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# Mei conserved quantity of the Nielsen equation for a non-Chetaev-type non-holonomic system*

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The Mei symmetry and Mei conserved quantity of the Nielsen equation for a non-Chetaev-type non-holonomic non-conservative system are studied. The differential equations of motion of the Nielsen equation for the system, the definition and the criterion of Mei symmetry and the condition and the form of Mei conserved quantities deduced directly from the Mei symmetry for the system are obtained. Finally, an example is given to illustrate the application of the results.

**Keywords:** non-Chetaev-type non-holonomic system, Nielsen equation, Mei symmetry, Mei conserved quantity

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## 1. Introduction

In recent years, the theoretical research on symmetries and conserved quantities has aroused the interest of many scholars in the area of analytical mechanics. Great development has been achieved in the fields of Noether symmetry,[1,2] Lie symmetry,[3−6] Mei symmetry[7,8] and the Noether conserved quantity,[1,2] Hojman conserved quantity,[9] and Mei conserved quantity.[10,11] Many achievements have been made in the three main symmetries and conserved quantities of mechanical systems.[12−23] At present, analytical mechanics has been widely applied in the fields of engineering and technology[16,17] and has facilitated the development of mathematical methods such as calculus of variations and the geometric method.[24−27] The relation between the form invariance of the Nielsen equation and the Noether symmetry in Ref.[28], the form invariance of the Nielsen equation of a non-conservative mechanical system in Ref.[29], a non-Noether conserved quantity constructed directly by using the form invariance for the Nielsen equation of a non-conservative mechanical system in Ref.[30], and the Mei symmetry and the Mei conserved quantity of Nielsen equation for a non-holonomic system in Ref.[31] have been studied.

In the present paper, the definition and the criterion of the Nielsen equation for a non-Chetaev-type non-holonomic non-conservative system are given. In addition, by studying the Mei symmetry, both the conditions and the expression of Mei conserved quantities of the Nielsen equation for a non-Chetaev-type non-holonomic non-conservative system are obtained. Finally, an example is given to illustrate the application of the results and some conclusions are drawn from the present study.

## 2. Equations of motion for the system

Assume the configuration of the system to be determined by $n$ generalized coordinates $q_s (s = 1, 2, \ldots, n)$ and the system to be subjected to $g$ non-Chetaev-type non-holonomic constraints

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad (\beta = 1, \ldots, g). \tag{1}$$

Consider the non-Chetaev condition, of which the constraint Eq.(1) imposed on virtual displacements is

$$f_{\beta s}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\delta q_s = 0, \quad (\beta = 1, \ldots, g), \tag{2}$$

where the unified subscript denotes the Einstein summation convention in the above equations as well as

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in the following text. In general, $f_{\beta s}$ and $\dfrac{\partial f_\beta}{\partial \dot{q}_s}$ have no relation. In particular, if

$$f_{\beta s} = \frac{\partial f_\beta}{\partial \dot{q}_s},$$

then the constraints will degenerate to Chetaev-type non-holonomic constraints. By using general Lagrangian multipliers, the differential equations of motion of the system are established according to the D'Alembert–Lagrange principle and Eq.(2) becomes

$$\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t} - 2\frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta f_{\beta s}, \quad (s = 1, \ldots, n), \quad (3)$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian function of the system; $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is a non-potential generalized force; and $\lambda_\beta$ are Lagrangian multipliers. Equation (3) is called the Nielsen equation for a non-Chetaev-type non-holonomic non-conservative system.

Introducing Mei operators

$$N_s = \frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s}, \quad (s = 1, \ldots, n), \quad (4)$$

then Eq.(3) can be expressed as

$$N_s(L) = Q_s + \lambda_\beta f_{\beta s}, \quad (s = 1, \ldots, n). \quad (5)$$

Before integrating the differential equations of motion of the system, from Eqs.(1) and (5) all values of $\lambda_\beta = \lambda_\beta\,(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ can be found. Therefore, Eq.(5) can also be expressed as

$$N_s(L) = Q_s + \Lambda_s, \quad (s = 1, \ldots, n), \quad (6)$$

where

$$\Lambda_s = \Lambda_s(t, q, \dot{q}) = \lambda_\beta f_{\beta s}, \quad (s = 1, \ldots, n) \quad (7)$$

with $\Lambda_s$ being called the generalized non-holonomic constraints. Equation (6) is the Nielsen equation for a holonomic non-conservative system corresponding to Eqs.(1) and (3) for the non-Chetaev-type non-holonomic non-conservative system. If the initial condition of motion satisfies Eq.(1) of the non-Chetaev-type non-holonomic constraints, the solution of Eq.(6) will give the orbit of motion of the system above. Considering the system to be non-singular and using Eq.(6), all generalized accelerations can be obtained as

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, \ldots, n). \quad (8)$$

# 3. Definition of Mei symmetry

Introducing the infinitesimal transformations of groups of time and the generalized coordinates

$$t^* = t + \Delta t, \quad \text{and} \quad q_s^*(t^*) = q_s(t) + \Delta q_s, \quad (9)$$

the transformation (9) can be extended to

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),$$

and

$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (10)$$

where $\varepsilon$ is an infinitesimal parameter; $\xi_0$ and $\xi_s$ are infinitesimal generators.

Introducing a vector $X^{(0)}$ of the infinitesimal generators

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} \quad (11)$$

and its first expansion $\tilde{X}^{(1)}$ along the trajectory of motion of the system

$$\tilde{X}^{(1)} = X^{(0)} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}, \quad (12)$$

the total derivative along the trajectory of the system with respect to $t$ in Eq.(12) is

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s}. \quad (13)$$

After transformations (10), the functions $L$, $Q_s$, $\Lambda_s$ and $f_\beta$ respectively become $L^*$, $Q_s^*$, $\Lambda_s^*$ and $f_\beta^*$. Taking the Taylor expansions of $L^*$, $Q_s^*$, $\Lambda_s^*$ and $f_\beta^*$ at a point of $(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ along the trajectory of motion of the system, we have

$$\begin{aligned} L^* &= L(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}) \\ &= L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{X}^{(1)}(L) + O(\varepsilon^2), \end{aligned} \quad (14)$$

$$\begin{aligned} Q_s^* &= Q_s(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}) \\ &= Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{X}^{(1)}(Q_s) + O(\varepsilon^2), \\ &(s = 1, \ldots, n), \end{aligned} \quad (15)$$

$$\begin{aligned} \Lambda_s^* &= \Lambda_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \\ &= \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{X}^{(1)}(\Lambda_s) + O(\varepsilon^2), \\ &(s = 1, \ldots, n), \end{aligned} \quad (16)$$

$$\begin{aligned} f_\beta^* &= f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \\ &= f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{X}^{(1)}(f_\beta) + O(\varepsilon^2), \\ &(\beta = 1, \ldots, g). \end{aligned} \quad (17)$$

**Definition 1** If the dynamic functions $L^*$, $Q_s^*$ and $\Lambda_s^*$, which are transformed by Eq.(6), are used respectively to replace the dynamic functions $L$, $Q_s$ and $\Lambda_s$ which are those before transformation, Eq.(6) remains unchanged in form, namely,

$$N_s(L^*) = Q_s^* + \Lambda_s^*, \quad (s = 1, \ldots, n). \tag{18}$$

Hence, this symmetry is called the Mei symmetry for a holonomic non-conservative system corresponding to the Nielsen equation, i.e. Eqs.(1) and (3), which belong to the non-Chetaev-type non-holonomic non-conservative system.

**Definition 2** If the dynamic functions $L^*$, $Q_s^*$, $\Lambda_s^*$ and $f_\beta^*$, which are transformed by Eq.(6), are used respectively to replace the dynamic functions $L$, $Q_s$, $\Lambda_s$ and $f_\beta$ which are those before transformations, Eqs.(1) and (6) remain unchanged in form, namely

$$f_\beta^* = f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = 0, \quad (\beta = 1, \ldots, g). \tag{19}$$

If Eqs.(18) and (19) hold true at the same time, this type is called the loose Mei symmetry of the Nielsen equation, i.e. Eqs.(1) and (3) which belong to the non-Chetaev-type non-holonomic non-conservative system.

It can be easily proved that restriction equations (2) of which the constraint equation (1) is imposed on the virtual displacement $\delta q_s$ can be rewritten in the following form

$$f_{\beta s}(\xi_s - \dot{q}_s \xi_0) = 0 \quad (\beta = 1, \ldots, g). \tag{20}$$

Equation (20) is called the additional restriction equation.

**Definition 3** If the dynamic functions $L^*$, $Q_s^*$, $\Lambda_s^*$ and $f_\beta^*$, which are transformed by Eq.(6), are used respectively to replace the dynamic functions $L$, $Q_s$, $\Lambda_s$ and $f_\beta$, which are those before transformations, Eqs.(1) and (6) remain unchanged in form. Also, if they require the additional restriction equations of non-Chetaev-type restriction equations under the infinitesimal transformation generators $\xi_0$ and $\xi_s$ to hold true, then this kind of symmetry will be called the strict Mei symmetry of the Nielsen equation, i.e. Eqs.(1) and (3) which belong to the non-Chetaev-type non-holonomic non-conservative system.

## 4. Criterion of Mei symmetry

Substituting Eqs.(14), (15) and (16) into Eq.(18) and neglecting terms that are of the order of $O(\varepsilon^2)$ or of higher orders, and using Eq.(6), we obtain

$$N_s[\tilde{X}^{(1)}(L)] = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(\Lambda_s), \quad (s = 1, \ldots, n). \tag{21}$$

Finding the total derivative of the dynamic function along the trajectory of the system with respect to time by using the Mei operators, the Mei operators of the system may change from expression (4) into the following expression:

$$\tilde{N}_s = \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s}, \quad (s = 1, \ldots, n), \tag{22}$$

where $\tilde{N}_s$ is called the generalized Mei operator. Hence, we have

$$\tilde{N}_s[\tilde{X}^{(1)}(L)] = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(\Lambda_s), \quad (s = 1, \ldots, n). \tag{23}$$

Substituting Eq.(17) into Eq.(19), neglecting terms that are of the order of $O(\varepsilon^2)$ or of higher orders, and using Eq.(1), we obtain

$$\tilde{X}^{(1)}(f_\beta) = 0, \quad (\beta = 1, \ldots, g). \tag{24}$$

Then we have the following criteria.

**Criterion 1** For the Nielsen equation (6), if the infinitesimal generators $\xi_0$ and $\xi_s$ under infinitesimal transformations (10) satisfy expression (22), then the relevant symmetry is called the Mei symmetry of the system.

Equation (23) is called the criterion equation of the Mei symmetry for a holonomic non-conservative system corresponding to the Nielsen equation, i.e. Eqs.(1) and (3) which belong to the non-Chetaev- type non-holonomic non-conservative system.

**Criterion 2** For the Nielsen equation (6), if the infinitesimal generators $\xi_0$ and $\xi_s$ under the infinitesimal transformations (10) satisfy Eqs.(22) and (23), then the relevant symmetry is called the loose Mei symmetry of the system.

Equations (23) and (24) are respectively called the criterion equation and the constraint equation of the Mei symmetry for a holonomic non-conservative system corresponding to the Nielsen equation, i.e. (1) and (3) which belong to the non-Chetaev-type non-holonomic non-conservative system.

**Criterion 3** For the Nielsen equation (6), if the infinitesimal generators $\boldsymbol{\xi_0}$ and $\xi_s$ under the infinitesimal transformations (10) satisfy Eqs.(20), (23) and (24), then the relevant symmetry is called the strict Mei symmetry of the system.

Equation (20) is called the additional restrictive equation of the Mei symmetry for a holonomic

non-conservative system corresponding to the Nielsen equation, i.e. Eqs.(1) and (3) which belong to the non-Chetaev-type non-holonomic non-conservative system.

## 5. Mei conserved quantity deduced from Mei symmetry

The Mei conserved quantity can be deduced directly from the Mei symmetry of the Nielsen equation for a non-Chetaev-type non-holonomic non-conservative non-singular system. The condition and the form of the Mei conserved quantity are deduced directly from the following proposition.

**Proposition 1** If the infinitesimal generators $\xi_0$ and $\xi_s$, and the gauge function $G_{\rm M}=G_{\rm M}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ of the Mei symmetry, i.e. the loose Mei symmetry and the strict Mei symmetry of the Nielsen equation for a non-singular non-Chetaev-type non-holonomic non-conservative system satisfy the following structural equation:

$$\tilde{X}^{(1)}(L)\frac{\bar{\rm d}}{{\rm d}t}\xi_0+\tilde{X}^{(1)}[\tilde{X}^{(1)}(L)]+\tilde{X}^{(1)}(Q_s+\Lambda_s)(\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\rm d}}{{\rm d}t}G_{\rm M}=0, \tag{25}$$

then the Mei conserved quantities deduced from the Mei symmetries are expressed as

$$I_{\rm M}=\tilde{X}^{(1)}(L)\xi_0+\frac{\partial\tilde{X}^{(1)}(L)}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G_{\rm M}={\rm const.} \tag{26}$$

**Proof**

Substituting Eq.(26) into expression (13), and using Eq.(25), we have

$$\begin{aligned}\frac{\bar{\rm d}}{{\rm d}t}I_{\rm M}=&\left[\frac{\partial}{\partial t}\tilde{X}^{(1)}(L)+\dot{q}_s\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L)+\alpha_s\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\right]\xi_0\\&+\tilde{X}^{(1)}(L)\frac{\bar{\rm d}}{{\rm d}t}\xi_0+\left[\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\right](\xi_s-\dot{q}_s\xi_0)\\&+\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\left[\frac{\bar{\rm d}}{{\rm d}t}\xi_s-\alpha_s\xi_0-\dot{q}_s\frac{\bar{\rm d}}{{\rm d}t}\xi_0\right]+\frac{\bar{\rm d}}{{\rm d}t}G_{\rm M}\\=&\tilde{X}^{(1)}[\tilde{X}^{(1)}(L)]-\xi_s\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L)+\dot{q}_s\xi_0\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L)\\&+\tilde{X}^{(1)}(L)\frac{\bar{\rm d}}{{\rm d}t}\xi_0+\frac{\bar{\rm d}}{{\rm d}t}G_{\rm M}+\left[\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\right](\xi_s-\dot{q}_s\xi_0)\\=&\left[\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)-\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L)-\tilde{X}^{(1)}(Q_s+\Lambda_s)\right](\xi_s-\dot{q}_s\xi_0).\end{aligned} \tag{27}$$

Substituting the criterion equation (21) into Eq.(27) yields

$$\frac{\bar{\rm d}}{{\rm d}t}I_{\rm M}=\left\{\left[\left(\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial}{\partial\dot{q}_s}-\frac{\partial}{\partial q_s}\right)-\tilde{N}_s\right]\tilde{X}^{(1)}(L)\right\}(\xi_s-\dot{q}_s\xi_0)=0. \tag{28}$$

## 6. Example

Consider a Lagrangian function of the system that is expressed as

$$L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2)-q_2, \tag{29}$$

and the non-holonomic constraint subjected to the system that is given as

$$f=\dot{q}_1-t\dot{q}_2=0. \tag{30}$$

Assuming that the constraint equation (30) is of the non-Chetaev type and the generalized non-potential forces equal zero, then the virtual displacement will be

$$\delta q_1-\delta q_2=0. \tag{31}$$

Now we come to study the Mei conserved quantity deduced directly from the Mei symmetry of the Nielsen equation for a non-Chetaev-type non-

holonomic non-conservative system.

Performing the expansion of Eq.(2) and comparing with Eq.(31), we have

$$f_{11} = 1 \quad \text{and} \quad f_{12} = -1. \tag{32}$$

From Eq.(3), we have

$$\ddot{q}_1 = \lambda, \quad \text{and} \quad \ddot{q}_2 = -(\lambda + 2). \tag{33}$$

From Eqs.(30) and (32), we obtain

$$\lambda = \frac{\dot{q}_2 - 2t}{1 + t}. \tag{34}$$

Taking notice of Eqs.(7) and (8) and substituting Eq.(34) into Eq.(33), the differential equations of motion of the system are obtained as follows:

$$\ddot{q}_1 = \alpha_1 = \frac{\dot{q}_2 - 2t}{1 + t}, \quad \ddot{q}_2 = \alpha_2 = -\frac{\dot{q}_2 + 2}{1 + t},$$

and

$$\Lambda_1 = -\Lambda_2 = \alpha_1. \tag{35}$$

Take the infinitesimal generators as follows:

$$\xi_0 = 0, \quad \text{and} \quad \xi_1 = \xi_2 = \dot{q}_1 + \dot{q}_2 + 2t, \tag{36}$$

then we will have

$$\begin{aligned} &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 = 0, \quad \tilde{X}^{(1)}(f) = 0, \\ &\tilde{X}^{(1)}(L) = -(\dot{q}_1 + \dot{q}_2 + 2t), \\ &\tilde{X}^{(1)}(\Lambda_1) = \tilde{X}^{(1)}(\Lambda_2) = 0, \\ &\tilde{N}_1\left[\tilde{X}^{(1)}(L)\right] = \tilde{N}_2\left[\tilde{X}^{(1)}(L)\right] = 0. \end{aligned} \tag{37}$$

Using Eq.(37), we can easily verify the criterion equation (23), the constraint equation (24) and the additional restriction equations (20). By virtue of criteria 1 to 3, we know that the infinitesimal generators in Eq.(36) are of Mei, loose Mei and strict Mei symmetries of the Nielsen equation for a non-Chetaev-type non-holonomic system.

From the structural equation (25), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{M}} = 0. \tag{38}$$

Selecting

$$G_{\mathrm{M}} = 0, \tag{39}$$

and using expression (26), the non-Noether conserved quantity i.e. the Mei conserved quantity deduced directly from the Mei symmetry of the Nielsen equation for a non-Chetaev-type non-holonomoc system will be obtained as

$$I_{\mathrm{M}} = \dot{q}_1 + \dot{q}_2 + 2t = \text{const.} \tag{40}$$

# 7. Conclusions

In the present paper we adopt a method of finding the total derivative along the trajectory of the system with respect to $t$, and we also present the definition and the criterion of the Mei symmetry of the Nielsen equation for a non-Chetaev-type non-holonomic non-conservative mechanical system. From the mathematical angle, it is helpful to obtain more infinitesimal generators of the Mei symmetry by using the above method. However, whether new non-ordinary Mei symmetry can be obtained using this method needs further investigation. In the present paper, the Mei conserved quantity is studied and both the condition and the expressions of the Mei conserved quantity deduced directly from the Mei symmetry of the Nielsen equation for a non-Chetaev-type non-holonomic non-conservative system are also obtained. The main results are criteria 1 to 3 and proposition 1.

From Eqs.(3), (5), (6), (18), (23) and (25) it follows that if the values of generalized non-potential force $Q'_s$ are all equal to zero, the obtained conclusions in this paper will turn into the conclusions of the Mei symmetry of the Nielsen equation for a non-Chetaev-type non-holonomic conservative system; if the values of the generalized non-holonomic constraint $\Lambda_s$ are all equal to zero, the obtained conclusions in this paper will be applicable to the Mei symmetry of the Nielsen equation for a holonomic non-conservative system; if all the values of generalized non-potential force $Q'_s$ and also all the values of generalized non-holonomic constraint $\Lambda_s$ are equal to zero, the obtained conclusions in this paper will hold for the Mei symmetry of the Nielsen equation for a holonomic conservative system. Therefore, the obtained conclusions above are of more universal significance.

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