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Evolution analysis of the states of the EZ model*

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Based on suitable choice of states, this paper studies the stability of the equilibrium state of the EZ model by regarding the evolution of the EZ model as a Markov chain and by showing that the Markov chain is ergodic. The Markov analysis is applied to the EZ model with small number of agents, the exact equilibrium state for $N = 5$ and numerical results for $N = 18$ are obtained.

Keywords: EZ model, Markov chain, stationary distribution, equilibrium state
PACC: 8900

1. Introduction

Eguíluz and Zimmermann[1] presented a dynamic stochastic model (henceforth referred to as the EZ model) for opinion cluster formation and information dispersal by agents in a network. It is a dynamical model in which continual grouping and re-grouping of agents form clusters. This model is simple but significant, and it has received much attention these years.[2−11]

The basic model considers a population of N agents. An agent can be connected to any of the $N-1$ other agents. The connectivity has the following properties. If agent A is connected to B, then B is connected to A. If agent A is connected to B and agent B is connected to C, then A is connected to C. The connected agents form one cluster. At the beginning, all agents are not connected. At each step, an agent A is selected at random according to uniform distribution. With probability a, the connections in the cluster containing A are broken and all agents in this cluster become isolated agents. With probability $1-a$, another agent B is selected at random according to uniform distribution. If agents A and B belong to two different clusters, then all the agents in the two clusters are connected together to form a bigger cluster. If agents A and B happen to belong to the same cluster, no action is carried out and we go to the next step.[2]

In the EZ model, a cluster of agents carry out the same action, i.e., buy or sell, with probability a and the cluster dissolves after a transaction is made. For large N and appropriate a, EZ model can mimic the crowding effect in financial markets which arises when groups of agents share information and capture the herd behaviour in financial market.[1,3] Herding assumes some degree of coordination between groups of agents. This coordination may arise from the interaction and communication among agents. EZ model is an extension of the static percolation model proposed by Cont and Bouchaud.[12]

D'Hulst and Rodgers[3] developed a mean-field analysis on the EZ model and found an analytic expression for the size distribution of the clusters; the expression is just the steady state of a master equation. Their analytical expression is valid when N is sufficiently large and $a \gg 1/\sqrt{N}$.[2] Xie *et al* studied finite-size effect of EZ model for sufficiently large N and $a \ll 1/N \ln N$.[2] In the mean-field analysis one deals with a (series of) master equation, which describes the evolution of states in the model. It is difficult to obtain the exact solution of the master equation. One has to introduce some additional conditions on a and N (for example N sufficiently large and $a \ll 1/N \ln N$[2]), so that some approximate equations are valid.

The EZ model with numerous agents is used to describe the herd behaviour in financial markets. But the EZ model with small number of agents (between 5 and 30) can be used to model some businesses in

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which some big companies' behaviour plays an essential role, for example the reinsurance market, bidding, transactions involving famous sports players, etc. One can consider the efficiency of the business and the self-organization among the agents.

In the present work, we study the stability of the steady state of the master equation describing the evolution of the EZ model, which is always assumed to be true in all simulation of the model. This is achieved by regarding the evolution of the EZ model as a Markov chain. For fixed N, if we select the states from EZ model suitably, the stochastic process is a Markov chain with finite states. The evolution of the Markov chain is determined by the state transition matrix and the initial state.[13,14] According to the ergodic theorem of the irreducible Markov chain, the stationary distribution of a Markov chain is determined by its state transition matrix. It is interesting to note that we can obtain the exact solution of the master equation in mean-field analyses from stationary distribution of the Markov chain. We present the transition matrix of EZ model for fixed N. We obtain the analytical expression of stationary distribution and equilibrium state of the EZ model when $N = 5$. We can also calculate the stationary distribution and equilibrium state for $N = 18$. This kind of calculation enables us to know how far the stationary distribution which we obtained from the exact stationary distribution. It is difficult for the previous analyses by mean-field theory and simulation to obtain the error estimation. Our method can be applied to all the modifications of the EZ model.[4−6,9]

The organization of this paper is as follows. In Section 2, we specify the microstates of the EZ model and describe the number of microstates in the system for different N. In Section 3 and Section 4, we consider the transition matrix of the system and give the stationary distribution. In Sections 5 and 6, we give the analytical solution for the case $N = 5$ and some numerical results for $N = 18$ respectively. We summarize and discuss our results in Section 7.

2. The description of states

At each step, the system presents some clusters including one or many agents. This state can be written as $[n_1, n_2, \ldots, n_i, \ldots, n_N]$, where n_i $(0 < i \leq N)$ is the number of clusters of size i. The states are confined to $\sum_{i=1}^{N} in_i = N$.

The number of states of the system can be obtained from known mathematical results in number theory. Consider the Diophantine equation (an equation to be solved in nonnegative integers)

$$\sum_{i=1}^{N} x_i = N, \quad x_i \in [0, N] \quad (i = 1, \ldots, N). \tag{1}$$

Any solution of Eq.(1) corresponds to a state of the system, and also corresponds to a partition of N. Suppose $[x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N]$ is a solution of Eq.(1), $[x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N]$ is also a solution of Eq.(1) because $\{x_i\}_{i=1}^{N}$ are symmetry in this equation. These two solutions are regarded as the same solution. They induce the same partition of N, correspond to the same microstate in EZ model.

For example, for $N = 5$, equation (1) has seven solutions as follows:

Solution of equation (1)	State of EZ model
$1, 1, 1, 1, 1$	$[5, 0, 0, 0, 0]$
$1, 1, 1, 2$	$[3, 1, 0, 0, 0]$
$1, 2, 2$	$[1, 2, 0, 0, 0]$
$1, 1, 3$	$[2, 0, 1, 0, 0]$
$1, 4$	$[1, 0, 0, 1, 0]$
$2, 3$	$[0, 1, 1, 0, 0]$
5	$[0, 0, 0, 0, 1]$.

For given N, equation (1) admits finite distinct solutions. We denote the number of distinct solutions of Eq.(1), i.e. the number of microstates in the EZ model, by $k := S(N)$. Table 1 presents the number of states for different agents N.

Table 1. The number of microstates for different N.

N	2	3	4	5	6	7
k	2	3	5	7	11	15
N	8	9	10	11	12	13
k	22	30	42	56	77	101
N	14	15	16	17	18	19
k	135	176	231	297	385	490
N	20	30	40	50	100	
k	627	5604	37338	204226	190569292	

From Table 1, we can see that the number of states grows quickly as N increases. In fact, there is an approximated formula for large N[15],

$$k \approx \frac{1}{4N\sqrt{3}} \exp\left(\pi\sqrt{\frac{2N}{3}}\right) \quad \text{as} \quad N \to \infty.$$

3. The evolution of states

During the evolution of the system, old cluster may be dissolved and new cluster will be formed. The system evolves from one state to another state. Suppose the current state is $[n_1, n_2, \ldots, n_N]$. An agent A is selected randomly and A belongs to a cluster including i $(i = 1, 2, \ldots, N)$ agents. An agent B is selected randomly and B belongs to a cluster including j $(j = 1, 2, \ldots, N)$ agents. The evolution of the system can be described by the following five cases:

- $[n_1, n_2, \cdots, n_N]$, if $i = 1$ and the first cluster is dissolved, or the first cluster is not dissolved and A, B belong to the same cluster, then the probability is
$$\frac{n_1}{N}a + \sum_{i=1}^{N} \frac{in_i}{N}\frac{i}{N}(1-a).$$
- $[n_1 + i, n_2, \cdots, n_i - 1, \cdots, n_N]$, if $i \neq 1$ and the first cluster is dissolved, then the probability is
$$\frac{in_i}{N}a.$$
- $[n_1, \cdots, n_i - 2, \cdots, n_{2i} + 1, \cdots, n_N]$, if the first cluster is not dissolved and A and B belong to different cluster with length $i = j$, then the probability is
$$\frac{in_i}{N}\frac{i(n_i-1)}{N}(1-a).$$
- $[n_1, \cdots, n_i - 1, \cdots, n_j - 1, \cdots, n_{i+j} + 1, \cdots, n_N]$, if the first cluster is not dissolved and $(i \neq j)$, then the probability is
$$\frac{in_i}{N}\frac{jn_j}{N}(1-a).$$
- Otherwise, the probability is 0.

If we know the state of system in step n, the possible state in step $n + 1$ and the probabilities can be calculated by the above five cases. So the evolution of the system satisfies the Markov property. And the states evolution of EZ model is Markov chain with finite states, although the number of states is often very large.

The evolution of Markov chain can be described by its initial state and its state transition matrix. For fixed N, there are k states. The state transition matrix is a $k \times k$ matrix. And the matrix can be computed from the above five cases.

Suppose the state transition matrix of the system is
$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1k} \\ m_{21} & m_{22} & \cdots & m_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k1} & m_{k2} & \cdots & m_{kk} \end{bmatrix},$$
where m_{ij} is the transition probability from state s_i to state s_j. An easy observation from the above five cases is $M(a) = M_0 a + M_1$, where M_0 and M_1 are matrix independent of the parameter a.

4. Stationary distribution

For given N and a, the evolution process of the EZ model can be described by a Markov chain with finite states. We conclude that the Markov chain is *irreducible*, which implies that it is *ergodic* and admits a unique stationary distribution. Let s_1 be the state which consists of N isolated agents. As the evolution of the EZ model, any state s_i can be formed by s_1 in finite steps. So there exists a positive integer r such that $m_{1i}^r > 0$, i.e. after r steps, the transition probability from state s_1 to state s_i is positive. On the other hand, the state s_j can be dissolved into s_1 with positive probability. It follows that for any pair of s_i and s_j we have $s_j \to s_1$ and $s_1 \to s_j$, i.e. any pair of states in EZ model are communicated. So the Markov chain is irreducible. According to the ergodic theorem of Markov chain,[13] every irreducible Markov chain with finite states is ergodic and admits a unique stationary distribution $[p_1, p_2, \ldots, p_k]$, where $p_1, p_2, \ldots, p_k$ $(\sum_{i=1}^{k} p_i = 1)$ are average frequencies of the occurrence of the states $s_1, s_2, \ldots, s_k$ respectively. In fact, the stationary distribution $[p_1, p_2, \ldots, p_k]$ is the unique nonnegative solution of the following linear equation:
$$[p_1, p_2, \ldots, p_k] \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1k} \\ m_{21} & m_{22} & \ldots & m_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k1} & m_2 & \ldots & m_{kk} \end{bmatrix} = [p_1, p_2, \ldots, p_k].$$

The equilibrium state $[n_1{}^*, n_2{}^*, \ldots, n_N{}^*]$ can be derived from the stationary distribution of the Markov chain. Let $n_j^{(i)}$ be the number of clusters with j agents in the i-th state s_i, we conclude that
$$[n_1{}^*, \ldots, n_j^*, \ldots] = \left[\sum_{i=1}^{k} p_i n_1^{(i)}, \ldots, \sum_{i=1}^{k} p_i n_j^{(i)}, \ldots\right]. \quad (2)$$

It is the exact solution of the master equation in Xie's paper.[2]

5. The special case for N=5

In this section, we present a completed Markov analysis for special case $N = 5$. In this case, there are 7 states in the system:

$s_1 = [5,0,0,0,0]$, $s_2 = [3,1,0,0,0]$, $s_3 = [1,2,0,0,0]$, $s_4 = [2,0,1,0,0]$, $s_5 = [0,1,1,0,0]$, $s_6 = [1,0,0,1,0]$, $s_7 = [0,0,0,0,1]$.

According to the five cases in Section 3, we calculated the transition matrix for $N = 5$, which is

$$M(a) = M_0 a + M_1,$$

where

$$M_0 = \begin{bmatrix} \frac{4}{5} & \frac{-4}{5} & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{8}{25} & \frac{-6}{25} & \frac{-12}{25} & 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{-4}{25} & 0 & \frac{-8}{25} & \frac{-8}{25} & 0 \\ \frac{3}{5} & 0 & 0 & \frac{-1}{25} & \frac{-2}{25} & \frac{-12}{25} & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} & \frac{-13}{25} & 0 & \frac{-12}{25} \\ \frac{4}{5} & 0 & 0 & 0 & 0 & \frac{-12}{25} & \frac{-8}{25} \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$M_1 = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{25} & \frac{6}{25} & \frac{12}{25} & 0 & 0 & 0 \\ 0 & 0 & \frac{9}{25} & 0 & \frac{8}{25} & \frac{8}{25} & 0 \\ 0 & 0 & 0 & \frac{11}{25} & \frac{2}{25} & \frac{12}{25} & 0 \\ 0 & 0 & 0 & 0 & \frac{13}{25} & 0 & \frac{12}{25} \\ 0 & 0 & 0 & 0 & 0 & \frac{17}{25} & \frac{8}{25} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The stationary distribution of the Markov chain can be calculated as

$$p_1 = -5\frac{(31a^4-84a^3+966a^2+196a+2016)(3a+2)a}{T},$$
$$p_2 = 50\frac{(33a^4+328a^3+1068a^2+1248a+448)(a-1)a}{T},$$
$$p_3 = 75\frac{(33a^3+196a^2+284a+112)(a-1)^2a}{T},$$
$$p_4 = -200\frac{(9a^3+228a^2+292a+96)(a-1)^2a}{T},$$
$$p_5 = 200\frac{(3a+2)(3a+22)(a-1)^3a}{T},$$
$$p_6 = 150\frac{a(23a^2+354a+248)(a-1)^3}{T},$$
$$p_7 = -48\frac{(41a^3+457a^2-162a-336)(a-1)^3}{T}.$$

There is $T = (192a^6 - 332a^5 - 13095a^4 + 56110a^3 - 29320a^2 - 75552a - 16128)$.

See Fig.1 for the curves of p_1, p_2, p_3, p_4, p_5, p_6, p_7. One can see that p_1 is increasing, p_7 is decreasing, and the others first increase and then decrease finally.

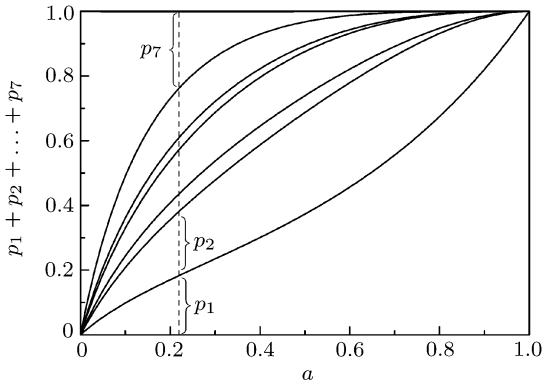


Fig.1. The stationary distribution for $N = 5$. $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1$. p_1 is creasing, p_7 is decreasing. p_i $(i = 2, 3, 4, 5, 6)$ admits a maximum $p_i(a_i)$, is increasing if $a \le a_i$ and is decreasing if $a > a_i$.

Now we calculate the equilibrium state of the system from the stationary distribution by Eq.(2). We obtained the equilibrium state of the system for $N = 5$ (see Table 2 for details):

$${n_1}^* = -125\frac{a}{12a^2-29a-8},$$
$${n_2}^* = -250\frac{(6a^3+25a^2+62a+32)(a-1)(a-6)a}{T},$$
$${n_3}^* = -1000\frac{(33a^2+64a+28)(a-1)^2a}{T},$$
$${n_4}^* = 150\frac{a(23a^2+354a+248)(a-1)^3}{T},$$
$${n_5}^* = -48\frac{(41a^3+457a^2-162a-336)(a-1)^3}{T},$$

where $T = (192a^6 - 332a^5 - 13095a^4 + 56110a^3 - 29320a^2 - 75552a - 16128)$.

Table 2. The calculation of equilibrium state for N=5.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	equilibrium state
n_1	5	3	1	2	0	1	0	$n_1^* = 5p_1 + 3p_2 + p_3 + 2p_4 + p_6$
n_2	0	1	2	0	1	0	0	$n_2^* = p_2 + 2p_3 + p_5$
n_3	0	0	0	1	1	0	0	$n_3^* = p_4 + p_5$
n_4	0	0	0	0	0	1	0	$n_4^* = p_6$
n_5	0	0	0	0	0	0	1	$n_5^* = p_7$

One can check that

$$n_1^* + 2n_2^* + 3n_3^* + 4n_4^* + 5n_5^* = 5,$$

and see Fig.2 for the curves.

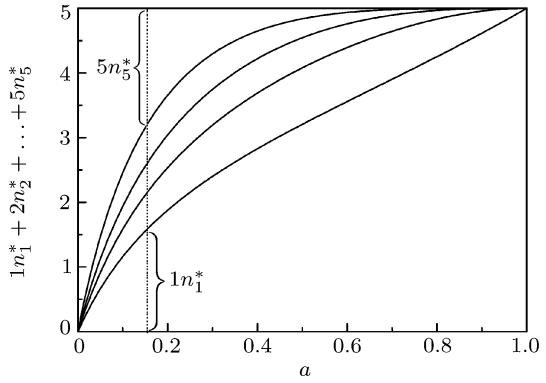


Fig.2. The equilibrium state for $N = 5$. It follows that $n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 = 5$. Most agents belong to the clusters with one or five agents, especially when a is small or a is large.

6. The numerical results for $N = 18$

For the case $N = 18$, there are 385 states in the EZ model. The 385×385 state transition matrix is too big to present here. It is difficult to obtain the analytic result like $N = 5$. On the other hand, even if we can obtain the analytical expression of the stationary distribution, no one will be interested with it because the expression is too complex. So it is interesting to present some numerical results for specific a. If a is fixed, the state transition matrix is a 385×385 matrix without any parameter. We can *calculate* its stationary distribution and *obtain* the equilibrium state for it.

Following the five cases in last section, we calculated the state transition matrix $M(a)$. For specific a, we substitute a into the matrix and obtain a numerical matrix. Denote the 385×385 state transition matrix by $M(a)$. The stationary distribution of $M(a)$ is the unique nonnegative solution of the linear equation $pM(a) = p$, where $p = (p_1, p_2, \ldots, p_{385})$ is a row vector. We use iteration method to approximate the stationary distribution (row vector) p, and stop if the error (defined as $\frac{1}{385}\sum_{i=1}^{385} |p_i - q_i|$ for $N = 18$, where $q_i = (pM(a))_i$ is the i-th element in the row vector $pM(a)$) is less than 10^{-11}. Note that if the error is 0, then $pM(a) = p$, which implies that p is just the stationary distribution. So the row vector p is very close to the stationary distribution if the difference between p and $pM(a)$ is small enough.[13] We can use the approximate stationary distribution to calculate the equilibrium state of the system by Eq.(2). The equilibrium state is a good approximation of the solution of the master equation in mean-field analysis because the stationary distribution p is stable.

We calculated the equilibrium state of the EZ model for $N = 18$ with different a. We present some of them in the following figures.

In Figs.3 and 4, we can see that it is more liable to obtain clusters with more agents if the transaction probability is $a = 0.0001$. In this case, there are few transactions in the market, but the transaction will significantly affect the market because the volume is great. It presents heavy tails and leads to power law. We can see herd behaviour.

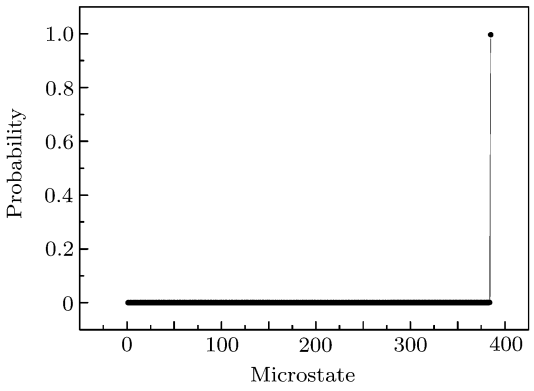


Fig.3. The stationary distribution for $N = 18$ and $a = 0.0001$.

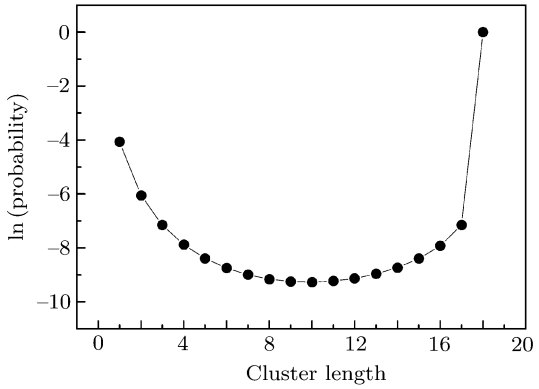


Fig.4. The equilibrium state for $N = 18$ and $a = 0.0001$.

In Figs.5 and 6, the transaction probability is $a = 0.2$. In this case, there are more transactions in the market, and only few transactions will affect the market significantly because the probability of big trade volume is relatively small. We can see self-organization of the agents.

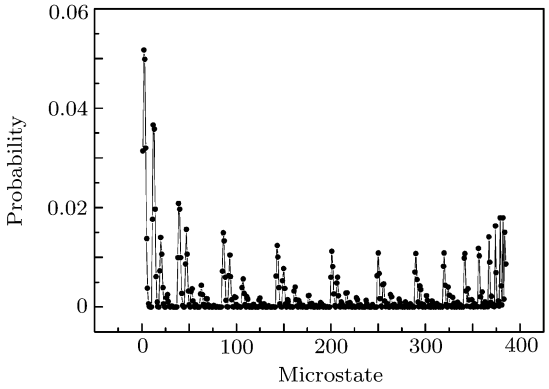


Fig.5. The stationary distribution for $N = 18$ and $a = 0.2$.

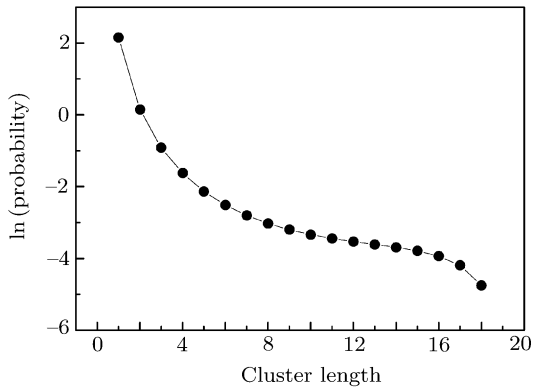


Fig.6. The equilibrium state for $N = 18$ and $a = 0.2$.

In Figs.7 and 8, the transaction probability is $a = 0.8$. We can see that it is extremely difficult for the system to form long cluster, the typical phenomenon are clusters with few (mostly one or two) agents. In this case, most agents want to transact. All transactions play little effect on the market because the trade volume is very small. There is almost no organization in the market.

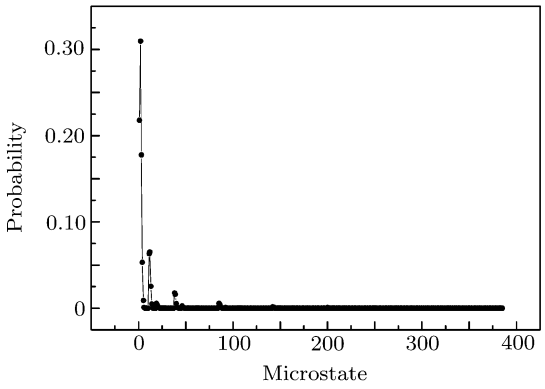


Fig.7. The stationary distribution for $N = 18$ and $a = 0.8$.

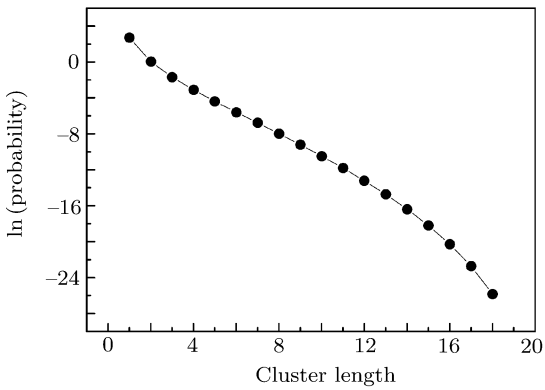


Fig.8. The equilibrium state for $N = 18$ and $a = 0.8$.

In Fig.9, we present the equilibrium state for different a. One can see the effect of the change of a more intuitively.

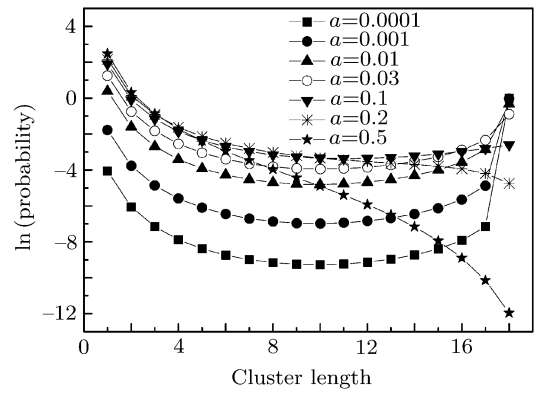


Fig.9. The equilibrium state for $N = 18$ and $a = 0.0001, 0.001, 0.01, 0.03, 0.1, 0.2, 0.5$.

7. Discussion

Different agents in the EZ model are distinguished by information. The system evolves with information communication, which is different from general physical system and evolves according to the exchange of substance and energy.

There is no stable microstate in the EZ model because the evolution of the system will never stop, and the current state will change from time to time. The stationary state is just the equilibrium state, which can be obtained from the stationary distribution by Eq.(2). On the other hand, one can also get the equilibrium state from time average because the EZ model is ergodic, which is used in many works based on simulation. In other words, the long time average of the system will approach the stationary state. Since the obtained equilibrium state is also the solution of the master equation in mean-field analysis, we can conclude that the results obtained by simulation are stable if the simulation evolves long enough. It is difficult to decide when the simulation should stop because there is no good convergence rate estimation currently. Compare with many simulation results, our results are more reliable because we have error bound (less than 10^{-11}).

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