frac{1}{\alpha} A^{(2)}, \\ r[2] &= -\alpha x - \frac{1}{\beta} \frac{\cosh \xi_1 (-\sqrt{2} \sinh \xi_1 + \cosh \xi_1)}{\lambda_1 \cosh(2\xi_1) - \dfrac{\lambda_1}{\sqrt{2}} \sinh(2\xi_1)} - \frac{1}{\beta} B^{(2)}, \end{aligned} \tag{36}$$

where

$$A^{(2)} = \frac{\bar{h}_{1[1]}^{(2)^2} - \bar{h}_{2[1]}^{(2)^2}}{\lambda_2 \left(\bar{h}_{1[1]}^{(2)^2} + \bar{h}_{2[1]}^{(2)^2} \right)}, \quad B^{(2)} = \frac{2 \bar{h}_{1[1]}^{(2)} \bar{h}_{2[1]}^{(2)}}{\lambda_2 \left(\bar{h}_{1[1]}^{(2)^2} + \bar{h}_{2[1]}^{(2)^2} \right)},$$

with

$$\begin{aligned} \bar{h}_{1[1]}^{(2)} &= \frac{\dfrac{1}{\lambda_2} + \dfrac{1}{\lambda_1}}{\left[h^{(1)}, h^{(1)} \right]} \begin{vmatrix} h_1^{(2)} & \left[h^{(2)}, h^{(1)} \right] \\ h_1^{(1)} & \left[h^{(1)}, h^{(1)} \right] \end{vmatrix}, \\ \bar{h}_{2[1]}^{(2)} &= \frac{\dfrac{1}{\lambda_2} + \dfrac{1}{\lambda_1}}{\left[h^{(1)}, h^{(1)} \right]} \begin{vmatrix} h_2^{(2)} & \left[h^{(2)}, h^{(1)} \right] \\ h_2^{(1)} & \left[h^{(1)}, h^{(1)} \right] \end{vmatrix}. \end{aligned}$$

The two-soliton solution (36) is plotted in Fig.2.

With the help of expression (30), we can obtain an N-soliton solution of Eq.(1). By means of computer, we can give only the figured three-soliton solution (see Fig.3).

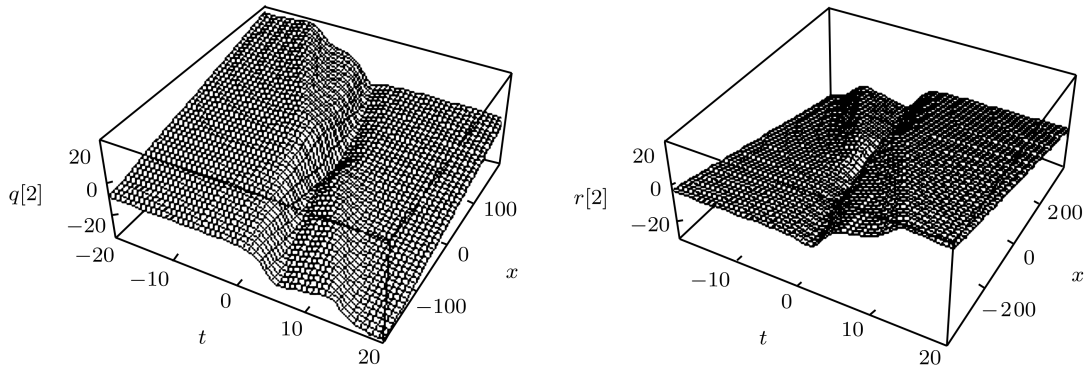


Fig.2. Two-soliton solution (36) with $\alpha = \beta = 0.1, \lambda_1 = 1, \lambda_2 = 1.1$, and $\xi_1^{(0)} = \xi_2^{(0)} = 0$.

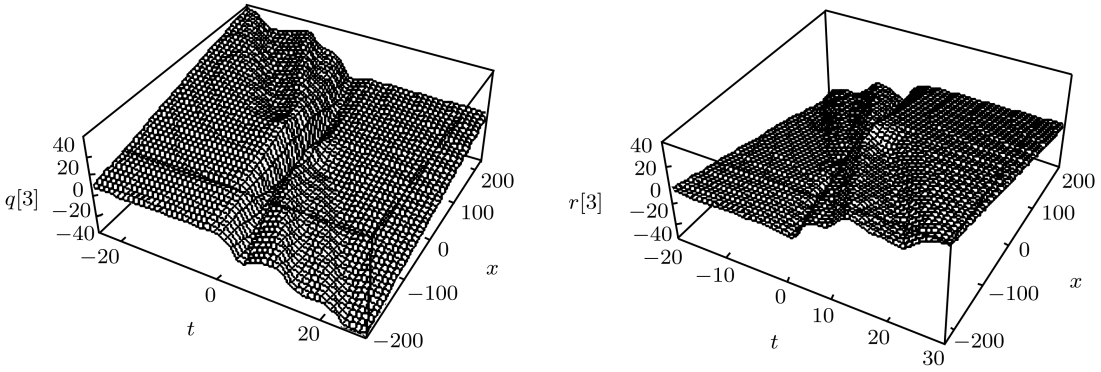


Fig.3. Three-soliton solution with $\alpha = \beta = 0.1, \lambda_1 = 1, \lambda_2 = 1.1, \lambda_3 = 1.2$, and $\xi_1^{(0)} = \xi_2^{(0)} = \xi_3^{(0)} = 0$.

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