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Gain/loss effect on a bright solitary wave in a cigar-shaped attractive condensate in the presence of an expulsive parabolic potential*

Wang Feng-Jiao(王凤姣)[a], Ding Jian-Wen(丁建文)[a],
Yan Xiao-Hong(颜晓红)[b], and Wang Deng-Long(王登龙)[a]

[a] *Department of Physics & Institute for Nanophysics and Rare-earth Luminescence, Xiangtan University, Xiangtan* 411105, *China*

[b] *College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing* 210016, *China*

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Taking into account both gain/loss and time-dependent atomic scattering length, this paper analytically derives an exact bright solitary wave in a cigar-shaped attractive condensate in the presence of an expulsive parabolic potential. Due to the balance of the scattering length and gain/loss, the bright solitary wave is shown to have constant amplitude. Especially, it is found that the bright solitary wave is accelerated by expulsive force, whose velocity can be modulated by changing the axial and transverse angular frequencies. The results are in good agreement with the experimental observations by Khaykovich *et al* (2002 Science **296** 1290).

Keywords: Bose–Einstein condensate, bright solitary wave, gain/loss
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1. Introduction

Matter wave solitons in Bose–Einstein condensates (BECs) have been the subject of intense theoretical and experimental interest in recent years. Dark and bright matter wave solitons have been experimentally observed in a repulsive and attractive condensate, respectively.[1−4] For the dark matter wave solitons, theoretically, the physical mechanisms have been well understood.[5−8] For a bright matter wave soliton, some interesting phenomena are newly observed, such as the emergence of the narrow window of atom number, constant amplitude, and the accelerated motion.[4] This greatly stimulates the investigating interest of both theoretical and experimental workers. In the presence of a tanh-shaped potential, for example, the window of atom number was estimated analytically, and the bright soliton was indicated to be static.[9] Experimentally, the bright soliton was observed to be accelerated,[4] which was thought to be due to the small offset between the maximum of the expulsive potential and the initial position of the atoms.[4,9] Also, it is simply explained by the motion of the soliton when it is projected to a slope of the expulsive potential.[4] This simple classical theory fits well the experimental data. However, it should be necessary to explore the problem based on the governed Gross–Pitaevskii (GP) equation. Here we present an alternative idea by analytical investigation, which may be helpful for the understanding of the accelerated motion mechanism and provide an instruction for the experimental workers.

At very low temperatures, weak interatomic interactions are usually dominated by s-wave scattering. Considering the ideal binary elastic collisions, the scattering length was usually taken to be a constant in previous works.[9,10] In fact, both the magnitude and the sign of the atomic scattering length can be changed in the current experiment by adjusting the external magnetic field via Feshbach resonance.[4] In a cigar-shaped attractive condensate, the time-dependent atomic scattering length was considered to explore the dynamics of a bright soliton.[11−14] In a true condensate system, in addition, there always exists a gain or loss in the condensate due to the two-body inelastic collision or three-body interactions be-

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tween the condensate and the thermal cloud.[15−17] In the presence of gain/loss, the dynamical equation of the BEC system is usually termed the so-called GP gain equation,[17] which is extensively used to explore soliton behaviours in a BEC system under certain conditions. Kengne and Talla investigated the bright solitary wave solution of the GP gain equation with constant scattering length and analysed the gain effect on the bright soliton.[18] By an appropriate transformation, the GP gain equation with time-dependent scattering length was reduced to that with constant scattering length.[17] Beginning with the GP gain equation with time-dependent scattering length, bright soliton trains were theoretically obtained in the presence of a confining oscillator potential[16] and expulsive parabolic potential[19] respectively. The result is closely related with the experimental observations in Ref.[3]. In Ref.[19], two families of solitary wave solutions of GP gain equation were also obtained by a direct method.

In this paper, by developing a simpler transformation, we derive the exact bright solitary wave in a cigar-shaped attractive condensate in the presence of a slight expulsive parabolic potential. The bright solitary wave is found to exhibit constant amplitude and accelerated motion, which are in good agreement with the experimental observations in Ref.[4].

2. Model and formulae

As is well known, the three-dimensional GP equation based on the mean field approximation governs the evolution of the macroscopic wave function of BECs at zero temperature. For a true cigar-shaped condensate, the macroscopic wave function in the axial direction obeys the following one-dimensional GP gain equation:[16,17]

$$\mathrm{i}\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{2}\lambda^2 z^2 \Psi + \frac{2a_{\mathrm{s}}(t)}{a_{\mathrm{B}}} |\Psi|^2 \Psi + \mathrm{i}\Gamma\Psi, \quad (1)$$

where a_{B} is the Bohr radius. The parameter $\lambda = |\omega_z|/\omega_\perp$ is very small for the limiting case of the cigar-shaped condensate with $\omega_z(\omega_\perp)$ the angular frequency of the harmonic trap in the axial (transverse) direction. It should be noted that in Eq.(1), the time t is measured in units of $1/\omega_\perp$, and the coordinate variable z is measured in units of the transverse harmonic-oscillator length $a_\perp$ with $a_\perp = [\hbar/(m\omega_\perp)]^{1/2}$ and m being the atom mass.

To consider the gain/loss effect on a condensate, we add the last feeding term at the right hand side of Eq.(1), where the small gain (loss) feeding coefficient $\Gamma > 0$ ($\Gamma < 0$) means that the condensate absorbs (releases) some atoms from (into) the background thermal cloud.[16,17] Due to the adjusting external magnetic field, the atomic scattering length may be time-dependent. Here, the atomic scattering length $a_{\mathrm{s}}(t)$ in Eq.(1) is taken to be $a_{\mathrm{s}}(t) = a_0 \exp(\gamma t)$[11,12] with γ the growing (decaying) coefficient for $\gamma > 0$ ($\gamma < 0$) and a_0 an initial atomic scattering length. It is generally believed that there exists a bright (dark) soliton in an attractive (repulsive) condensate. Here we are devoted to the study of bright solitary wave behaviours, so we focus on an attractive condensate with a negative atomic scattering length ($a_0 < 0$).

In the absence of the feeding term ($\Gamma = 0$), bright soliton solutions of Eq.(1) were obtained by Darboux transformation[12] and gauge transformation[11] respectively. In the presence of the feeding term ($\Gamma \neq 0$), Kengne and Talla presented a family of bright solitary wave solutions in the limiting case of constant scattering length ($\gamma = 0$).[18] These results show that, under a safe range of parameters, the bright soliton can be compressed into very high local matter densities by increasing the absolute value of the atomic scattering length or the value of the gain feeding coefficient. In the presence of both the time-dependent scattering length and gain/loss, we analytically derive the bright solitary wave solution of Eq.(1) by a direct and simple transformation. It is found that the peak amplitude of a bright solitary wave depends strongly on both the time-dependent atomic scattering length and gain/loss effect. Due to the balance of the two factors, the bright solitary wave is shown to have constant amplitude.

In order to obtain the exact solitary wave solution of Eq.(1), firstly we present an appropriate transformation

$$\Psi(z,t) = \sqrt{-\frac{a_{\mathrm{B}}}{2a_0}}\,\Phi(\xi,\eta)\exp\left(\left(\frac{1}{2}\lambda + \Gamma\right)t - \frac{\mathrm{i}}{2}\lambda z^2\right) \quad (2)$$

with

$$\xi = z\exp(\lambda t), \quad (3)$$

$$\eta = \frac{1}{2\lambda}\exp(2\lambda t). \quad (4)$$

Substituting Eqs.(2)–(4) into Eq.(1), we directly obtain the standard nonlinear Schrödinger equation (NLSE) of $\Phi(\xi, \eta)$ as follows,

$$\mathrm{i}\frac{\partial \Phi}{\partial \eta} + \frac{1}{2}\frac{\partial^2 \Phi}{\partial \xi^2} + |\Phi|^2 \Phi = 0. \tag{5}$$

Equation (5) is always self-consistent with a constraint of $\gamma = \lambda - 2\Gamma$. Obviously, this condition can be experimentally realized by adjusting the angular frequencies ω_z and/or $\omega_\perp$ or by changing the external magnetic field.

Now one only needs to solve the standard NLSE Eq.(5) instead of Eq.(1). Equation (5) can be integrated by the inverse scattering transform method (IST),[20] which has a bright solitary wave solution of the type

$$\begin{aligned}\Phi =\ & A\mathrm{sech}\left[A(\xi + v\eta - \xi_0)\right] \\ & \times \exp\left[-\mathrm{i}v\xi + \frac{\mathrm{i}}{2}(A^2 - v^2)\eta + \mathrm{i}\varphi_0\right],\end{aligned} \tag{6}$$

where A represents the amplitude as well as the width of the bright solitary wave, and φ_0 represents the phase constant. The arbitrary integration constants v and ξ_0 denote the velocity and centre of solitary wave, respectively. From Eqs.(2) and (6), the probability density of the condensate can be derived by

$$\begin{aligned}&|\Psi|^2 \\ =\ & -\frac{a_\mathrm{B}}{2a_0}A^2 \exp\left[(\lambda + 2\Gamma)t\right] \\ & \times \mathrm{sech}^2\left\{A\left[z\exp(\lambda t) + \frac{1}{2\lambda}v\exp(2\lambda t) - \xi_0\right]\right\}.\end{aligned} \tag{7}$$

The central position of bright solitary wave is then obtained by

$$z_\mathrm{c} = z_0 \exp(-\lambda t) - \frac{v}{\lambda}\sinh(\lambda t), \tag{8}$$

where $z_0 = z_\mathrm{c}(t = 0) = \xi_0 - v/(2\lambda)$ denotes the initial position.

The solitary wave velocity can also be derived from Eq.(7) by

$$v_\mathrm{c} = -\xi_0 \lambda \exp(-\lambda t) - \frac{v}{2}\exp(\lambda t). \tag{9}$$

For a cigar-shaped condensate, $|\omega_z| \ll \omega_\perp$ and thus $\lambda = |\omega_z|/\omega_\perp \ll 1$. So the case of $\lambda = 1$ is impossible in the cigar-shaped condensate. However, it is still interesting to note that in the case of $\gamma = 0$ and $\lambda = 1$, Eq.(7) is reduced into the result obtained in an optical fibre by symmetry group analysis.[21] From the methodology, this further verifies the validity of our method. Substituting $-\lambda$ for λ in Eqs.(2)–(4), in addition, a similar bright solitary wave can be obtained. In this case, the constraint is given by $\gamma = -\lambda - 2\Gamma$, which is not presented in Ref.[21].

3. Discussions

From Eq.(7), obviously, the functional form of probability density keeps the hyperbolic secant unchanged. The width of the bright solitary wave decreases by $W(t) = \exp(-\lambda t)/A$, while the peak amplitude varies by $A_\mathrm{m}(t) = -a_\mathrm{B}A^2\exp[(\lambda + 2\Gamma)t]/(2a_0)$. Especially, it is found that at $\Gamma = -\lambda/2$ and thus $\gamma = 2\lambda$, the peak amplitude keeps unchanged, $A_\mathrm{m} = -a_\mathrm{B}A^2/(2a_0)$. To further explore the feeding effect on the bright solitary wave, Fig.1 shows the evolution of the peak amplitude at a given λ but various feeding coefficients Γ. In the limiting case of $\lambda = 0.1$[4] with the constraint of $\gamma = \lambda - 2\Gamma$, it is seen from Fig.1 that at $\Gamma > -0.05$, the peak amplitude increases with time (see the upper two curves of $\Gamma = -0.02$ and 0.01) while it decreases at $\Gamma < -0.05$ (see the lower two curves of $\Gamma = -0.11$ and -0.08). Obviously, the critical feeding coefficient is obtained to be $\Gamma = -\lambda/2 = -0.05$ and $\gamma = 2\lambda = 0.2$ at $\lambda = 0.1$, corresponding to a constant peak amplitude (see the middle curve of $\Gamma = -0.05$). The result is in good agreement with the experimental observations.[4] As for the bright solitary wave solution of $\lambda \to -\lambda$, similar results can be obtained. In this case, the critical condition of a constant peak amplitude is obtained to be $\Gamma = \lambda/2$ and thus $\gamma = -2\lambda$. The existence of the critical situation can be understood by the following consideration.

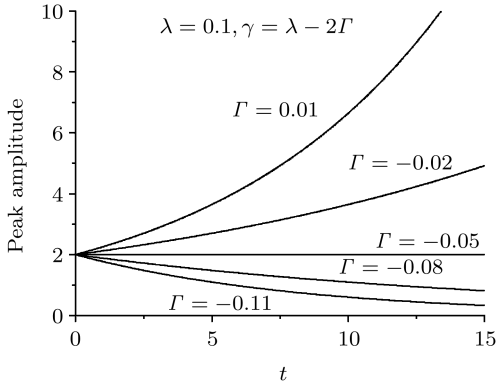


Fig.1. Peak amplitude of probability density as a function of propagation time at $\lambda = 0.1$ and various Γ in the constraint of $\gamma = \lambda - 2\Gamma$. t is in units of $1/\omega_\perp$ and z in units of $a_\perp$. The other parameters are taken to be $a_0 = -0.21$ nm and $A = 4.0$.

The gain/loss will lead to a fluctuation of the density and energy of the atoms.[15] It is found that the bright soliton can be compressed into very high local matter densities by increasing the absolute value of the atomic scattering length[12] or the value of the gain feeding coefficient.[18] Therefore, the peak amplitude tends to decrease (increase) at a loss (gain) effect $\Gamma < 0$ ($\Gamma > 0$), while it tends to increase (decrease) at a growing (decaying) atomic scattering length $\gamma > 0$ ($\gamma < 0$). In the presence of both time-dependent atomic scattering length and the gain/loss effect, there exists competition between the two factors. The peak amplitude increases at $\Gamma > -\gamma/4$ and decreases at $\Gamma < -\gamma/4$. Importantly, there exists a balance at $\Gamma = -\gamma/4$ between a loss (gain) effect and a growing (decaying) atomic scattering length, which results in a constant peak amplitude. Therefore, it is shown that the peak amplitude can be controlled by adjusting the external magnetic field or changing the background cloud.

In the absence of both expulsive parabolic potential ($\lambda = 0$) and gain/loss ($\Gamma = 0$),[10,22] the dispersion and the attractive force can be balanced with each other. For a constant scattering length a_0 ($\gamma = 0$), a stable bright soliton will appear and propagate with a constant velocity. Taking into account the feeding term ($\Gamma \neq 0$) and time-dependent scattering length ($\gamma \neq 0$), in Fig.2 we further explore the dynamical properties of the bright solitary wave in the presence of the expulsive parabolic potential ($\lambda \neq 0$). In the calculations, we have restricted our study to a safe range of parameters as used in the experiment.[4] Here we take $\lambda = 0.1$, corresponding to the experimental parameters $\omega_z = \mathrm{i}2\pi\times70$ Hz and $\omega_\perp = 2\pi\times710$ Hz. For the real time and spatial variables, a unity of time, $\Delta t = 1$ in the dimensionless variables corresponds to $1/\omega_\perp = 0.224$ ms, and a unity of coordinate, $\Delta x = 1$ in the dimensionless variables corresponds to $a_\perp = [\hbar/(m\omega_\perp)]^{1/2} = 1.4$ μm. In the experiment, there exists a small offset of about 50 μm between the maximum of the expulsive potential and the initial position of the atoms, corresponding to a length of about 35 in the dimensionless units, so we take $z_0 = -35$. From Fig.2, a bright solitary wave is found to propagate towards one longitudinal direction, differing from the oscillating property of the growing condensate in a confinable harmonic potential.[23] Especially, it is shown that the velocity increases with time at a varying acceleration, while $v_\mathrm{c} \approx 0$ at $t = 0$.[9] The central position z_c of the solitary wave is obtained to be -0.08, -0.43, -0.67 and -1.05 mm at $t = 2$, 6, 7 and 8 ms respectively. The results are different from the stable nonlinear solitary waves[24] but in good agreement with the experimental observations.[4] Under a safe range of parameters, in fact, the attractive interactions together with gain/loss effect can balance the effects of both the dispersion and the expulsive parabolic potential. As a result, there exists a local energy minimum, which supports a macroscopic quantum bound state.[4] For an expulsive parabolic potential, an expulsive force exists. Due to a potential difference, the soliton may be accelerated, just like a particle. The velocity of the bright solitary wave depends on the ratio of the axial frequency to the transverse frequency but is independent of both the gain/loss and interatomic interactions.

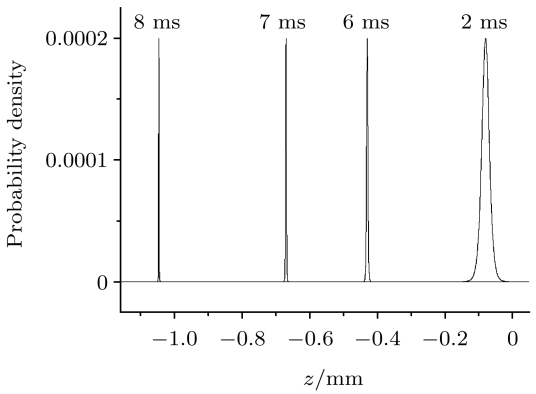


Fig.2. Propagation of a bright solitary wave at λ=0.1 with $\Gamma = -0.05$ and $\gamma = 0.2$. The parameters are taken to be $a_0 = -0.21$ nm, $A = 0.04$, $v = 4.2$ and $z_0 = -35$.

The bright solitary wave propagates with constant peak and the varying width can be explained as follows. In the absence of gain/loss ($\Gamma = 0$), from Eq.(7), the probability density peak increases exponentially with time and the width decreases, but the number of atoms in the bright solitary wave keeps invariant. This verifies that the time-dependent nature of the atomic scattering length does not change the number of atoms in a bright soliton if there is no reservoir.[17] If the feeding coefficient Γ takes a nonzero value, exchange of the atoms between the bright solitary wave and the background thermal cloud occurs. In the case of $\Gamma = -\gamma/4$, as discussed above, the peak of the probability density keeps a constant due to the balance between the atomic scattering length and the gain/loss. The negative (positive) feeding coefficient $\Gamma = -\lambda/2$ ($\Gamma = \lambda/2$) means that the condensate releases (absorbs) some atoms into (from) the background thermal cloud, and thus the width

of the bright solitary wave decreases (increases) with time.

4. Conclusions

Taking into account both the gain/loss and time-dependent atomic scattering length, we have studied analytically the bright solitary wave in a cigar-shaped attractive condensate in the presence of an expulsive parabolic potential. It is shown that the peak amplitude of the bright solitary wave depends strongly on both the time-dependent atomic scattering length and gain/loss. Due to the balance of the scattering length and gain/loss, the bright solitary wave is shown to have constant amplitude. Especially, in the presence of an expulsive parabolic potential, it is found that the bright solitary wave is accelerated and set into motion, and the accelerated speed depends only on the ratio of axial frequency to transverse frequency. The results are in good agreement with the experimental observations. The method and results can be helpful to develop effective control for bright solitary wave behaviours in future BEC experiments.

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