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Normal ordering and antinormal ordering of the operator $(fQ+gP)^n$ and some of their applications*

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In this paper by virtue of the technique of integration within an ordered product (IWOP) of operators and the intermediate coordinate-momentum representation in quantum optics, we derive the normal ordering and antinormal ordering products of the operator $(fQ+gP)^n$ when n is an arbitrary integer. These products are very useful in calculating their matrix elements and expectation values and obtaining some useful mathematical formulae. Finally, the applications of some new identities are given.

Keywords: normal ordering, antinormal ordering, IWOP technique, intermediate coordinate-momentum representation

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1. Introduction

As is well known, normal ordering products of the operators are very useful in calculating their expectation values, which are helpful to theoretical prediction of many properties of atoms and molecules such as oscillator strength and radiation life-times. It is the main reason that the matrix elements of the normally ordered operator function $:f\left(a^{\dagger},a\right):$ in coherent states $|\alpha\rangle$ yields[1,2]

$$\langle\alpha'|:f\left(a^{\dagger},a\right):|\alpha''\rangle=f\left(\alpha'^{*},\alpha''\right), \tag{1}$$

where the symbol $::$ denotes normal ordering, and $a^{\dagger}$, a are the bosonic creation and annihilation operators satisfying the commutative relation $\left[a,a^{\dagger}\right]=1$. In Refs.[3, 4] using the IWOP technique, Fan *et al* have derived the normal ordering forms of some operators. For example, using the completeness relations of the eigenvectors $|q\rangle$ of the coordinate operator Q[5,6]

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}q\,|q\rangle\langle q| &= \pi^{-1/2}\int_{-\infty}^{\infty}\mathrm{d}q:\exp\left[-\left(q-Q\right)^2\right]:\\ &= 1,\end{aligned} \tag{2}$$

and the following integral formula

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}\mathrm{e}^{-(x-y)^2}\mathrm{H}_n(x)=(2y)^n, \tag{3}$$

where $\mathrm{H}_n(x)$ is the single-variable Hermite polynomials defined as

$$\mathrm{H}_n(x)=\sum_{k=0}^{[n/2]}(-)^k\frac{n!}{k!\,(n-2k)!}(2x)^{n-2k}, \tag{4}$$

the normal ordering form of the coordinate operator function $\mathrm{H}_n(Q)$ is immediately obtained,

$$\begin{aligned}\mathrm{H}_n(Q) &= \int_{-\infty}^{\infty}\mathrm{d}q\mathrm{H}_n(q)\,|q\rangle\langle q|\\ &= \pi^{-1/2}\int_{-\infty}^{\infty}\mathrm{d}q\mathrm{H}_n(q):\exp\left[-\left(q-Q\right)^2\right]:\\ &= 2^n:Q^n:.\end{aligned} \tag{5}$$

In addition, for the antinormal ordering forms of the operators, we usually use them to derive some useful mathematical formulae and the Weyl ordering products of the operators.[7,8] Therefore, it is an important topic to obtain the normal and antinormal products of the operators in quantum optics theory.

In this paper, we use the intermediate coordinate-momentum representation and the IWOP technique to derive the normal and antinormal ordering expansions of the operator $(fQ+gP)^n$, with n being an arbitrary integer. Motivations to pay attention to the work lie in the following two facts: 1) the combinational operator $(fQ+gP)^n$ is complex, its normal and antinormal ordering products are more ecumenical and not reported

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in literature before because one cannot find the appropriate representations. 2) The normal and antinormal ordering forms have some applications, for instance, obtaining the matrix elements of $(fQ+gP)^n$, Q^n and P^n in the coherent states and Fock states.

This paper is organized as follows. In Section 2 we first obtain the normal ordering of $(fQ+gP)^n$ when n is a positive integer via the intermediate coordinate-momentum representation, then using the formula in Ref.[8] we derive its antinormal form. In Section 3 we gain the normal ordering product of $\frac{1}{(fQ+gP)}$ by means of the same method employed in Section 2 and check the validity of it. In Section 4 we prove a new identity of $(fQ+gP)^n$ and further obtain the normal ordering and antinormal ordering of $(fQ+gP)^{-n}$. In Section 5 the applications of some new identities are briefly discussed.

2. Normal ordering and antinormal ordering of $(fQ+gP)^n$

As described in Ref.[9], there exists the intermediate coordinate-momentum representation $|q\rangle_{f,g}$, defined as

$$|q\rangle_{f,g} = \frac{1}{[\pi(f^2+g^2)]^{\frac{1}{4}}} \exp\left[-\frac{q^2}{2(f^2+g^2)} + \sqrt{2}a^\dagger \frac{q}{f-\mathrm{i}g} - \frac{f+\mathrm{i}g}{2(f-\mathrm{i}g)}a^{\dagger 2}\right]|0\rangle, \quad (6)$$

where f and g are two real parameters and the state $|q\rangle_{f,g}$ becomes the coordinate (or momentum) eigenstate for $f=1$ and $g=0$ (or $f=0$ and $g=1$), for this reason we assign the state $|q\rangle_{f,g}$ this name. Noting

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}}\left(a+a^\dagger\right), \\ P &= \frac{1}{\sqrt{2}\mathrm{i}}\left(a-a^\dagger\right), \\ [Q,P] &= \mathrm{i}\hbar, \end{aligned} \quad (7)$$

we see that the state $|q\rangle_{f,g}$ is the eigenvector of the operator $(fQ+gP)$ corresponding to the eigenvalue q, i.e., $(fQ+gP)|q\rangle_{f,g} = q|q\rangle_{f,g}$. The set of the states $|q\rangle_{f,g}$ makes up a complete quantum mechanical representation, since using the IWOP technique we can prove

$$\int_{-\infty}^{\infty} \mathrm{d}q\, |q\rangle_{f,g\ f,g}\langle q| = \frac{1}{[\pi(f^2+g^2)]^{\frac{1}{2}}} \int_{-\infty}^{\infty} \mathrm{d}q : \exp\left\{\frac{-1}{(f^2+g^2)}\left[q-(fQ+gP)\right]^2\right\} := 1. \quad (8)$$

For Eq.(8) by using the following integral formula[6]

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{\pi}} x^l \mathrm{e}^{-\sigma(x-y)^2} = \frac{1}{\sqrt{\sigma^{l+1}}} \sum_{k=0}^{[l/2]} \frac{l!}{2^{2k}k!(l-2k)!}\left(\sigma^{\frac{1}{2}}y\right)^{l-2k}, \quad (\mathrm{Re}\sigma>0), \quad (9)$$

we obtain

$$\begin{aligned} (fQ+gP)^n &= \int_{-\infty}^{\infty} \mathrm{d}q\, (fQ+gP)^n |q\rangle_{f,gf,g}\langle q| \\ &= \frac{1}{[\pi(f^2+g^2)]^{\frac{1}{2}}} \int_{-\infty}^{\infty} \mathrm{d}q q^n : \exp\left\{\frac{-1}{f^2+g^2}\left[q-(fQ+gP)\right]^2\right\} : \\ &= \left(f^2+g^2\right)^{n/2} \sum_{k=0}^{[n/2]} \frac{n!}{2^{2k}k!(n-2k)!} : \left[\sqrt{\frac{1}{f^2+g^2}}(fQ+gP)\right]^{n-2k} :, \end{aligned} \quad (10)$$

which is the normal ordering form of $(fQ+gP)^n$ when n is a positive integer. Further, using Eq.(4) we can rewrite Eq.(10) as

$$(fQ+gP)^n = \left(-\mathrm{i}\sqrt{\frac{f^2+g^2}{4}}\right)^n : \mathrm{H}_n\left[\mathrm{i}\sqrt{\frac{1}{f^2+g^2}}(fQ+gP)\right] : . \quad (11)$$

If letting $\mu=\dfrac{f-\mathrm{i}g}{\sqrt{2}}$, $\nu=\dfrac{f+\mathrm{i}g}{\sqrt{2}}$, we can identify Eq.(11) as

$$\left(\mu a+\nu a^{\dagger}\right)^{n}=\left(-\mathrm{i}\sqrt{\frac{\mu\nu}{2}}\right)^{n}:\mathrm{H}_{n}\left(\mathrm{i}\sqrt{\frac{\mu}{2\nu}}a+\mathrm{i}\sqrt{\frac{\nu}{2\mu}}a^{\dagger}\right):, \tag{12}$$

which is the normal ordering of a new operator $\left(\mu a+\nu a^{\dagger}\right)^{n}$.

In order to obtain the antinormal ordering of the operator $(fQ+gP)^{n}$, we recall the following formula for converting an operator A into its antinormal ordering form[6]

$$A=\int\frac{\mathrm{d}^{2}\alpha}{\pi}\left\langle-\alpha\right|A\left|\alpha\right\rangle\ \vdots\exp\left(\left|\alpha\right|^{2}+\alpha^{*}a-\alpha a^{\dagger}+a^{\dagger}a\right)\vdots,\quad\alpha=\alpha_{1}+\mathrm{i}\alpha_{2}, \tag{13}$$

where the symbol $\vdots\ \vdots$ denotes the antinormal ordering. Inserting Eq.(12) into Eq.(13) and carrying out the integration via the IWOP technique, we have

$$\begin{aligned}(fQ+gP)^{n}&=\left(\mu a+\nu a^{\dagger}\right)^{n}\\&=\left(-\mathrm{i}\sqrt{\frac{\mu\nu}{2}}\right)^{n}\int\frac{\mathrm{d}^{2}\alpha}{\pi}\left\langle-\alpha\right|:\mathrm{H}_{n}\left(\mathrm{i}\sqrt{\frac{\mu}{2\nu}}a+\mathrm{i}\sqrt{\frac{\nu}{2\mu}}a^{\dagger}\right):\left|\alpha\right\rangle\times\ \vdots\ \exp\left(\left|\alpha\right|^{2}+\alpha^{*}a-\alpha a^{\dagger}+a^{\dagger}a\right)\ \vdots\\&=\left(-\mathrm{i}\sqrt{\frac{\mu\nu}{2}}\right)^{n}\frac{\partial^{n}}{\partial t^{n}}\int\frac{\mathrm{d}^{2}\alpha}{\pi}\vdots\ \ \exp\left\{\mathrm{i}\left(\sqrt{\frac{2\mu}{\nu}}\alpha-\sqrt{\frac{2\nu}{\mu}}\alpha^{*}\right)t-t^{2}-\left|\alpha\right|^{2}+\alpha^{*}a-\alpha a^{\dagger}+a^{\dagger}a\right\}|_{t=0}\vdots\\&=\left(-\mathrm{i}\sqrt{\frac{\mu\nu}{2}}\right)^{n}\frac{\partial^{n}}{\partial t^{n}}\vdots\ \ \exp\left\{t^{2}+\mathrm{i}t\left(a\sqrt{\frac{2\mu}{\nu}}+a^{\dagger}\sqrt{\frac{2\nu}{\mu}}\right)\right\}|_{t=0}\vdots\\&=\left(\sqrt{\frac{\mu\nu}{2}}\right)^{n}\ \vdots\ \ \mathrm{H}_{n}\left(\sqrt{\frac{\mu}{2\nu}}a+\sqrt{\frac{\nu}{2\mu}}a^{\dagger}\right)\ \vdots,\end{aligned} \tag{14}$$

which is the antinormal ordering form of $(fQ+gP)^{n}$. Where we have used the generating function of H_{n}[10]

$$\mathrm{H}_{n}(x)=\frac{\partial^{n}}{\partial t^{n}}\mathrm{e}^{2xt-t^{2}}|_{t=0} \tag{15}$$

and the integral formula[6]

$$\int\frac{\mathrm{d}^{2}z}{\pi}\exp\left(\zeta\left|z\right|^{2}+\xi z+\eta z^{*}+fz^{2}+gz^{*2}\right)=\frac{1}{\sqrt{\zeta^{2}-4fg}}\exp\left[\frac{-\zeta\xi\eta+\xi^{2}g+\eta^{2}f}{\zeta^{2}-4fg}\right] \tag{16}$$

with $\mathrm{Re}(\zeta+f+g)<0$, $\mathrm{Re}\left(\dfrac{\zeta^{2}-4fg}{\zeta+f+g}\right)<0$.

3. Normally ordered expansion of $\dfrac{1}{(fQ+gP)}$

From Eqs.(8) and (9), we have the following integral

$$\begin{aligned}\frac{1}{(fQ+gP)}&=\int_{-\infty}^{\infty}\mathrm{d}q\frac{1}{q}\left|q\right\rangle_{f,gf,g}\left\langle q\right|\\&=\frac{1}{\left[\pi\left(f^{2}+g^{2}\right)\right]^{\frac{1}{2}}}\int_{-\infty}^{\infty}\mathrm{d}q\frac{1}{q}:\exp\left\{\frac{-1}{\left(f^{2}+g^{2}\right)}\left[q-(fQ+gP)\right]^{2}\right\}:.\end{aligned} \tag{17}$$

Since the integral in Eq.(17) has a singular point at $q=0$ on the contour, according to the basic mathematical theory we obtain the Cauchy principal value integral as follows:

$$\frac{1}{(fQ+gP)}=\frac{1}{\left[\pi\left(f^{2}+g^{2}\right)\right]^{\frac{1}{2}}}:\lim_{\varepsilon\to\infty,\epsilon\to0+}\left\{\int_{-\varepsilon}^{-\epsilon}\mathrm{d}q\frac{\exp\left\{\dfrac{-1}{\left(f^{2}+g^{2}\right)}\left[q-(fQ+gP)\right]^{2}\right\}}{q}\right.$$

$$+\int_{\epsilon}^{\varepsilon}\mathrm{d}q\frac{\exp\left\{\dfrac{-1}{(f^2+g^2)}\left[q-(fQ+gP)\right]^2\right\}}{q}:\Bigg\}. \tag{18}$$

If setting $q=-q$ in the first term, then equation (18) becomes

$$\begin{aligned}\frac{1}{(fQ+gP)} &= \frac{1}{\left[\pi\left(f^2+g^2\right)\right]^{\frac{1}{2}}}\lim_{\varepsilon\to\infty,\epsilon\to 0+}\Bigg\{:\int_{\varepsilon}^{\epsilon}\mathrm{d}q\frac{\exp\left\{\dfrac{-1}{(f^2+g^2)}\left[q+(fQ+gP)\right]^2\right\}}{q}\\ &\quad+\int_{\epsilon}^{\varepsilon}\mathrm{d}q\frac{\exp\left\{\dfrac{-1}{(f^2+g^2)}\left[q-(fQ+gP)\right]^2\right\}}{q}:\Bigg\}\\ &= \frac{1}{\left[\pi\left(f^2+g^2\right)\right]^{\frac{1}{2}}}\lim_{\varepsilon\to\infty,\epsilon\to 0+}\int_{\epsilon}^{\varepsilon}\mathrm{d}q\\ &\quad:\frac{-\exp\left\{\dfrac{-1}{(f^2+g^2)}\left[q+(fQ+gP)\right]^2\right\}+\exp\left\{\dfrac{-1}{(f^2+g^2)}\left[q-(fQ+gP)\right]^2\right\}}{q}:\\ &= \frac{1}{\left[\pi\left(f^2+g^2\right)\right]^{\frac{1}{2}}}\int_0^{+\infty}\mathrm{d}q\exp\left[\frac{-1}{(f^2+g^2)}q^2\right]\\ &\quad:\frac{\exp\left[\dfrac{2q\left(fQ+gP\right)}{(f^2+g^2)}\right]-\exp\left[\dfrac{-q\left(fQ+gP\right)}{(f^2+g^2)}\right]}{q}\exp\left[\frac{-1}{(f^2+g^2)}\left(fQ+gP\right)^2\right]:.\end{aligned} \tag{19}$$

By using the following integral formula[10]

$$\int_0^{+\infty}\mathrm{d}x\mathrm{e}^{-x}x^{t-1}=\Gamma\left(t\right),\quad \mathrm{Re}\left(t\right)>0, \tag{20}$$

we can calculate the integral (19) which converges at values between $q=0$ and $q\to\infty$ as follows:

$$\begin{aligned}\frac{1}{(fQ+gP)} &= :\frac{1}{\left[\pi\left(f^2+g^2\right)\right]^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{2^{2k+2}}{(2k+1)!}\left(\frac{1}{f^2+g^2}\right)^{2k+1}\left(\int_0^{+\infty}\mathrm{d}q\mathrm{e}^{-\frac{1}{f^2+g^2}q^2}q^{2k}\right)\\ &\quad\times\left(fQ+gP\right)^{2k+1}\exp\left[\frac{-1}{(f^2+g^2)}\left(fQ+gP\right)^2\right]:\\ &= :\frac{1}{\left[\pi\left(f^2+g^2\right)\right]^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{2^{2k+2}}{(2k+1)!}\left(\frac{1}{f^2+g^2}\right)^{2k+1}\Gamma\left(k+\frac{1}{2}\right)\\ &\quad\times\left(fQ+gP\right)^{2k+1}\exp\left[\frac{-1}{(f^2+g^2)}\left(fQ+gP\right)^2\right]:.\end{aligned} \tag{21}$$

Then using the following well-known combinatorial formula[11]

$$\sum_{k=0}^{n}\left(-\right)^k\frac{x}{k+x}\binom{n}{k}=\frac{1}{\binom{n+x}{n}} \tag{22}$$

and the rearranging double summation

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A_kB_n=\sum_{m=0}^{\infty}\sum_{n=0}^{m}A_{m-n}B_n, \tag{23}$$

we are further able to simplify Eq.(21) as

$$\begin{aligned}\frac{1}{(fQ+gP)} &= :\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-)^m\frac{2}{(2k+1)\,k!m!}\left(\frac{1}{f^2+g^2}\right)^{k+m+1/2}(fQ+gP)^{2k+2m+1}:\\ &= :2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-)^{n-k}}{(2k+1)\,k!\,(n-k)!}\left(\frac{1}{f^2+g^2}\right)^{n+1/2}(fQ+gP)^{2n+1}:\\ &= :2\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-)^k\frac{1/2}{k+1/2}\binom{n}{k}\right]\frac{(-)^n}{n!}\left(\frac{1}{f^2+g^2}\right)^{n+1/2}(fQ+gP)^{2n+1}:\\ &= :2\sum_{n=0}^{\infty}\frac{1}{\binom{n+1/2}{n}}\frac{(-)^n}{n!}\left(\frac{1}{f^2+g^2}\right)^{n+1/2}(fQ+gP)^{2n+1}:\\ &= :\sqrt{\pi}\sum_{n=0}^{\infty}\left(\frac{1}{f^2+g^2}\right)^{n+1/2}\frac{(-)^n}{\Gamma\left(n+\frac{3}{2}\right)}(fQ+gP)^{2n+1}:, \end{aligned}\tag{24}$$

where we have employed $\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}2^{-k}\,(2k-1)!!$. In particular, in Eq.(24) for $f=1$ and $g=0$, we obtain the normal ordering of the inverse of the coordinate operator

$$\frac{1}{Q} =: \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-)^n}{\Gamma\left(n+\frac{3}{2}\right)}Q^{2n+1}:. \tag{25}$$

Similarly, for $f=0$ and $g=1$, we have

$$\frac{1}{P} =: \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-)^n}{\Gamma\left(n+\frac{3}{2}\right)}P^{2n+1}:. \tag{26}$$

Equations (25) and (26) are consistent with the results obtained in Ref.[11]. Based on Eqs.(25) and (26) we can judge that the result in Eq.(24) is more valid and common, namely the normally ordered expansions of the operators $\frac{1}{Q}$ and $\frac{1}{P}$ are only the particular illustrations of Eq.(24).

4. Normal ordering and antinormal ordering of $(fQ+gP)^{-n}$

In order to obtain the normal ordering expansion of the operator $(fQ+gP)^{-n}$, first let us prove that there exists a new identity of $(fQ+gP)^n$. Using the completeness relations of eigenvectors $|q\rangle_{f,g}$, we have the normal form of the operator $\mathrm{H}_n\left[\sqrt{\frac{1}{f^2+g^2}}\,(fQ+gP)\right]$, i.e.,

$$\begin{aligned}\mathrm{H}_n\left[\sqrt{\frac{1}{f^2+g^2}}\,(fQ+gP)\right] &= \int_{-\infty}^{\infty}\mathrm{d}q\mathrm{H}_n\left(\sqrt{\frac{1}{f^2+g^2}}q\right)|q\rangle_{f,gf,g}\,\langle q|\\ &= \frac{1}{\left[\pi\left(f^2+g^2\right)\right]^{\frac{1}{2}}}\int_{-\infty}^{\infty}\mathrm{d}q\mathrm{H}_n\left(\sqrt{\frac{1}{f^2+g^2}}q\right)\ :\ \exp\left\{\frac{-1}{\left(f^2+g^2\right)}\left[q-(fQ+gP)\right]^2\right\}:\\ &= \left(\frac{4}{f^2+g^2}\right)^{n/2}:(fQ+gP)^n:. \end{aligned}\tag{27}$$

Using Eq.(27) and the recurrence relation of Hermite polynomial $\mathrm{H}_n(x)$[10]

$$\mathrm{H}_n'(x) = 2n\mathrm{H}_{n-1}(x), \tag{28}$$

we find that there is a differential relation of the operator $(fQ+gP)^n$

$$\frac{\mathrm{d}}{\mathrm{d}\,(fQ+gP)}:(fQ+gP)^n: = \left(\frac{4}{f^2+g^2}\right)^{-n/2}\frac{\mathrm{d}}{\mathrm{d}\,(fQ+gP)}\mathrm{H}_n\left[\sqrt{\frac{1}{f^2+g^2}}\,(fQ+gP)\right]$$

$$= n\left(\frac{4}{f^2+g^2}\right)^{\frac{1-n}{2}} \mathrm{H}_{n-1}\left[\sqrt{\frac{1}{f^2+g^2}}\,(fQ+gP)\right]$$
$$= n:(fQ+gP)^{n-1}:, \tag{29}$$

so there exists the following new identity within the normal ordering

$$\frac{\mathrm{d}}{\mathrm{d}\,(fQ+gP)}:(fQ+gP)^n: = :\frac{\mathrm{d}}{\mathrm{d}\,(fQ+gP)}\,(fQ+gP)^n:. \tag{30}$$

From Eqs.(24) and (30) we obtain the normal ordering of $(fQ+gP)^{-n}$

$$\begin{aligned}(fQ+gP)^{-n} &= \frac{(-)^{n-1}}{(n-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}\,(fQ+gP)}\right)^{n-1}\frac{1}{(fQ+gP)}\\ &= \frac{1}{(n-1)!}:\sqrt{\pi}\sum_{k=0}^{\infty}\left(\frac{1}{f^2+g^2}\right)^{k+1/2}\frac{(-)^{k+n-1}}{\Gamma\left(k+\frac{3}{2}\right)}\times\left(\frac{\mathrm{d}}{\mathrm{d}\,(fQ+gP)}\right)^{n-1}(fQ+gP)^{2k+1}:\\ &= \sqrt{\pi}\sum_{k=\left[\frac{n-1}{2}\right]}^{\infty}\left(\frac{1}{f^2+g^2}\right)^{k+1/2}\frac{(-)^{k+n-1}}{\Gamma\left(k+\frac{3}{2}\right)}\binom{2k+1}{n-1}:(fQ+gP)^{2k-n+2}:\\ &= \sqrt{\pi}\sum_{k=\left[\frac{n-1}{2}\right]}^{\infty}\left(\frac{1}{f^2+g^2}\right)^{k-1/2}\frac{(-)^{k+n}}{\Gamma\left(k+\frac{1}{2}\right)}\binom{2k-1}{n-1}:(fQ+gP)^{2k-n}:. \end{aligned}\tag{31}$$

In particular, if in Eq.(31) setting $f=1$ and $g=0$ (or $f=0$ and $g=1$), we are also able to obtain the normal product of the operator Q^{-n} (or P^{-n}). They are just the conclusions given in Ref.[11].

Similarly, from Eqs.(13) and (31) using the following mathematical formula[6]

$$\int\frac{\mathrm{d}^2z}{\pi}z^n z^{*m}\exp\left(|z|^2+\xi z+\eta z^*\right)=\exp\left(-\xi\eta/\zeta\right)\sum_{l=0}^{\min(m,n)}\frac{m!n!\xi^{m-l}\eta^{n-l}}{l!\,(m-l)!\,(n-l)!\,(-\zeta)^{m+n-l+1}},\quad \mathrm{Re}\zeta<0, \tag{32}$$

we also have the antinormal ordering of $(fQ+gP)^{-n}$

$$\begin{aligned}(fQ+gP)^{-n} &= \sqrt{\pi}\sum_{k=\left[\frac{n-1}{2}\right]}^{\infty}\left(\frac{1}{f^2+g^2}\right)^{k-1/2}\frac{(-)^{k+n}}{\Gamma\left(k+\frac{1}{2}\right)}\binom{2k-1}{n-1}\\ &\quad\times\int\frac{\mathrm{d}^2\alpha}{\pi}\langle-\alpha|:(fQ+gP)^{2k-n}:|\alpha\rangle\ \vdots\exp\left(|\alpha|^2+\alpha^*a-\alpha a^\dagger+a^\dagger a\right)\vdots\\ &= \sum_{k=\left[\frac{n-1}{2}\right]}^{\infty}\sum_{i=0}^{2k-n}\sqrt{\pi}\left(\frac{1}{f^2+g^2}\right)^{k-1/2}\frac{(-)^{k-i}\,(2k-1)!\,(2k-n)!}{\Gamma\left(k+\frac{1}{2}\right)(n-1)!i!}\mu^i\nu^{2k-n-i}\\ &\quad\times\int\frac{\mathrm{d}^2\alpha}{\pi}\ \vdots\,\alpha^i\alpha^{*2k-n-i}\exp\left(-|\alpha|^2+\alpha^*a-\alpha a^\dagger+a^\dagger a\right)\vdots\\ &= \sum_{k=\left[\frac{n-1}{2}\right]}^{\infty}\sum_{i=0}^{2k-n}\sum_{l=0}^{\min(i,2k-n-i)}\sqrt{\pi}\left(\frac{1}{f^2+g^2}\right)^{k-1/2}\frac{(-)^{k+n+l}\,\mu^i\nu^{2k-n-i}}{\Gamma\left(k+\frac{1}{2}\right)!}\\ &\quad\times\frac{(2k-1)!}{(n-1)!l!\,(2k-n-i-l)!\,(i-l)!}\ \vdots\,a^{i-l}a^{\dagger 2k-n-i-l}\exp\left(a^\dagger a\right)\vdots. \end{aligned}\tag{33}$$

In particular, when $n=1$, we have

$$\frac{1}{(fQ+gP)}=\frac{\sqrt{\pi\,(f^2+g^2)}}{\Gamma\left(\frac{1}{2}\right)!}\sum_{n=0}^{\infty}\ \vdots\ \frac{\left(a^\dagger a\right)^n}{n!}\vdots \tag{34}$$

5. Applications of Eqs.(10) and (31)

In this section we want to discuss some applications of the above new identities. Recently, for a harmonic oscillator some general calculation formulae[12] or general recurrence relations[13,14] among the matrix elements of some operators such as coordinate operator, momentum operator and radial operator, etc. are widely discussed. However, in these papers they always use some special functions to obtain some very lengthy and complicated formulae among the matrix elements, and it is very much difficult to understand the whole process of calculation. Here we can see that it is easy to use the new identity (10) to obtain the matrix elements of the coordinate operator and momentum operator in coherent states and Fock states. According to Eq.(10) we can immediately obtain the matrix elements of $(fQ+gP)^n$ in coherent states

$$\begin{aligned}&\langle\beta|\left(fQ+gP\right)^n|\alpha\rangle\\ &=\sqrt{(2\mu\nu)^n}\sum_{k=0}^{[n/2]}\frac{n!}{2^{2k}k!\left(n-2k\right)!}\left[\sqrt{\frac{1}{2\mu\nu}}\left(\mu\alpha+\nu\beta^*\right)\right]^{n-2k}\times\exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\alpha\beta^*\right].\end{aligned}\tag{35}$$

By recasting the overcomplete relation of coherent state $|\alpha\rangle$[1] into Eq.(35), we have the operator $(fQ+gP)^n$ matrix elements of Fock states for harmonic oscillator

$$\begin{aligned}\langle m'|\left(fQ+gP\right)^n|n'\rangle &=\int\frac{\mathrm{d}^2\alpha\mathrm{d}^2\beta}{\pi^2}\langle m'|\beta\rangle\langle\beta|\left(fQ+gP\right)^n|\alpha\rangle\langle\alpha|n'\rangle\\ &=\frac{1}{\sqrt{m'!n'!}}\sum_{k=0}^{[n/2]}\sum_{i=0}^{n-2k}\frac{n!}{2^k k!i!\left(n-2k-i\right)!}\mu^{k+i}\nu^{n-k-i}\\ &\quad\times\int\frac{\mathrm{d}^2\alpha\mathrm{d}^2\beta}{\pi^2}\exp\left[-|\alpha|^2-|\beta|^2+\alpha\beta^*\right]\alpha^i\alpha^{*n'}\beta^{m'}\left(\beta^*\right)^{n-2k-i},\end{aligned}\tag{36}$$

where we have employed the expansion of the binomial[10]

$$\left(\alpha+\beta\right)^l=\sum_{k=0}^{\infty}\left(\begin{array}{c}l\\k\end{array}\right)\alpha^k\beta^{l-k}\tag{37}$$

and the coherent state representation in single-mode Fock space

$$\langle n|\alpha\rangle=\frac{\alpha^n}{\sqrt{n!}}\exp\left(-\frac{|\alpha|^2}{2}\right).\tag{38}$$

Further, we introduce the following two mathematical formulae[6]

$$\int\frac{\mathrm{d}^2\alpha}{\pi}\alpha^{*n}\alpha^k\mathrm{e}^{\lambda|\alpha|^2+\mu\alpha}=\left(-\right)^{n+1}\lambda^{-(n+1)}\frac{n!}{(n-k)!}\mu^{n-k},\quad(\mathrm{Re}\lambda<0,k\leq n)\tag{39}$$

and

$$\int\frac{\mathrm{d}^2\alpha}{\pi}\alpha^{*n}\alpha^k\mathrm{e}^{\lambda|\alpha|^2}=\delta_{n,k}\left(-\right)^{n+1}\lambda^{-(n+1)}k!\,;\tag{40}$$

hence we can prove

$$\langle m'|\left(fQ+gP\right)^n|n'\rangle=\sum_{k=0}^{[n/2]}\frac{\mu^{\frac{n'+n-m'}{2}}\nu^{\frac{m'+n-n'}{2}}n!\sqrt{m'!n'!}}{2^k k!\left(\frac{m'-n'+n}{2}-k\right)!\left(\frac{m'+n'-n}{2}+k\right)!\left(\frac{n'-m'+n}{2}-k\right)!},\tag{41}$$

which are the matrix elements of the operator $(fQ+gP)^n$ of Fock states and the power n is an arbitrary positive integer. Using Eqs.(10) and (41), if letting $f=0$, $g=1$, i.e., $\mu=\nu=\dfrac{1}{\sqrt{2}}$, we immediately obtain the normal ordering form and matrix elements of the operator Q^n for Fock states respectively,

$$Q^n=\sum_{k=0}^{[n/2]}\frac{n!}{2^{\frac{n}{2}+k}k!\,(n-2k)!}\ :\ \left(a+a^{\dagger}\right)^{n-2k}:, \tag{42}$$

$$\left\langle m'\right|Q^n\left|n'\right\rangle=\sum_{k=0}^{[n/2]}\frac{n!\sqrt{m'!n'!}}{2^{k+\frac{n}{2}}k!\left(\dfrac{m'-n'+n}{2}-k\right)!\left(\dfrac{m'+n'-n}{2}+k\right)!\left(\dfrac{n'-m'+n}{2}-k\right)!}. \tag{43}$$

In the same way if letting $f=0$, $g=1$, i.e., $\mu=-\nu=\dfrac{1}{\sqrt{2}\mathrm{i}}$ in Eqs.(10) and (41), we can also deduce the normal ordering form and matrix elements of the operator P^n for Fock states

$$P^n=\sum_{k=0}^{[n/2]}\frac{\mathrm{i}^{2k-n}n!}{2^{\frac{n}{2}+k}k!\,(n-2k)!}:\left(a-a^{\dagger}\right)^{n-2k}:, \tag{44}$$

$$\left\langle m'\right|P^n\left|n'\right\rangle=\sum_{k=0}^{[n/2]}\frac{(-)^{\frac{n}{2}-k}n!\sqrt{m'!n'!}}{2^{k+\frac{n}{2}}k!\left(\dfrac{m'-n'+n}{2}-k\right)!\left(\dfrac{m'+n'-n}{2}+k\right)!\left(\dfrac{n'-m'+n}{2}-k\right)!}. \tag{45}$$

From the above discussion we can see that the result in Eq.(41) is valid and common, and equations (43) and (45) are only the particular illustrations of the result in Eq.(41). Moreover, the values of $\left\langle m'\right|Q^n\left|n'\right\rangle$ and $\left\langle m'\right|P^n\left|n'\right\rangle$ only have important relations to the Fock states and the power n of the operator.

As another application we consider a harmonic oscillator perturbed by a potential $V=\lambda\left(fQ+gP\right)^{-n}$, where λ is small. Then using Eq.(31) the energy correction to the unperturbed ground state $\left|0\right\rangle$ is obtained

$$\begin{aligned}&\left\langle 0\right|\lambda\left(fQ+gP\right)^{-n}\left|0\right\rangle\\ &=\lambda\sqrt{\pi}\left(\frac{1}{f^2+g^2}\right)^{(n-1)/2}\frac{(-)^{3n/2}}{\Gamma\left(\dfrac{n+1}{2}\right)}.\end{aligned} \tag{46}$$

In summary, in this paper by virtue of the IWOP technique and the intermediate coordinate-momentum representation in quantum optics, we have obtained the normal ordering and antinormal ordering products of the operator $(fQ+gP)^n$ where n is an arbitrary integer. The paper demonstrates the usefulness of the IWOP technique once again. Equations (12), (14), (31) and (33) are the new fundamental operator identities in quantum mechanics, which are very useful results in obtaining their expectation values and some useful mathematical formulae, as well as the Weyl ordering forms of the operators. Moreover, we briefly discussed some applications of Eqs.(10) and (31) in quantum optics..

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