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Chaotification for a class of nonlinear systems*

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More and more attention has been focused on effectively generating chaos via simple physical devices. The problem of creating chaotic attractors is considered for a class of nonlinear systems with backlash function in this paper. By utilizing the Silnikov heteroclinic and homoclinic theorems, some sufficient conditions are established to guarantee that the nonlinear system has horseshoe-type chaos. Examples and simulations are given to verify the effectiveness of the theoretical results.

Keywords: chaos, backlash function, heteroclinic orbit, chaos generation
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1. Introduction

Over the last decade, as chaos can actually be useful and can be well controlled, the intensive study of chaotic dynamics has evolved from the traditional trend of understanding and analysing chaos to the new attempt of controlling and utilizing it. Chen[1] summarized the methods of generating chaos simply either in discrete systems or in continuous systems. Anti-control of chaos was studied for uncertain nonlinear systems based on the method of sliding mode control in Ref.[2]. The Melnikov chaotic solution and the chaotic region of parameter space were found by using the direct perturbation method in Ref.[3]. Meanwhile, some researchers used a switching piecewise linear controller to create chaotic attractors.[4−8] The chaotication problem of driving a continuous-time system to a chaotic state by using an impulsive control input was studied in Ref.[9]. The controller was designed to ensure that the controlled orbit was bounded and, meanwhile, had positive Lyapunov exponents.

Dissipativity is a notion which can be used in many areas of science, and it allows the control engineer to relate a set of efficient mathematical tools to well known physical phenomena. A homoclinic orbit refers to a bounded trajectory of a system that is doubly asymptotic to an equilibrium point of the system, i.e. the trajectory converges to some equilibrium point as t approaches $\pm\infty$. A heteroclinic orbit, on the other hand, is similarly defined except that there are two distinct saddle foci that are connected by the orbit: one is corresponding to the forward asymptotic time, and the other to the reverse asymptotic time limit. Silva[10] suggested the homoclinic Silnikov method and the heteroclinic Silnikov method to make a system with horseshoe-type chaos. The existence of Silnikov chaos in the Lorenz-family system has been investigated by Wang *et al.*[11]

In the present paper, we use the Silnikov criteria to guarantee that a class of nonlinear systems with backlash function has horseshoe-type chaos via parameter analysis. The remainder of the present paper is organized as follows. In Section 2 the concept of backlash function is introduced. In Section 3, some characteristics of the nonlinear system are presented and some parameters are discussed. In Section 4 the existence of heteroclinic orbit is analysed in detail and an example is given to prove the effectiveness of the theorem. In Section 5, a theorem about the existence of homoclinic-type chaos, as well as an example and simulation are presented. Finally, some concluding remarks are given in Section 6.

2. Backlash function description

In this section the concept of backlash function is reviewed and some of its fundamental characteristics are also presented. It is well known that backlash function is one of the typical nonlinear functions, such as the stair function, the saturated function and the

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hysteretic function. It is depicted as follows:

$$f(x)=\begin{cases} k(x-q),\ \dot{x}>0,\ x>-(p-2q); \\ k(a-q),\ \dot{x}<0,\ x>(p-2q); \\ k(x+q),\ \dot{x}<0,\ x<(p-2q); \\ k(-p+q),\ \dot{x}>0,\ x<-(p-2q), \end{cases} \tag{1}$$

where $k>0$ is the slope of the backlash function, and p and q are real parameters. The backlash function is a piecewise-linear continuous function and it can switch on the state space $x=\pm(p-2q)$.

3. Parameter analysis

Consider the following nonlinear system:

$$\dot{X}=\tilde{A}X+F(X) \tag{2}$$

with

$$\tilde{A}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix},$$

where $X=[x,y,z]^{\mathrm{T}}$ are state variables, $F(X)=[0,0,f(x)]^{\mathrm{T}}$ with $f(x)$ is given in Eq.(1) and k,p,q are real parameters.

For a chaotic system, its trajectory can fold and stretch itself repeatedly. Especially, its volume can run in a defined area but cannot extend into the whole space. So, we can use the dissipative theory to analyse equation system (2).

Considering system (2), the time variation rate of the volume $V(t)$ or the trace of the system Jacobian (divergence of the vector fields) is

$$\frac{1}{V}\frac{\mathrm{d}V}{Dt}=\mathrm{div}V=\frac{\partial\dot{x}}{x}+\frac{\partial\dot{y}}{y}+\frac{\partial\dot{z}}{z}=c.$$

As attractors are in a special equilibrium state, they exist only in dissipative systems. Therefore, system (2) is dissipative and its trajectory converges at an exponential contraction rate e^{ct}.

Remark 1 A volume element V_0 is contracted by the flow into a volume element $V_0\mathrm{e}^{ct}$ in time t, that is to say, each volume containing the system orbit shrinks to zero as t approaches to ∞ at an exponential rate e^{ct}, when $c<0$, which is independent of x,y,z. Accordingly, all system orbits will ultimately converge into a bounded area and form an attractor asymptotically.

In system (2), we can obtain that if $\dot{x}>0$ and $x>-(p-2q)$, its equilibrium point is $x_{\mathrm{e}}^1(0,0,kq/(a+k))$; if $\dot{x}<0$ and $x>(p-2q)$, the equilibrium point is $x_{\mathrm{e}}^2(0,0,k(p-q)/a)$; if $\dot{x}<0$ and $x<(p-2q)$, the equilibrium point is $x_{\mathrm{e}}^3(0,0,-kq/(a+k))$; if $\dot{x}>0$ and $x<-(p-2q)$, the equilibrium point is $x_{\mathrm{e}}^4(0,0,-k(p-q)/a)$.

For the equilibrium points x_{e}^1 and x_{e}^3, the system Jacobian J is

$$J=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a+k & b & c \end{pmatrix}.$$

The characteristic equation is

$$\lambda^3-c\lambda^2-b\lambda-(a+b)=0. \tag{3}$$

To calculate the characteristic value, letting $\lambda=\mu+\dfrac{c}{3}$ yields

$$\mu^3+\alpha\mu+\beta=0, \tag{4}$$

where $\alpha=-b-\dfrac{c^2}{3}$, and $\beta=-(a+k)-\dfrac{bc}{3}+\dfrac{2c^2}{27}$. For simplicity, let

$$\Delta=\left(\frac{\beta}{2}\right)^2+\left(\frac{\alpha}{3}\right)^3.$$

Then, calculation yields

$$\begin{aligned}\Delta=&\frac{1}{4}(a+k)^2-\frac{1}{108}b^2c^2+\frac{1}{27}c^3(a+k)\\&+\frac{1}{6}(a+k)bc-\frac{1}{27}b^3.\end{aligned}$$

Equation (4) has a unique real root along with one pair of complex conjugate roots; or equivalently, the original algebraic equation (3) has one real root and one pair of complex conjugate roots: $\lambda_1=\dfrac{c}{3}+A-B$, and $\lambda_{2,3}=\dfrac{c}{3}-\dfrac{1}{2}(A-B)\pm\dfrac{\sqrt{3}}{2}(A+B)\mathrm{i}$, where $A=\sqrt[3]{-\dfrac{\beta}{2}+\sqrt{\Delta}}$, $B=\sqrt[3]{\dfrac{\beta}{2}+\sqrt{\Delta}}$ and $\mathrm{i}=\sqrt{-1}$.

For the equilibrium points x_{e}^2 and x_{e}^4, the system Jacobian J' is

$$J'=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}.$$

The characteristic equation is

$$\lambda^3-c\lambda^2-b\lambda-a=0. \tag{5}$$

To calculate the characteristic value, letting $\lambda=\mu+\dfrac{c}{3}$ yields

$$\mu^3+\alpha'\mu+\beta'=0, \tag{6}$$

where $\alpha' = -b - \dfrac{c^2}{3}$, and $\beta' = -a - \dfrac{bc}{3} + \dfrac{2c^2}{27}$. For simplicity, let

$$\Delta' = \left(\frac{\beta'}{2}\right)^2 + \left(\frac{\alpha}{3}\right)^3,$$

then, calculation will yield

$$\Delta' = \frac{1}{4}a^2 - \frac{1}{108}b^2c^2 + \frac{1}{27}c^3a + \frac{1}{6}abc - \frac{1}{27}b^3.$$

Similarly, Eq.(6) has a unique real root along with one pair of complex conjugate roots; or equivalently, the original algebraic equation (5) has one real root and one pair of complex conjugate roots: $\lambda_1' = \dfrac{c}{3} + A' - B'$, and $\lambda_{2,3}' = \dfrac{c}{3} - \dfrac{1}{2}(A' - B') \pm \dfrac{\sqrt{3}}{2}(A' + B')\mathrm{i}$, where $A' = \sqrt[3]{-\dfrac{\beta'}{2} + \sqrt{\Delta'}}$, $B' = \sqrt[3]{\dfrac{\beta'}{2} + \sqrt{\Delta'}}$ and $\mathrm{i} = \sqrt{-1}$.

4. Chaos generation of Silnikov heteroclinic orbit type

Lemma 1[10] (Silnikov heteroclinic theorem)

Suppose that two distinct equilibrium points of system (2), denoted by x_e and x_e' respectively, are saddle foci, whose characteristic values γ_k and $\sigma_k \pm \mathrm{i}\omega_k (k = 1, 2)$ satisfy the following Silnikov inequality:

$$|\gamma_k| > |\sigma_k| > 0, \ k = 1, 2,$$

under constraint

$$\sigma_1\sigma_2 > 0, \quad \text{or} \quad \gamma_1\gamma_2 > 0.$$

If there exists a heteroclinic orbit jointing x_e and x_e', then

(I) the Silnikov map, defined in a neighbourhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics;

(II) for arbitrary sufficiently small C^1-perturbation g of f, the perturbed system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x), x \in R^3,$$

has at least a finite number of Smale horseshoes in the discrete Silnikov map defined near the heteroclinic orbit;

(III) both the original system (2) and the perturbed system have horseshoe-type chaos.

Remark 2 x_e and x_e' must be one pair of symmetric equilibrium points. Meanwhile, at the two equilibrium points x_e and x_e', we have identical characteristic polynomials of the Jacobian matrices. The three corresponding eigenvalues have the following properties: one is negative (e.g. $\lambda_1 < 0$ in Eq.(4)), and the other two are one pair of complex conjugate values with positive real parts (e.g. $\mathrm{Re}[\lambda_{2,3}] > 0$ in Eq.(4)). Thus, for system (2), we can have that $a + k < 0$, as $-(a + k) = (-1)^3\lambda_1\lambda_2\lambda_3 > 0$ given from Eq.(3) (or $a < 0$, as $-a = (-1)^3\lambda_1\lambda_2\lambda_3 > 0$ given from Eq.(5)). Outside trajectories are attracted into the vicinity of its steady state and they alternatively swirl between the two equilibrium points in the chaotic attractor, if a system has a two-scroll chaotic attractor.

Remark 3 As is well known, a Smale map has two main features: contraction and prolongation. In the Silnikov theorems, the equilibrium points are of saddle-foci type, which is important since it has some basic properties of Smale horseshoes, i.e. the negative characteristic value corresponds to the contraction direction while the positive real part of the conjugate pair of complex characteristic values corresponds to the prolongation direction in the Smale map, and the contraction rate should be bigger than the prolongation rate. The combination of contraction and prolongation results in the Smale horseshoe structure of the Silnikov-type chaos.

Theorem 1 Suppose that there are two distinct equilibrium points x_e^1 and x_e^3 (or x_e^2 and x_e^4) for system (2), which are saddle foci. If $\Delta > 0$, $\beta > 0$, and $\dfrac{1}{2}(A - B) < \dfrac{c}{3} < \dfrac{1}{4}(A - B)$ (or $\Delta' > 0$, $\beta' > 0$, and $\dfrac{1}{2}(A' - B') < \dfrac{c}{3} < \dfrac{1}{4}(A' - B')$) hold true, then there is a heterclinic orbit that connects these equilibrium points and system (2) possesses horseshoe-type chaos.

Proof To start with, we consider the equilibrium points x_e^1 and x_e^3 of system (2). From the assumption of theorem 1, we can find that

$$\lambda_1\mathrm{Re}[\lambda_{2,3}] < 0, \quad \text{and} \quad \mathrm{Im}[\lambda_{2,3}] = \frac{\sqrt{3}}{2}(A + B) \neq 0. \quad (7)$$

As $\beta > 0$ and $\Delta > 0$, then we can find that

$$A - B = \sqrt[3]{-\frac{\beta}{2} + \sqrt{\Delta}} - \sqrt[3]{\frac{\beta}{2} + \sqrt{\Delta}} < 0. \quad (8)$$

Furthermore,

$$\lambda_1 = \frac{c}{3} + A - B < 0, \quad (9)$$

and $c < 0$ from the above discussion. From expressions (7) and (9) it follows that

$$\mathrm{Re}[\lambda_{2,3}] = \frac{c}{3} - \frac{1}{2}(A - B) > 0. \quad (10)$$

Therefore,

$$\begin{aligned}&|\lambda_1| - |\mathrm{Re}[\lambda_{2,3}]|\\ &= -\left(\frac{c}{3} + A - B\right) - \left(\frac{c}{3} - \frac{1}{2}(A-B)\right)\\ &= -2\left(\frac{c}{3} - \frac{A-B}{4}\right).\end{aligned}$$

Because $\dfrac{c}{3} < \dfrac{A-B}{4}$, we can obtain that

$$|\lambda_1| - |\mathrm{Re}[\lambda_{2,3}]| = -2\left(\frac{c}{3} - \frac{A-B}{4}\right) > 0.$$

By using Lemma 1, for the equilibrium points x_{e}^1 and x_{e}^3 of system (2), there must exist a heteroclinic orbit that joints the two equilibrium points.

Similarly, for the equilibrium points x_{e}^2 and x_{e}^4, the proof is the same. If $\Delta' > 0$, $\beta' > 0$, and $\dfrac{1}{2}(A'-B') < \dfrac{c}{3} < \dfrac{1}{4}(A'-B')$ hold true, then there is a heteroclinic orbit connecting the two equilibrium points x_{e}^2 and x_{e}^4, and system (2) has horseshoe-type chaos.

The proof is thus completed.

Example 1 Let the parameters in system (2) be $a = -3.25$, $b = 3.60$, $c = -1.8$, $p = 3.5$, $q = 1.5$, and $k = 2.5$, and we can find the corresponding system to be chaotic, namely, there is a heteroclinic orbit connecting the equilibrium points. On the other hand, by using the method of Wolf reconstruction，the maximum Lyapunov exponent of this attractor, LE, is calculated to be 0.2153, i.e. LE $= 0.2153 > 0$, and it further proves that the system is chaotic under the control of backlash function. The attractor is shown in Fig.1.

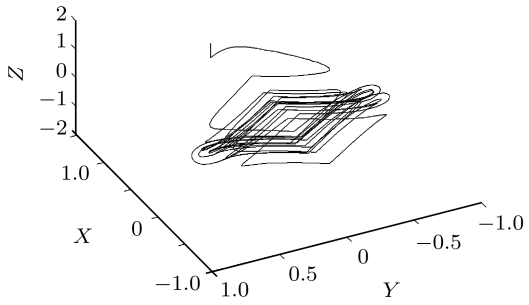


Fig.1. Two-scroll chaotic attractor.

5. Chaos generation of Silnikov homoclinic orbit type

Lemma 2[10] (Silnikov homoclinic theorem)

Let x_{e} be an equilibrium point of system (2) and make the following assumptions:

(1) the equilibrium point x_{e} is a saddle focus and satisfies Silnikov inequality, that is

$$|\gamma_k| > |\sigma_k| > 0;$$

(2) there exists a homoclinic orbit connected at x_{e}, then the following points will be obtained:

(1) the Silnikov map, defined in a neighbourhood of the homoclinic orbit of the system, possesses a countable number of Smale horseshoes in its corresponding discrete system;

(2) for arbitrary sufficiently small C^1-perturbation g of f, the perturbed system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x), x \in R^3$$

has at least a finite number of Smale horseshoes in the corresponding system of the Silnikov map defined near the homoclinic orbit;

(3) both the original system (2) and the perturbed system possess horseshoe chaos.

Theorem 2 Suppose that there is a corresponding equilibrium point x_{e}^1 or x_{e}^3, (x_{e}^2 or x_{e}^4) for system (2) respectively, which is a saddle focus. If $\Delta > 0$, $\beta > 0$, and $-\dfrac{1}{4}(A-B) < \dfrac{c}{3} < 0$ (or $\Delta' > 0$, $\beta' > 0$, and $-\dfrac{1}{4}(A'-B') < \dfrac{c}{3} < 0$) hold true, then there is a homoclinic orbit based at the equilibria and system (2) can have horseshoe-type chaos.

Proof Similarly, we consider the equilibrium point x_{e}^1 or x_{e}^3 first. From the assumption of theorem 2 it follows that

$$\lambda_1 \mathrm{Re}[\lambda_{2,3}] < 0,$$

and

$$\mathrm{Im}[\lambda_{2,3}] = \frac{\sqrt{3}}{2}(A+B) \neq 0. \tag{11}$$

In this case, as $\beta > 0$ and $\Delta > 0$, it follows that

$$\mathrm{Re}[\lambda_{2,3}] = \frac{c}{3} - \frac{1}{2}(A-B) < 0. \tag{12}$$

From inequalities (11) and (12) it follows that

$$\lambda_1 = \frac{c}{3} + A - B > 0. \tag{13}$$

Therefore,

$$\begin{aligned}&|\lambda_1| - |\mathrm{Re}[\lambda_{2,3}]|\\ &= \left(\frac{c}{3} + A - B\right) + \left(\frac{c}{3} - \frac{1}{2}(A-B)\right)\\ &= \frac{2c}{3} + \frac{A-B}{2}.\end{aligned}$$

If $-\dfrac{A-B}{4} < \dfrac{c}{3}$, then it follows that

$$|\lambda_1| - |\mathrm{Re}[\lambda_{2,3}]| > 0.$$

By using Lemma 2, it can be concluded that there is a homoclinic orbit based at the equilibrium point x_{e}^1 or x_{e}^3.

For the equilibrium point x_{e}^2 or x_{e}^4, the proof is similar. If $\Delta' > 0$, $\beta' > 0$, and $-\dfrac{1}{4}(A'-B') < \dfrac{c}{3} < 0$ hold true, then there is a homoclinic orbit based at the equilibrium point and system (2) can have horseshoe-type chaos.

The proof is thus completed.

Example 2 Let the parameters in system (2) be $a = 4,\ b = 2,\ c = -18,\ p = -6,\ q = -0.5,$ and $k = 0.5$, we can find the corresponding system to be chaotic, namely, there is a homoclinic orbit based at the equilibrium point.

The simulation results are presented in Figs.2 and 3. Besides, the maximum Lyapunov exponent of the attractor, LE, is calculated to be 0.2691, i.e. LE $= 0.2691 > 0$, which further proves the effectiveness of theorem 2.

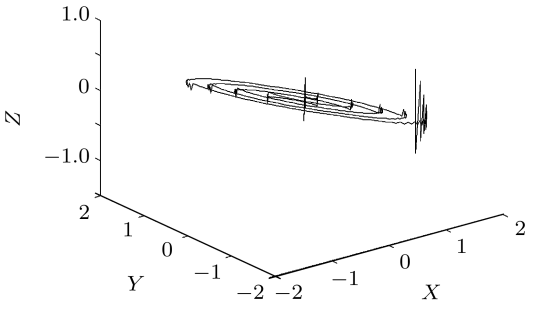


Fig.2. One-scroll chaotic attractor.

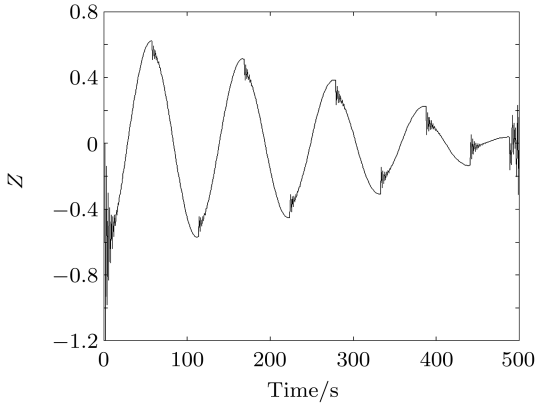


Fig.3. Phase trajectory of vector Z.

6. Conclusions

In the present paper, a new method to generate chaos for a class of nonlinear systems with backlash function is presented. Based on Silnikov theorems, it is verified that chaos can exist in nonlinear systems of that kind, provided that the parameters satisfy some conditions. It is also demonstrated that abundant complex dynamical behaviours can be generated if the system is designed appropriately.

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