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Stochastic resonance in a single-mode laser driven by quadratic pump noise and amplitude-modulated signal*

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This paper investigates the phenomenon of stochastic resonance in a single-mode laser driven by quadratic pump noise and amplitude-modulated signal. A new linear approximation approach is advanced to calculate the signal-to-noise ratio. In the linear approximation only the drift term is linearized, the multiplicative noise term is unchangeable. It is found that there appears not only the standard form of stochastic resonance but also the broad sense of stochastic resonance, especially stochastic multiresonance appears in the curve of signal-to-noise ratio as a function of coupling strength λ between the real and imaginary parts of the pump noise.

Keywords: quadratic pump noise, new linear approximation, stochastic resonance
PACC: 0540, 4260K

1. Introduction

The notion of the cross-correlation between the real and imaginary parts of complex noise was first introduced to the Langevin equation (LE) for a single-mode laser by Zhou *et al* in 1996.[1] Zhang *et al*[2−5] did further research after that, but all these works were limited to the cross-correlation between the real and imaginary parts of the quantum noise, since the cross-correlation between the real and imaginary parts of the pump noise was studied less possibly. In our latest work,[6−9] the cross-correlation between the real and imaginary parts of coloured pump noise was successfully introduced to the laser model by virtue of nonlinear external noise, however the response of a system driven by nonlinear external noise to an external periodic force has not been investigated.

This paper has two purposes. One is to seek an approximate approach to process the nonlinear Langevin equation (NLE), the approach is desired to ensure that the obtained approximate equation not only retains the essence of the multiplicative noise but also can be solved accurately. We can only linearize the drift term of NLE, the multiplicative noise term is still unchangeable, thus the obtained approximate NLE can achieve this aim. Another one is to reveal the response of a system driven by nonlinear external noise to an amplitude-modulated periodic force.

The rest of this paper is organized as follows. In Section 2, the approximate NLE is derived by means of a new linear approximation approach. The signal-to-noise ratio (SNR) of the system driven by an amplitude-modulated signal is then calculated. In Section 3, the effects of the pump noise intensity Q, the correlation coefficient λ, the net gain coefficient a, and the modulated frequency Ω of signal amplitude on SNR are discussed, and conclusions are drawn.

2. Theoretical approach

We consider a cubic Lamb laser model[10] with quadratic coloured pump noise described by the following Langevin equation (LE):[6,7]

$$\dot{E} = aE - A\left|E\right|^2 E + [2\sqrt{a}\zeta_c(t) + \zeta_c^2(t)]E, \qquad (1)$$

where E is the complex laser field, a and A are the net gain and the self-saturation coefficient, respectively. The random variable $\zeta_c(t) = \zeta_{c\,R}(t) + \mathrm{i}\zeta_{c\,I}(t)$ represents the complex pump noise, the noise term is assumed to be an O-U process, whose statistical prop-

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erties are

$$\langle \zeta_{c\,i}(t)\rangle = 0, \quad i = I,\ R, \tag{2}$$

$$\langle \zeta_{c\,R}(t)\zeta_{c\,R}(t')\rangle = \langle \zeta_{c\,I}(t)\zeta_{c\,I}(t')\rangle = \frac{Q}{\tau}\mathrm{e}^{-\frac{|t-t'|}{\tau}}, \tag{3}$$

$$\langle \zeta_{c\,R}(t)\zeta_{c\,I}(t')\rangle = \langle \zeta_{c\,I}(t)\zeta_{c\,R}(t')\rangle = \frac{\lambda Q}{\tau}\mathrm{e}^{-\frac{|t-t'|}{\tau}}, \tag{4}$$

where I and R represent the real and imaginary parts of pump noise, respectively, λ is the cross-correlation coefficient between the real and imaginary parts of pump noise, $-1 \le \lambda \le 1$, Q is the pump noise intensity, τ is the pump noise correlation time.

Corresponding to the Eq.(1), we find that the NLE of the laser intensity variable ($I = |E|^2$) reads

$$\dot{I} = 2aI - 2AI^2 + 2I(2\sqrt{a}\zeta_R(t) - 2\sqrt{1-\lambda^2}\lambda\zeta_I(t)\zeta_R(t)), \tag{5}$$

where the cross-correlated noise $\zeta_c(t)$ is replaced by the non-correlation coloured noise $\zeta(t) = \zeta_R(t) + \mathrm{i}\zeta_I(t)$,[11] the statistical properties of $\zeta(t)$ are $\langle\zeta(t)\rangle = 0$, and $\langle\zeta_a(t)\zeta_b(t')\rangle = \delta_{a,\,b}(Q/\tau)\mathrm{e}^{-\frac{|t-t'|}{\tau}}$ ($a,\ b = I,\ R$), $\delta_{a,\,b}$ is Kronecker symbol. Due to the coloured nature of $\zeta(t)$, the intensity I undergoes a non-Markovian process. The Markovian approximation is performed to Eq.(5) on condition that $\sqrt{Q\tau} \ll 1$,[12] and the approximate Fokker–Planck equation is obtained:

$$\frac{\partial P(I,t)}{\partial t} = -\frac{\partial}{\partial I}f(I)P(I,t) + D_0\frac{\partial}{\partial I}g(I)\frac{\partial}{\partial I}g(I)P(I,t) \tag{6}$$

with

$$f(I) = 2aI - 2AI^2, \quad g(I) = 2I,$$
$$D_0 = 4aQ + \frac{4Q^2}{\tau}(\lambda^2 - \lambda^4).$$

In this paper, only the above threshold operation is discussed, namely $a > 0$.

A Stratonovich intensity LE stochastically equivalent to Eq.(6) reads[13]

$$\dot{I} = 2(a + 2D_0)I - 2AI^2 + 2\sqrt{D_0}I\eta(t) + (B\cos\Omega t)\cos\bar{\omega}t, \tag{7}$$

where an amplitude-modulated signal $(B\cos\Omega t)\cdot\cos\bar{\omega}t$ is added, and $\eta(t)$ is Gaussian white noise with

$$\langle\eta(t)\rangle = 0, \quad \langle\eta(t)\eta(t')\rangle = 2\delta(t-t'). \tag{8}$$

It is deserved to point out that there are two different kinds of extent of linear approximation for Eq.(7). One is the so-called 'additive linearization' proposed by Gorcia in 1990.[14] In this approximation besides the drift term Eq.(7) is linearized, the laser intensity I of the multiplicative noise term $2\sqrt{D_0}I\eta(t)$ is also simplified to the deterministic steady-state intensity I_0, so the multiplicative noise term $2\sqrt{D_0}I\eta(t)$ is substituted by the additive noise term $2\sqrt{D_0}I_0\eta(t)$. Equation (7) is simplified to a linear LE which is easy to solve and make statistical average. This linear approximation can be used for Markovian process and for non-Markovian process as well, nevertheless this approximation may lose the important physical effect induced by multiplicative noise, for example noise-induced phase transition.[15] The other one is that the drift term of Eq.(7) is only linearized, the multiplicative noise does not change. In this paper we will adopt this new linear approximation which can avoid losing the multiplicative noise effect. The specific way is as follows: writing $I(t) = I_0 + \varepsilon(t)$ ($\varepsilon(t)$ is small disturbance around the deterministic steady-state intensity $I_0 = a/A$), and inserting it into Eq.(7), we have

$$\dot{\varepsilon}(t) = K\varepsilon(t) + 2\sqrt{D_0}\varepsilon(t)\eta(t) + 2\sqrt{D_0}I_0\eta(t) + (B\cos\Omega t)\cos\bar{\omega}t, \tag{9}$$

where $K = -2(a + 2D_0)$. Though equation (9) is still NLE,[13] the formal solution can be given by

$$\varepsilon(t) = \exp\Big\{\int_0^t [K + 2\sqrt{D_0}\eta(s')]\mathrm{d}s'\Big\} \times \Big\{\int_0^t [2\sqrt{D_0}I_0\eta(s) + (B\cos\Omega s)\cos\bar{\omega}s] \times \exp\Big\{-\int_0^s [K + 2\sqrt{D_0}\eta(s')]\mathrm{d}s'\Big\}\mathrm{d}s\Big\}. \tag{10}$$

Here when $t = 0$, $\varepsilon(t)$ is assumed to be $\varepsilon(0) = 0$, which does not affect the universality of the consequence.

The intensity correlation function is defined by

$$C(t) = \lim_{t'\to\infty}\overline{\langle I(t')I(t'+t)\rangle} = \lim_{t'\to\infty}\overline{\langle (I_0 + \varepsilon(t'))(I_0 + \varepsilon(t'+t))\rangle}, \tag{11}$$

where $\langle I(t')I(t'+t)\rangle$ of the dual average $\overline{\langle I(t')I(t'+t)\rangle}$ is the average for the realization of the noise, whereas $\overline{\langle I(t')I(t'+t)\rangle}$ is the periodical average of $\langle I(t')I(t'+t)\rangle$, that is

$$\overline{\langle I(t')I(t'+t)\rangle} = \frac{\Omega}{2\pi}\int_{t'}^{\frac{2\pi}{\Omega}+t'}\langle I(t')I(t'+t)\rangle \mathrm{d}t'. \quad (12)$$

From the definition of $C(t)$, if we get $\langle\varepsilon(t')\rangle$ and $\langle\varepsilon(t')\varepsilon(t'+t)\rangle$ by using the formal solution expressed by Eq.(10), the analytic expression of $C(t)$ can be calculated. Nevertheless in Eq.(10), the noise term appears in exponent, the more complicated average of nonlinear noise function, such as $\langle\eta(s)\exp -\int\zeta(s')\mathrm{d}s'\rangle$ also appears, the linear approximation addressed in this paper is still confronted with many difficulties. Fortunately equation (9) undergoes the Markovian process, that is, the noise in Eq.(9) is in the case of Gaussian white noise, the average formulas for Gaussian white noise have been derived in Refs.[16, 17] as follows:

$$\langle \mathrm{e}^{g\int_0^t\zeta(s')\mathrm{d}s'}\cdot\mathrm{e}^{g\int_0^{t+t'}\zeta(s')\mathrm{d}s'}\rangle = \langle \mathrm{e}^{2g\int_0^t\zeta(s')\mathrm{d}s'}\cdot\mathrm{e}^{g\int_t^{t+t'}\zeta(s')\mathrm{d}s'}\rangle = \mathrm{e}^{2g^2Qt}\cdot\mathrm{e}^{\frac{1}{2}g^2Qt'}, \quad (13)$$

$$\langle\eta(s)\eta(\bar{s})\mathrm{e}^{g\int_0^t\zeta(s')\mathrm{d}s'}\cdot\mathrm{e}^{g\int_{\bar{s}}^t\zeta(s')\mathrm{d}s'}\rangle = (D\delta(s-\bar{s})+\lambda^2QD)\mathrm{e}^{\frac{1}{2}g^2Q(\bar{s}-s)}\cdot\mathrm{e}^{2\gamma^2Q(t-\bar{s})}, \quad (14)$$

$$\langle\eta(s)\mathrm{e}^{-\int_s^{\bar{s}}\zeta(s')ds'}\rangle = -\frac{\lambda\sqrt{QD}}{2}\mathrm{e}^{\frac{1}{2}Q(\bar{s}-s)}. \quad (15)$$

So after tedious calculations, the intensity correlation function is given by

$$C(t) = M\mathrm{e}^{(K+2D_0)t} + \frac{B^2}{4}\cdot\frac{\cos(\Omega+\bar{\omega})t}{(K+2D_0)^2+(\Omega+\bar{\omega})^2} + \frac{B^2}{4}\cdot\frac{\cos(\Omega-\bar{\omega})t}{(K+2D_0)^2+(\Omega-\bar{\omega})^2}, \quad (16)$$

where

$$M = -\frac{2D_0I_0^2}{K+4D_0} - \frac{4D_0^2I_0^2}{(K+2D_0)^2} + \frac{16D_0^2I_0^2}{(K+2D_0)(K+6D_0)} + \frac{B^2}{4}\left[\frac{1}{(K+2D_0)^2+(\Omega+\bar{\omega})^2} + \frac{1}{(K+2D_0)^2+(\Omega-\bar{\omega})^2}\right].$$

By the Fourier transform of Eq.(16), the power spectrum reads

$$S(\omega) = \int_{-\infty}^{\infty}C(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t = S_1(\omega)+S_2(\omega), \quad (17)$$

where $S_1(\omega)$ is the power spectrum of the output signal ($S_1(\omega)$ defined for ω positive only).

$$S_1(\omega) = \frac{B^2}{8}\frac{\sqrt{2\pi}}{(K+2D_0)^2+(\Omega+\bar{\omega})^2}\delta(\omega-(\Omega+\bar{\omega})) + \frac{B^2}{8}\frac{\sqrt{2\pi}}{(K+2D_0)^2+(\Omega-\bar{\omega})^2}\delta(\omega-|\Omega-\bar{\omega}|), \quad (18)$$

$$S_2(\omega) = M\left(\frac{\sqrt{2/\pi}(K+2D_0)}{(K+2D_0)^2+\omega^2}\right). \quad (19)$$

The SNR is defined as

$$\mathrm{SNR} = \frac{P_s}{S_2(\omega=\Omega+\bar{\omega})+S_2(\omega=\Omega-\bar{\omega})}, \quad (20)$$

where $P_s = \int_0^{\infty}S_1(\omega)\mathrm{d}\omega$ is the output power of the signal, $S_2(\omega=\Omega\pm\bar{\omega})$ is the noise part of the spectrum at $\omega=\Omega\pm\bar{\omega}$.

3. Discussions and conclusions

Various forms of stochastic resonance (SR) are inherent to the analytical results of this paper, what is more, some phenomena are revealed first. The curve of SNR versus noise strength Q exists a resonance peak, which is the standard form of stochastic resonance, see Fig.1. Figure 2 depicts the curve of SNR versus correlation coefficient λ. The SNR $\sim\lambda$ curve exhibits multi-resonance peaks. This phenomenon is the new form of SR, which is not seen in the references heretofore. To trace the origin, it is the consequence of quadratic pump noise, SNR is a higher nonlinear function with respect to λ. The curve of SNR versus net gain coefficient a also exists a resonance peak, see Fig.3. Since a is not the noise parameter but the

parameter of system itself, the SR of Fig.3 belongs to SR in broad sense.[17] Figure 4 depicts the curve of

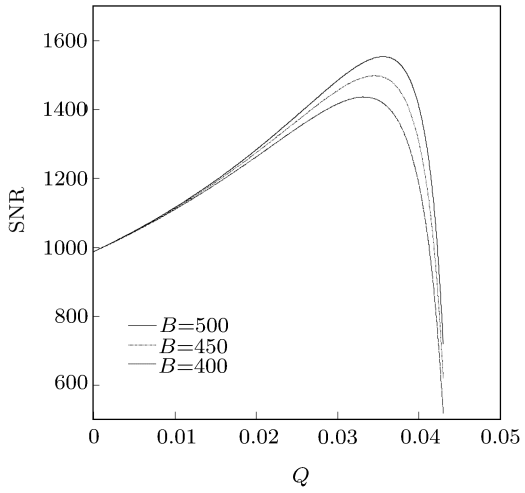


Fig.1. Signal-to-noise ratio as a function of the pump noise intensity Q. The parameters are chosen as follows: $a = 4$, $A = 3$, $\lambda = 0.7$, $\tau = 0.02$, $\bar{\omega} = 100$, and $\Omega = 1.3$. All parameters are dimensionless.

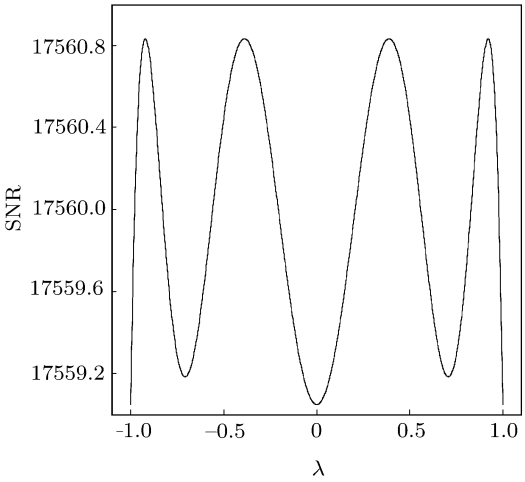


Fig.2. Signal-to-noise ratio as a function of the correlation coefficient λ. The parameters are chosen as follows: $a = 0.0575$, $A = 0.03$, $Q = 0.01$, $\tau = 0.4$, $\bar{\omega} = 50$, $B = 102$, and $\Omega = 1$. All parameters are dimensionless.

SNR versus the modulated frequency of signal amplitude Ω. The curve exhibits an intense suppression at $\Omega = \bar{\omega}$ (minimum), which is similar to 'reverse resonance' of mechanics. The cause of above phenomena is not entirely understood at present. These features still await a simple physical explanation. We hope that the further research will disentangle in great detail these complex novel features.

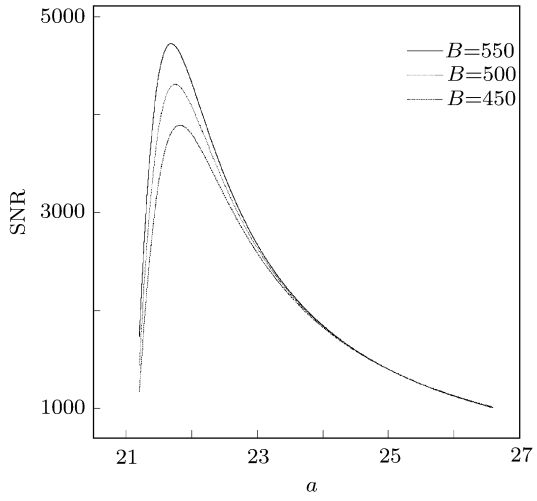


Fig.3. Signal-to-noise ratio as a function of the net gain coefficient a. The parameters are chosen as follows: $\lambda = 0.7$, $A = 3$, $Q = 0.03$, $\tau = 0.002$, $\bar{\omega} = 30$, and $\Omega = 13$. All parameters are dimensionless.

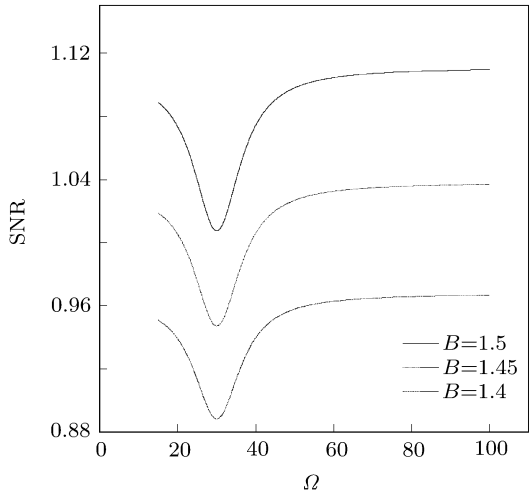


Fig.4. Signal-to-noise ratio as a function of the modulated frequency Ω of signal amplitude. The parameters are chosen as follows: $\lambda = 0.7$, $A = 3$, $Q = 0.01$, $\tau = 0.01$, $a = 4$, and $\bar{\omega} = 30$. All parameters are dimensionless.

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