Vol 18 No 4, April 2009
1674-1056/2009/18(04)/1512-05
Chinese Physics B
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# Single-mode photon number measurement for the squeezed two-mode number state*

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(Received 10 December 2008; revised manuscript received 20 January 2009)

Based on the fact that a two-mode squeezed number state is a two-variable Hermite polynomial excitation of the two-mode squeezed vacuum state, the result of one-mode $l$-photon measurement for the two-mode squeezed number state $S_2|m,n\rangle$ is discussed. It is found that a remaining field-mode simultaneously collapses into a number state $|n-m+l\rangle$ with the coefficient being a Jacobi polynomial of $n, m$ and $l$, which manifestly exhibits the entanglement between the two modes, i.e. it depends on the number-difference between the two modes. The second mode collapses into an excited coherent state when the first mode is measured as a coherent state.

**Keywords:** photon number measurement, two-mode squeezed number state, quantum entanglement
**PACC:** 4250, 0365

## 1. Introduction

Quantum entanglement has become more fascinating and important, for it has played a central role in quantum communication and quantum computation since Einstein, Podolsky and Rosen (EPR)[1] in 1935 published a paper arguing the incompleteness of quantum mechanics. By entanglement it is meant that a two-particle state is not factorized into a product of single particle states, but a sum of at least two terms, each of which is a product. When two particles are so entangled, neither particle individually has a state, because particles in entanglement do not independently have their own states or even some properties. Thus a two-particle system may be considered as a simple entity. The two-mode squeezed state itself is an entangled state which entangles an idle mode and a signal mode in a frequency domain as an outcome of a parametric down-conversion process (in a non-degenerate parametric amplifier).[2] Theoretically, a squeezed state is generated by operating the squeezing operator on the vacuum state.[3] Various entanglement measures have been invented to quantify the multiparticle entanglement.[4−11] Recently, the process where the second-mode field collapses into a single-mode squeezed state which is related to two independent parameters $\mu$ and $\nu$ is studied[12] when a single-mode quadrature-amplitude measurement, say $|x\rangle_{11}\langle x|$, with $|x\rangle_1$ being the eigenvector of the quadrature operator $\dfrac{a_1+a_1^\dagger}{\sqrt{2}}=X_1$, is performed on a generalized two-mode squeezed state.

In the present paper, the property of the second mode is investigated when a single-mode photon number measurement (with $l$ photons being detected) is performed on a squeezed two-mode number state $S_2|m,n\rangle$. The remainder of this paper is organized as follows. In Section 2, the result about a squeezed two-mode number state treated as a two-variable Hermite polynomial excited squeezed vacuum state is introduced. In Section 3, a single mode photon number measurement and the entanglement between the two modes are discussed. Finally, the discussion about the results obtained from the present study is presented in Section 4.

## 2. Squeezed two-mode number state treated as a two-variable Hermite polynomial excited squeezed vacuum state

Motivated by EPR's idea, Fan and Klauder have constructed a common eigenvector $|\xi\rangle$ of $X_1+X_2$ and

*Project supported by the National Natural Science Foundation of China (Grant No 10774108).
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$P_1 - P_2$, where $P_i$ is the momentum operator and found[13]

$$|\xi\rangle = \exp\left[-\frac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger\right]|00\rangle \equiv |\xi_1, \xi_2\rangle, \quad (1)$$

where $\xi = \xi_1 + \mathrm{i}\xi_2$ is a complex number, $|00\rangle$ is the two-mode vacuum state, $\left[a_i, a_j^\dagger\right] = \delta_{i,j}$, $i, j = 1, 2$, and $|\xi\rangle$ obeys

$$(X_1 + X_2)|\xi\rangle = \sqrt{2}\xi_1|\xi\rangle,$$

and

$$(P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle. \quad (2)$$

From expression (1), one can obtain

$$\left(a_1 + a_2^\dagger\right)|\xi\rangle = \xi|\xi\rangle,$$

and

$$\left(a_1^\dagger + a_2\right)|\xi\rangle = \xi^*|\xi\rangle, \quad (3)$$

so $|\xi\rangle$ is a bipartite entangled state. Its completeness relation and orthogonal property are respectively given as

$$\int \frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi| = 1,$$

and

$$\langle\xi'|\xi\rangle = \pi\delta(\xi' - \xi)\,\delta(\xi'^* - \xi^*) = \pi\delta^{(2)}(\xi' - \xi). \quad (4)$$

It is clear that the two-mode squeezing operator $S_2$ has a natural representation in terms of the entangled state[13]

$$S_2 = \exp\left[\lambda\left(a_1 a_2 - a_1^\dagger a_2^\dagger\right)\right] = \int \frac{\mathrm{d}^2\xi}{\pi\mu}|\xi/\mu\rangle\langle\xi|$$

with

$$\mu = \mathrm{e}^\lambda, \quad (5)$$

where $S_2$ leads to the two-mode squeezing transform

$$\begin{aligned} S_2 a_1 S_2^\dagger &= a_1\cosh\lambda + a_2^\dagger\sinh\lambda, \\ S_2 a_2 S_2^\dagger &= a_2\cosh\lambda + a_1^\dagger\sinh\lambda, \end{aligned} \quad (6)$$

and

$$S_2^\dagger|\xi\rangle = \mu|\mu\xi\rangle. \quad (7)$$

The $\langle\xi|$ representation can be used to demonstrate that a two-mode squeezed number state

$$S_2|m, n\rangle = \int \frac{\mathrm{d}^2\xi}{\pi\mu}|\xi/\mu\rangle\langle\xi|\,m, n\rangle \quad (8)$$

is just a two-variable Hermite polynomial excitation of the two-mode squeezed vacuum state. Recalling the generating function formula of the two-variable Hermite polynomial $\mathrm{H}_{m,n}(\xi, \xi^*)$,[14]

$$\sum_{m,n=0}^{\infty}\frac{z^m z'^n}{m!n!}\mathrm{H}_{m,n}(\xi, \xi^*) = \exp\{-zz' + z\xi + z'\xi^*\}, \quad (9)$$

where

$$\begin{aligned} \mathrm{H}_{m,n}(\xi, \xi^*) &= \sum_{l=0}^{\min(m,n)}\frac{m!n!}{l!\,(m-l)!\,(n-l)!} \\ &\quad \times (-1)^l\,\xi^{(m-l)}\xi^{*(n-l)} \\ &= \frac{\partial^{m+n}}{\partial t^m\partial t'^n}\exp\left(-tt' + t\xi + t'\xi^*\right)|_{t=t'=0}, \end{aligned} \quad (10)$$

the entangled state $|\xi\rangle$ can be expressed in two-mode Fock state as

$$|\xi\rangle = \sum_{m,n=0}^{\infty}\mathrm{e}^{-\frac{1}{2}|\xi|^2}\frac{1}{\sqrt{m!n!}}\mathrm{H}_{m,n}(\xi, \xi^*)|m, n\rangle, \quad (11)$$

then the overlap between $\langle\xi|$ and the two-mode Fock state is

$$\langle\xi|\,m, n\rangle = \mathrm{e}^{-|\xi|^2/2}\mathrm{H}^*_{m,n}(\xi, \xi^*)/\sqrt{m!n!}. \quad (12)$$

Substituting expression (12) into expression (8), and using the mathematical formula[14]

$$\begin{aligned} &\int \frac{\mathrm{d}^2 z}{\pi}\exp(\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) \\ &= \frac{1}{\sqrt{\zeta^2 - 4fg}}\exp\left[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right] \end{aligned} \quad (13)$$

with

$$\mathrm{Re}\,(\zeta \pm f \pm g) < 0, \text{ and } \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\zeta \pm f \pm g}\right) < 0, \quad (14)$$

on obtains

$$\begin{aligned} S_2|m, n\rangle &= \int \frac{\mathrm{d}^2\xi}{\pi\mu}|\xi/\mu\rangle\,\mathrm{e}^{-|\xi|^2/2}\mathrm{H}^*_{m,n}(\xi, \xi^*)/\sqrt{m!n!} \\ &= \frac{1}{\sqrt{m!n!}}\frac{\partial^{m+n}}{\partial t^m\partial t'^n}\int \frac{\mathrm{d}^2\xi}{\mu\pi}\exp\{-\frac{\mu^2 + 1}{2\mu^2}|\xi|^2 + \xi\left(\frac{a_1^\dagger}{\mu} + t'\right) \end{aligned}$$

$$
\begin{aligned}
&+\xi^{*}\left(\frac{a_2^{\dagger}}{\mu}+t\right)-tt'-a_1^{\dagger}a_2^{\dagger}\}\left|00\right\rangle|_{t=t'=0}\\
&=\frac{\mathrm{sech}\lambda}{\sqrt{m!n!}}\frac{\partial^{m+n}}{\partial t^{m}\partial t'^{n}}\exp\left[tt'\tanh\lambda+ta_1^{\dagger}\mathrm{sech}\lambda+t'a_2^{\dagger}\mathrm{sech}\lambda-\tanh\lambda a_1^{\dagger}a_2^{\dagger}\right]\left|00\right\rangle|_{t=t'=0}\\
&=\frac{\mathrm{sech}\lambda\left(-\tanh\lambda\right)^{\frac{m+n}{2}}}{\sqrt{m!n!}}\mathrm{H}_{m,n}\left(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\sinh 2\lambda}}a_1^{\dagger},\frac{\sqrt{2}}{\mathrm{i}\sqrt{\sinh 2\lambda}}a_2^{\dagger}\right)\exp\left[-\tanh\lambda a_1^{\dagger}a_2^{\dagger}\right]\left|00\right\rangle, \qquad (15)
\end{aligned}
$$

which is just the two-variable Hermite polynomial excitation of the two-mode squeezed vacuum state,[15] i.e. the two-mode squeezed vacuum state excited by the two-variable Hermite polynomial operator.

## 3. Single-mode photon number measurement

In this section, the $l$-photon measurement for the two-mode squeezed number state is considered. Using the differential expression (15), one has

$$
\begin{aligned}
{}_1\left\langle l\right|S_2\left|m,n\right\rangle&=\frac{\mathrm{sech}\lambda}{\sqrt{m!n!}}\frac{\partial^{m+n}}{\partial t^{m}\partial t'^{n}}\exp\left[tt'\tanh\lambda+t'a_2^{\dagger}\mathrm{sech}\lambda\right]_1\\
&\quad\times\left\langle l\right|\exp\left[ta_1^{\dagger}\mathrm{sech}\lambda-\tanh\lambda a_1^{\dagger}a_2^{\dagger}\right]\left|00\right\rangle|_{t=t'=0}\\
&=\frac{\mathrm{sech}\lambda}{\sqrt{m!n!}}\frac{\partial^{m+n}}{\partial t^{m}\partial t'^{n}}\exp\left[tt'\tanh\lambda+t'a_2^{\dagger}\mathrm{sech}\lambda\right]\\
&\quad\times\ {}_1\left\langle 0\right|\frac{a_1^{l}}{\sqrt{l!}}\exp\left[ta_1^{\dagger}\mathrm{sech}\lambda-\tanh\lambda a_1^{\dagger}a_2^{\dagger}\right]\left|00\right\rangle|_{t=t'=0}. \qquad (16)
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
&\exp\left[-\left(t\,\mathrm{sech}\lambda-\tanh\lambda a_2^{\dagger}\right)a_1^{\dagger}\right]a_1\exp\left[\left(t\mathrm{sech}\lambda-\tanh\lambda a_2^{\dagger}\right)a_1^{\dagger}\right]\\
&=a_1+t\,\mathrm{sech}\lambda-a_2^{\dagger}\tanh\lambda, \qquad (17)
\end{aligned}
$$

and

$$
{}_1\left\langle 0\right|\exp\left[\left(t\,\mathrm{sech}\lambda-\tanh\lambda a_2^{\dagger}\right)a_1^{\dagger}\right]={}_1\left\langle 0\right|, \qquad (18)
$$

expression (16) can be rewritten as

$$
\begin{aligned}
{}_1\left\langle l\right|S_2\left|m,n\right\rangle&=\frac{\mathrm{sech}\lambda}{\sqrt{m!n!l!}}\frac{\partial^{m+n}}{\partial t^{m}\partial t'^{n}}\exp\left[tt'\tanh\lambda+t'a_2^{\dagger}\mathrm{sech}\lambda\right]\\
&\quad\times\ {}_1\left\langle 0\right|\left(a_1+t\,\mathrm{sech}\lambda-a_2^{\dagger}\tanh\lambda\right)^{l}\left|00\right\rangle|_{t=t'=0}\\
&=\frac{\mathrm{sech}\lambda}{\sqrt{m!n!l!}}\frac{\partial^{m+n}}{\partial t^{m}\partial t'^{n}}\exp\left[tt'\tanh\lambda+t'a_2^{\dagger}\mathrm{sech}\lambda\right]\left(t\,\mathrm{sech}\lambda-a_2^{\dagger}\tanh\lambda\right)^{l}\left|0\right\rangle|_{t=t'=0}\\
&=\frac{\mathrm{sech}\lambda}{\sqrt{m!n!l!}}\sum_{k=0}^{l}\left(-\right)^{l-k}\binom{l}{k}\tanh^{l-k}\lambda\sum_{i,j=0}^{\infty}\frac{\tanh^{i}\lambda}{i!j!}a_2^{\dagger j+l-k}\mathrm{sech}^{j+k}\lambda\frac{\partial^{m+n}}{\partial t^{m}\partial t'^{n}}t^{i+k}t'^{i+j}\left|0\right\rangle|_{t=t'=0}\\
&=\frac{\sqrt{m!n!}}{\sqrt{l!}}\left(-\right)^{l}\tanh^{m+l}\lambda\,\mathrm{sech}^{n-m+1}\lambda\sum_{k=0}^{l}\left(-\right)^{k}\binom{l}{k}\frac{\left(\mathrm{sech}\lambda/\tanh\lambda\right)^{2k}}{(m-k)!\,(n-m+k)!}a_2^{\dagger n-m+l}\left|0\right\rangle. \qquad (19)
\end{aligned}
$$

Using the following formula:

$$
\sum_{k=0}^{l}\left(-\right)^{k}\binom{l}{k}\binom{n}{m-k}\left(x\right)^{k}=\binom{n}{m}{}_2\mathrm{F}_1\left[-l,-m,n-m+1,-x\right], \qquad (20)
$$

where ${}_2\mathrm{F}_1$ is the Jacobi polynomial of $n,m$ and $l$, whose definition is

$$
{}_2\mathrm{F}_1\left[-n,\beta,\gamma,z\right]=\sum_{k=0}^{n}\frac{\left(-n\right)_k\left(\beta\right)_k}{k!\left(\gamma\right)_k}z^{k} \qquad (21)
$$

with

$$(\beta)_k = \frac{\Gamma(\beta+k)}{\Gamma(\beta)},$$

and

$$\begin{aligned}\binom{\alpha}{n} &= \frac{\alpha!}{n!(\alpha-n)!} = \frac{(-1)^n\Gamma(n-\alpha)}{n!\Gamma(-\alpha)},\\ &\to \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)} = (-1)^n\frac{\alpha!}{(\alpha-n)!}.\end{aligned} \tag{22}$$

Expression (19) can be further rewritten as

$$\begin{aligned}{}_1\langle l|S_2|m,n\rangle &= \frac{\sqrt{m!}}{\sqrt{n!l!}}(-)^l\tanh^{m+l}\lambda\,\mathrm{sech}^{n-m+1}\lambda\\ &\quad\times\left[\sum_{k=0}^{l}(-)^k\binom{l}{k}\binom{n}{m-k}(\mathrm{sech}\lambda/\tanh\lambda)^{2k}\right]a_2^{\dagger n-m+l}|0\rangle\\ &= \frac{(-)^l\sqrt{n!}\sqrt{(n-m+l)!}\tanh^{m+l}\lambda\,\mathrm{sech}^{n-m+1}\lambda}{\sqrt{m!l!}(n-m)!}\\ &\quad\times{}_2\mathrm{F}_1\left[-l,-m,n-m+1,-\mathrm{sech}^2\lambda/\tanh^2\lambda\right]|n-m+l\rangle,\end{aligned} \tag{23}$$

from which one can see that the second mode collapses into the number state $|n-m+l\rangle$, which depends on the number-difference between the two modes. Particularly, when $m=n$, the second mode is also an $l$-photon number state,

$${}_1\langle l|S_2|n,n\rangle = (-)^l\tanh^{n+l}\lambda\,\mathrm{sech}\lambda\,{}_2\mathrm{F}_1\left[-l,-n,1,-\mathrm{sech}^2\lambda/\tanh^2\lambda\right]|l\rangle. \tag{24}$$

On the other hand, when the measurement basis is a coherent state,[16] $|z\rangle_1\,{}_1\langle z|$, according to expression (15), we can directly obtain the result measured as

$$\begin{aligned}{}_1\langle z|S_2|m,n\rangle &= \frac{\mathrm{sech}\lambda(-\tanh\lambda)^{\frac{m+n}{2}}}{\sqrt{m!n!}}\mathrm{H}_{m,n}\left(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\sinh 2\lambda}}z_1^*,\frac{\sqrt{2}}{\mathrm{i}\sqrt{\sinh 2\lambda}}a_2^\dagger\right)\\ &\quad\times\exp\left[-\frac{|z_1|^2}{2}-z_1^*a_2^\dagger\tanh\lambda\right]|0\rangle\\ &= \frac{\mathrm{sech}\lambda(-\tanh\lambda)^{\frac{m+n}{2}}}{\sqrt{m!n!}}\exp\left[-\frac{|z_1|^2}{2}\mathrm{sech}^2\lambda\right]\\ &\quad\times\mathrm{H}_{m,n}\left(\frac{\sqrt{2}}{\mathrm{i}\sqrt{\sinh 2\lambda}}z_1^*,\frac{\sqrt{2}}{\mathrm{i}\sqrt{\sinh 2\lambda}}a_2^\dagger\right)|z_1^*\tanh\lambda\rangle.\end{aligned} \tag{25}$$

Especially when $m=n$,

$$\mathrm{H}_{n,n}\left(\frac{\sqrt{2}z_1^*}{\mathrm{i}\sqrt{\sinh 2\lambda}},\frac{\sqrt{2}a_2^\dagger}{\mathrm{i}\sqrt{\sinh 2\lambda}}\right) = (-)^n n!\mathrm{L}_n\left(-\frac{2z_1^*a_2^\dagger}{\sinh 2\lambda}\right), \tag{26}$$

expression (25) becomes

$${}_1\langle z|S_2|m,n\rangle = \frac{\mathrm{sech}\lambda(-\tanh\lambda)^{\frac{m+n}{2}}}{\sqrt{m!n!}}\exp\left[-\frac{|z_1|^2}{2}\mathrm{sech}^2\lambda\right](-)^n n!\mathrm{L}_n\left(-\frac{2z_1^*a_2^\dagger}{\sinh 2\lambda}\right)|z_1^*\tanh\lambda\rangle. \tag{27}$$

Thus when the first mode is measured as a coherent state, the second mode collapses into a Laguerre polynomial excitation coherent state, which again exhibits the quantum entanglement.

## 4. Discussion

Based on the fact that the two-mode squeezed number state is a two-variable Hermite polynomial excitation of the two-mode squeezed vacuum state, the result of one-mode $l$-photon measurement for the two-mode squeezed number state $S_2 |m, n\rangle$ is obtained. It is found that the remaining field-mode simultaneously collapses into the number state $|n - m + l\rangle$ with the coefficient being a Jacobi polynomial of $n, m$ and $l$, which manifestly exhibits the entanglement between the two modes, i.e. it depends on the number-difference between the two modes. The second mode collapses into an excited coherent state when the first mode is measured as a coherent state. These results may be useful for quantum teleportation.

## Acknowledgment

It is a pleasure to thank Hu Li-Yun, Hao Xiang and Ren Jie for their fruitful discussions about the topic.

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