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Synchronization of time-delay chaotic systems on small-world networks with delayed coupling*

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By using the well-known Ikeda model as the node dynamics, this paper studies synchronization of time-delay systems on small-world networks where the connections between units involve time delays. It shows that, in contrast with the undelayed case, networks with delays can actually synchronize more easily. Specifically, for randomly distributed delays, time-delayed mutual coupling suppresses the chaotic behaviour by stabilizing a fixed point that is unstable for the uncoupled dynamical system.

Keywords: chaotic attractor, Ikeda model, scale-free network
PACC: 0545

1. Introduction

Over the past decade, synchronization of complex dynamical networks has received a great deal of attention from the nonlinear dynamical community. Attention has been most often focused on the synchronization of chaotic dynamical systems with relatively low dimensional chaotic attractor, for example using the models of Rössler or Lorenz.[1] It is of considerable interest to study the synchronization between hyperchaotic systems, i.e., the systems with a chaotic attractor, such that there are at least two positive Lyapunov exponents for the restriction of the system on the attractor.[2−8] A class of hyperchaotic systems, which will be considered here, is modelled by delay-differential equations (DDEs). As is well known, a simple nonlinear scalar DDE with a single fixed time lag $\tau, \dot{x}(t) = f(x(t),\, x(t-\tau))$ gives an infinite dimensional dynamical systems on the phase space $C(\tau, 0)$ of continuous functions on the interval $(\tau, 0)$.[9,10] Large τ usually implies high-dimensional chaotic attractor.[11−13] Besides the theoretical interest, the models given by DDEs often appear in applications, for example in biology, nonlinear optics, or secure communications.

Typically, the coupling between hyperchaotic units has a time delay due to the finiteness of signal transmission and switching speeds. The counterintuitive phenomenon has been extensively studied. For example, Voss[15] has observed that in the master-slave configuration there is a type of synchronization such that the slaved system anticipates the states of the driver. Later, Shahverdiev *et al*[16] applied the same methods to two Ikeda systems in the master-slave configuration with a particular form of the unidirectional delayed coupling, and studied lag and anticipating synchronization. Anticipating synchronization was also studied numerically in (master-slave) two Lang–Kobayashi laser in Refs.[17, 18]. Synchronization of three hyperchaotic bidirectionally delayed coupled Ikeda systems was studied in Ref.[19].

In this paper, we study synchronization of chaotic time-delay systems on small-world (SW) networks with delayed coupling, focusing on the influence of time delay. Two different situations will be considered: when the delay time in the coupling is the same for all the units; when the delay time is different for the different units. By taking the well-known Ikeda model as a node of linearly coupled SW network, we find that delayed coupling enhances the synchronization parameter region of the network. In particular, in the case of distributed delays, time-delayed coupling might suppress chaotic dynamics of the individual system by stabilizing an unstable steady states.

The rest of this paper is organized as follows. In Section 2, the SW dynamical network model with time

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delays is introduced. In Section 3, the synchronization of chaotic Ikeda systems on SW network is studied, both fixed delays and distributed delays cases. Finally, end this paper by a short conclusion in Section 4.

2. The SW dynamical network model

2.1. The SW network model

The SW networks have an intermediate connectivity between regular and random networks. They are characterized by a very small mean path length as in random networks while at the same time having a high clustering coefficient as in regular networks. In this paper, SW coupling is done as in the Newman–Watts algorithm.[20] We start with an N-node regular ring in which every node is connected to its k neighbours ($k/2$ on each size). Then with a probability p ($0 < p \ll 1$) we add new long-range edges (shortcuts) at random into the network. Duplicate edges are avoided. For $p = 0$, it reduces to the originally nearest-neighbour coupled system. For $p = 1$, it becomes a globally coupled system.

2.2. The SW dynamical network model

We consider an SW dynamical network consisting of N identical linearly and diffusively coupled nodes. Each node stands for an m-dimensional dynamical system, and a link connecting two nodes represents the coupling between them. The state equations of the network are

$$\dot{x_i}(t) = f(x_i(t), t) + \frac{\varepsilon}{k_i} \sum_{j=1}^{N} a_{ij}(x_j(t - \tau_{ij}) - x_i(t)),$$
$$i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{im}) \in R^m$ are the state variables of nodes i, $f(\cdot)$ describes the dynamics of each individual node. The matrix $A = (a_{ij}) \in R^{N \times N}$ defines the connectivity of the network, where $a_{ij} = a_{ji} = 1$ if there is a connection between the i-th and j-th nodes, and zero otherwise. ε is the coupling strength and τ_{ij} is the delay time in the interaction between the i-th and the j-th nodes. The sum in Eq.(1) runs over the k_i nodes which are coupled to the i-th node ($k_i = \sum_j a_{ij}$). The normalized prefactor $1/k_i$ means that each node receives the same total input from its neighbours.

In this paper, we choose the well-known Ikeda model as the local dynamics, described by

$$f(x(t), t) = -x(t) + \mu \sin(x(t - \tau_0)), \tag{2}$$

as a result of the existence of the delay time τ_0, the dynamical system (2) is an infinite-dimensional system. The Ikeda model was introduced to describe the dynamics of an optical bistable resonator and well-known for delay-induced chaos behaviour.[21] The character of chaos, in particular the number of positive Lyapunov exponents and their values, for a single Ikeda equation has been thoroughly studied.[12–14] For the parameter values $\tau_0 = 30$ and $\mu = 3$, a hyperchaotic attractor, with at least two positive Lyapunov exponents (Fig.1) occurs for the uncoupled system.[22] Embedded in this attractor, there are three fixed points $x = 0, \pm 2.27886$.

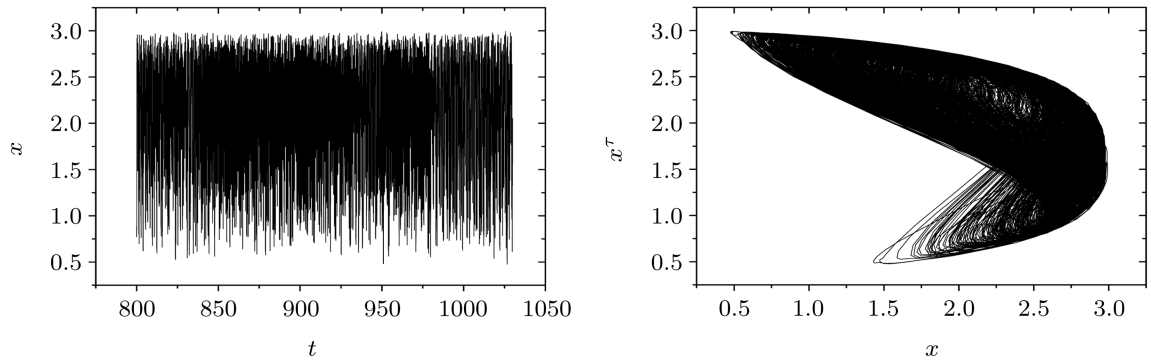


Fig.1. Chaotic attractor of the free running Ikeda system at $\tau = 30.0$, $\mu = 3.0$.

So the whole coupled system is given by

$$\begin{aligned}\dot{x_i}(t) &= -x_i(t) + \mu \sin(x_i(t-\tau_0)) \\ &\quad + \frac{\varepsilon}{k_i}\sum_{j=1}^{N} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)), \\ i &= 1, 2 \ldots, N. \end{aligned} \tag{3}$$

The system of DDE (3) has different time lags τ_0 and τ_{ij}, which makes its analysis more difficult than for the DDE with only one time lag. Furthermore, they appear in different ways and play quite different roles. The time lag τ_0 is an intrinsic parameter of each of the units and controls the complexity of the dynamics of the uncoupled unit, and the time lag τ_{ij} measures the time needed for the transfer of information between the i-th and the j-th units.

3. Numerical results

3.1. Synchronization in the fixed-delay case

First we study the effects of the same delay coupling on linearly coupled Ikeda systems

$$\begin{aligned}\dot{x_i}(t) &= -x_i(t) + \mu \sin(x_i(t-\tau_0)) \\ &\quad + \frac{\varepsilon}{k_i}\sum_{j=1}^{N} a_{ij}(x_j(t-\tau_1) - x_i(t)), \\ i &= 1, 2 \ldots, N \end{aligned} \tag{4}$$

where $\tau_{ij} = \tau_1$ is the time delay in the interaction between the units. All simulations are carried out for the network size $N = 1000$ and average degree $\langle k \rangle = 6$. Each data point is obtained by averaging 10 network realizations. Figure 2 displays the critical value of ε vs adding edges probability p for different delay time τ_1. We can see that the critical coupling strength ε_c, above which global synchronization occurs, decreases with the increase of p, i.e., synchronization is greatly improved by adding a small fraction of random shortcuts. But as the number of added shortcuts grows, the synchronization behaviours of SW and random network converge. On the other hand, it is also clear that for zero delay, synchronization can occur only for $p > 0.2$. With the increase of τ_1, the value of ε_c also decreases, that is to say, networks with delays can actually synchronize more easily than the case that networks without delays.

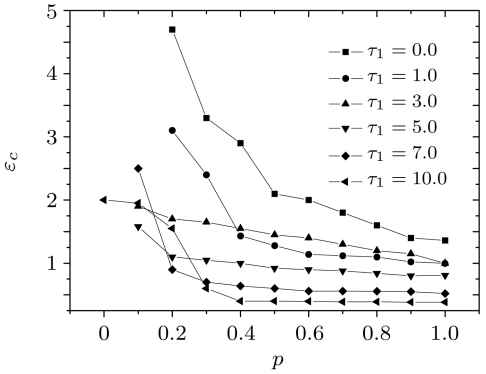


Fig.2. Critical coupling strength ε_c of synchronization on SW networks for different τ_1 and p.

To clearly exhibit the influence of the delay time τ_1 on synchronization, we plot the evolution of ε_c as a function of τ_1 for the SW network with $N = 1000$, $p = 0.3$, $\mu = 3.0$ and $\tau_0 = 30$ in Fig.3. It is evident that the network synchronization is very sensitive to τ_1. As the increase of τ_1, ε_c is observed to decrease, indicating the enhancement of synchronization.

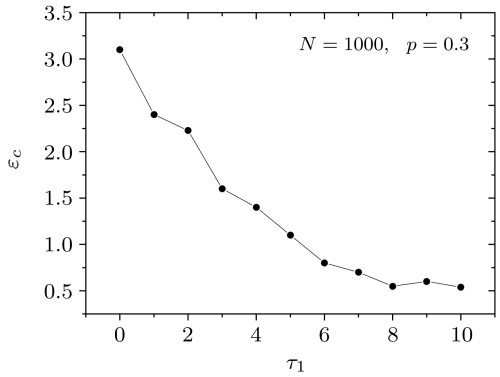


Fig.3. Critical coupling strength ε_c of synchronization on SW networks for different delay τ_1. Here $N = 1000$, $p = 0.3$.

In addition, we introduce a synchronization error as

$$\langle D \rangle = \frac{1}{(\triangle T)N}\sum_{i=2}^{N}\int_{T}^{T+\triangle T} \|x_i(t) - x_1(t)\| \mathrm{d}t. \tag{5}$$

In Fig.4 we investigate the behaviour of $\langle D \rangle$ vs the coupling strength ϵ under different coupling delay τ_1, and find that the networks with larger delay might be easier to synchronize.

These results undoubtedly point out that the presence of delays can indeed facilitate synchronization of time-delay chaotic systems.

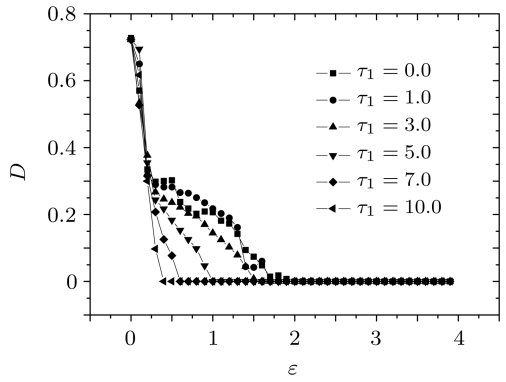


Fig.4. Synchronization error D as a function of coupling strength ε for different delay τ_1.

3.2. Synchronization in the case of random delays

However, actual delays in real systems are not necessarily the same for all the elements of the system, they may be distance dependent or randomly distributed. In populations of spatially separated neurons, the synaptic communications between them involve distributed delays. In computer networks, random delays arise from queueing times and propagation times.

In this section, we consider Gaussian distributed delays: $\tau_{ij} = \tau_0 + \text{near}(c\xi)$, where c is a parameter that allows varying the width of the delay distribution; ξ is Gaussian distributed with zero mean and standard deviation one; near denotes the nearest integer. Depending on τ_0 and c, we truncate the distribution of delays to avoid negative delays (see Fig.5).

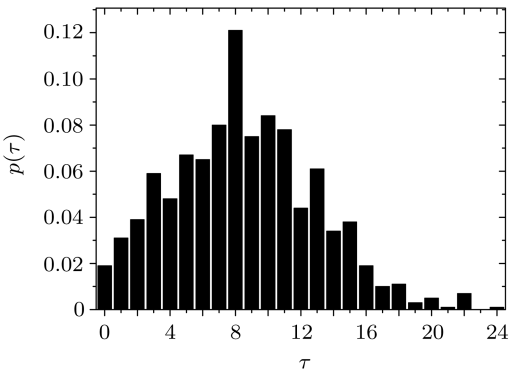


Fig.5. The coupling delays are Gaussian distributed with $\tau_0 = 8$, $c = 5.0$.

In Fig.6, the evolution of $x_i(t)$ $(i = 1, 2, \ldots, N)$ is plotted versus time. It is seen that the dynamical network successfully stabilizes to the unstable steady state $x = 2.27886$ of the uncoupled Ikeda system by $t \geq 40$.

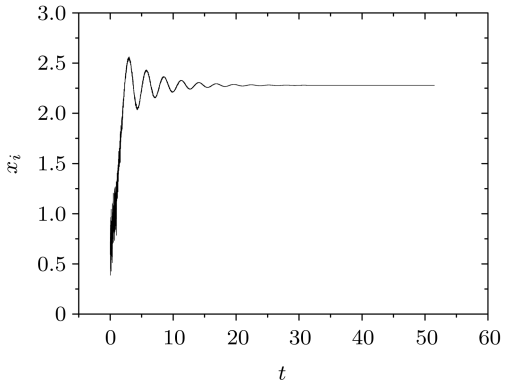


Fig.6. Time series of x_i of Ikeda systems on small-world networks for Gaussian distributed delays.

4. Conclusion

We have investigated the synchronization of linearly coupled time-delay systems on SW networks, both the fixed-delay and random delays cases. The hyperchaotic nature of the dynamics of the single unit is produced by an intrinsic time delay τ_0, using as the units of the Ikeda model, and the time delay τ_{ij} in the interaction between the units is taken explicitly. We found that for adequate coupling strength and time delay, the whole dynamical networks can not only synchronize in a spatially homogeneous state but also can lead to better synchronization than the undelayed case. Specifically, for random delay times, the network synchronizes in a homogeneous steady state, which is unstable for the uncoupled Ikeda system. The dynamical system described by DDEs differs from the system modelled by ordinary differential equations and is an infinite-dimensional system, so our findings shed some light on the collective dynamics of infinite-dimensional systems on complex networks.

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