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An exact inflationary solution in the chaotic model with non-minimal coupling*

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This paper presents a new exact inflationary solution to the non-minimally coupled scalar field. The inflation is driven by the evolution of a scalar field with inflation potential $V(\varphi) = (\lambda/4)\varphi^4 + b_1\varphi^2 + b_2 + b_3\varphi^{-2} + b_4\varphi^{-4}$. The spectral index of the scalar density fluctuations n_s is consistent with the result of WMAP3 (Wilkinson Microwave Anisotropy Probe 3) for ΛCDM (Lambda-Cold Dark Matter). This model relaxes the constraint to the quartic coupling constant. And it can enter smoothly into a radiation-dominated stage when inflation ends.

Keywords: exact solution, chaotic inflation, non-minimally coupling, spectral index
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1. Introduction

Inflation theory is one of the most successful cosmological theories. It gives us not only the solution of major puzzles of big bang theory such as the horizon and flatness problems, but also the generation of primordial perturbations. The first semi-realistic model of inflation was proposed by Starobinsky in 1979.[1] This model is based on the investigation of a conformal anomaly in quantum gravity, and on the assumption of a homogeneous and isotropic universe at the very beginning. Obviously, the assumption diverge the original goals of proposing inflation scenario. More achievements of gravitational anomaly see Refs.[2–4]. The 'old inflation'[5] and the 'new inflation'[6,7] represented a substantial but incomplete modification of the big bang theory. They both implied that the scalar field φ takes only a specific value at the initial time, and they still assumed that the universe was in a state of thermal equilibrium from the very beginning that it was relatively homogeneous and large enough to survive until the beginning of inflation, this seems somewhat artificial. Fortunately, the chaotic inflation scenario,[8] proposed by Linde in 1983, perfectly avoids these drawbacks. According to this scenario, a more natural ideal is that the scalar field φ can take a sort of values at the very beginning, and inflation may begin even if there was no thermal equilibrium in the early universe. Moreover, it has little limit to potential: chaotic inflation can occur in any theory where the potential has a sufficiently flat region which allows the existence of the slow-roll regime. The chaotic inflation scenario is now widely accepted.

The minimal coupling case of chaotic inflation is based on the assumption that the coupling parameter in the action describing the interaction between the gravitational field and the scalar field is zero (i.e. $\xi = 0$). Recently, the non-minimal coupling case (i.e. $\xi \neq 0$) has attracted a great deal of attention as a possible improvement of the minimal coupling one.[9] The researches on quantum field in curved space–time show that the non-minimal coupling is actually required when the space–time curvature is large.[10]

Most detailed studies of inflation were made by using numerical integration, or by employing some approximation schemes. The slow-roll approximation[7] is the one used most widely. This approximation neglects the most slowly changing terms in the equations of motion and works well in many minimal coupling cases. But it must eventually be failed because the inflation has to end. Moreover, even weak violations from the inflation would result in large deviations from the observable predications such as the spectrum of density perturbations. As to non-minimal case, it is more difficult and unreliable to achieve this approximation. Fortunately, slow-roll is not the only possibil-

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ity for successfully implementing models of inflation. A new way has been found. Generally, the Hubble parameter H cannot be an exact constant, but can vary along with the scalar field φ. A convenient approach is to express H directly as a function of φ instead of as a function of time t, namely $H = H(\varphi)$.[11] This method is creditable because it allows the full dynamical behaviour of field to be investigated in terms of the function $H(\varphi)$ thus does not have to assume that friction terms in the field equations dominate or that the field kinetic energy is negligible. This approach can be used to solve some exact inflationary solutions, see Refs.[12, 13] for example.

In this paper we will make use of the new approach to get an exact inflationary solution in the chaotic model to non-minimally coupled scalar field, and check the results with the observed data. The paper is organized as follows. In Section 2 we get an exact inflationary solution to non-minimally coupled scalar field. In Section 3 we calculate the spectral index of density fluctuations and the quartic coupling constant and compare the results with the observed values. The last section is our discussion.

2. Exact inflationary solution

Let us start from the action of non-minimal coupling[14]

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[\Big(\frac{1}{16\pi G} + \frac{1}{2}\xi\varphi^2\Big)R + \frac{1}{2}g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} - V(\varphi)\Big], \tag{1}$$

where G is the Newton gravitational constant, $V(\varphi)$ is a generic potential for the scalar field φ, and $\xi\varphi^2R/2$ is the coupled term between the scalar field and the space–time curvature. In inflationary theory it is assumed that the scalar field dominates the evolution of the universe and that no forms of matter other than the scalar field are included in the Lagrangian density. It is clearly that the action will decay to minimally coupling one when $\xi = 0$. In this paper we assume a homogeneous distribution for the scalar field, and the line element is taken to be that of a Robertson–Walker universe

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}x^2, \tag{2}$$

where $a(t)$ is the scale factor of universe, and the spatial curvature is set to be zero because it is unimportant in this context.

From Eqs.(1) and (2) we can get the dynamic equations of non-minimal coupling

$$\ddot{\varphi} + 3H\dot{\varphi} - 6\xi\varphi(\dot{H} + 2H^2) + V'(\varphi) = 0, \tag{3}$$

$$3H^2\Big(\frac{1}{8\pi G} + \xi\varphi^2\Big) = \Big[\frac{1}{2}\dot{\varphi}^2 + V(\varphi) - 6\xi H\varphi\dot{\varphi}\Big], \tag{4}$$

$$\dot{H}\Big(\frac{1}{8\pi G} + \xi\varphi^2\Big) = -\frac{1}{2}\dot{\varphi}^2 + \xi H\varphi\dot{\varphi} - \xi\dot{\varphi}^2 - \xi\varphi\ddot{\varphi}, \tag{5}$$

where $V'(\varphi) \equiv \mathrm{d}V(\varphi)/\mathrm{d}\varphi$. Obviously, according to the Bianchi identity, there are only two independent field equations. From the above three equations we can obtain the following functions, for more details see Ref.[15]:

$$V(\varphi) = \Big[\frac{3}{8\pi G} + \frac{3}{2k}\Big(1 + \frac{3}{k}\Big)\varphi^2\Big]H^2 - \frac{1}{2}\Phi^2(\varphi), \tag{6}$$

$$\dot{\varphi} = \frac{3}{k}\varphi H \pm \Phi(\varphi), \tag{7}$$

$$\Phi(\varphi) = C(\varphi)\varphi^{-(k+1)}, \tag{8}$$

$$C'(\varphi) = \mp\varphi^{k+1}\Big\{\Big[\frac{k}{4\pi G\varphi} + \Big(1 + \frac{3}{k}\Big)\varphi\Big]H' + 2\Big(1 + \frac{3}{k}\Big)H\Big\}, \tag{9}$$

where $k \equiv 1/(2\xi)$, and the primes denote the differentiation with respect to φ, and when $H' > 0$ choose the up signs of $\pm$ and $\mp$, otherwise, use the down ones. From the above four functions we see that once $H(\varphi)$ is given, $C(\varphi)$ can be solved from Eq.(9), and thus $\Phi(\varphi)$ is defined according to Eq.(8), then equations (7) and (6) can be solved. Based on this thinking, we won't start our work from $V(\varphi)$ as usually do, but from a specific $H(\varphi)$ to get an exact inflationary solution of the universe.

In this paper we set

$$H(\varphi) = \alpha\varphi - \frac{\beta}{\varphi}, \tag{10}$$

where α and β are positive constants. From Eqs.(9) and (10), we find

$$C = -\frac{3\alpha}{k}\varphi^{k+3} - \Big[km - \frac{(k+3)}{k(k+1)}\beta\Big]\varphi^{k+1} - kn\varphi^{k-1}, \tag{11}$$

where $m = \dfrac{\alpha}{4\pi G(k+1)}$, $n = \dfrac{\beta}{4\pi G(k-1)}$, and the integration constant is assumed to be zero. Substituting Eq.(11) into Eq.(8), we get

$$\Phi(\varphi) = -\frac{3\alpha}{k}\varphi^2 - \Big[km - \frac{(k+3)}{k(k+1)}\beta\Big] - kn\varphi^{-2}. \tag{12}$$

Substituting Eqs.(10) and (12) into Eq.(7), we get

$$(\varphi_0-\varphi)+q\ln\left[\frac{(\varphi_0-q)(\varphi+q)}{(\varphi-q)(\varphi_0+q)}\right]=\left(km+\frac{2}{k+2}\beta\right)t, \tag{13}$$

where $q=\sqrt{\left|\frac{kn}{km+2\beta/(k+1)}\right|}$, and φ_0 is the value of scalar field when $t=0$. From Eqs.(6), (10) and (12), we have

$$V(\varphi)=\frac{\lambda}{4}\varphi^4+b_1\varphi^2+b_2+b_3\varphi^{-2}+b_4\varphi^{-4}, \tag{14}$$

where

$$\lambda=\frac{6\alpha^2}{k}, \tag{15}$$

$$b_1=3(k-1)\alpha m-\frac{3(k+3)}{k(k+1)}\alpha\beta, \tag{16}$$

$$b_2=\frac{(3k^2+14k+14)}{2k(k+1)^2}\beta^2-\frac{1}{2}k^2m^2-\frac{(3k^3+5k^2+k+3)}{(k+1)^2}n\alpha, \tag{17}$$

$$b_3=\frac{(3k^2+2k+3)}{(k+1)}n\beta-k^2mn, \tag{18}$$

$$b_4=-\frac{1}{2}k^2n^2. \tag{19}$$

Finally, according to $H=\dot{a}/a$ and Eqs.(10), (13), we find

$$a=a_0[w\varphi^2+kn]^{(\beta+\alpha q^2)/2w}\exp(-\alpha\varphi^2/2w), \tag{20}$$

where $w=km+2\beta/(k+2)$, and a_0 is a positive constant. The equations (10) and (13)–(20) give an exact inflationary solution of the universe. From these equations we find that all physical variables are finite and analytic as t approaches to zero. Thus our solution can be free of initial singularity problem. In the next section we will see whether our solution is consistent with the observed data of the universe or not.

3. Scalar density fluctuation and quartic coupling constant

An accurate calculation of the perturbation spectra is in terms of the Hubble parameter following the conformal transformation $\hat{H}(\phi)$ and its derivatives, where ϕ is a newly defined scalar field

$$\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}\equiv\frac{1}{\sqrt{8\pi G}}\sqrt{\frac{2f(\varphi)+3[f'(\varphi)]^2}{2f^2(\varphi)}}\approx\Omega^{-1}\sqrt{1+6\xi}, \tag{21}$$

where

$$\Omega^2=1+8\pi G_N\xi\varphi^2, \tag{22}$$

and $f(\varphi)\equiv(1+8\pi G\xi\varphi^2)/8\pi G$, and the strong coupling condition $8\pi G\xi\varphi^2\gg1$ is assumed.[16] We make a particular conformal transformation[17]

$$\hat{g}^{\mu\nu}=\Omega^{-2}g^{\mu\nu}. \tag{23}$$

The line element of the conformal space–time is

$$\mathrm{d}\hat{s}^2=\Omega^2[\mathrm{d}t^2-a^2(t)\mathrm{d}x^2]. \tag{24}$$

If we further make the following change of variables:

$$\mathrm{d}\hat{t}=\Omega\mathrm{d}t,\quad \hat{a}=\Omega a, \tag{25}$$

then equation (23) can be brought into the same form as Eq.(2)

$$\mathrm{d}\hat{s}^2=\mathrm{d}\hat{t}^2-\hat{a}^2(\hat{t})\mathrm{d}\hat{x}^2. \tag{26}$$

Now we can rewrite the theory in terms of these new variables as a minimally coupled theory with a metric (26) and an action

$$S=\int\mathrm{d}^4\hat{x}\sqrt{-\hat{g}}\left[\frac{1}{16\pi G_N}\hat{R}+\frac{1}{2}\hat{g}^{\mu\nu}\phi_{;\mu}\phi_{;\nu}-\hat{V}(\phi)\right], \tag{27}$$

where

$$\hat{V}(\phi)=\Omega^{-4}V(\varphi). \tag{28}$$

In the new theory, the spectral index of the scalar density fluctuations is given by[18]

$$n_{\mathrm{s}}=1-4\varepsilon+2\eta-8(1+c)\varepsilon^2+2(3+5c)\varepsilon\eta-2c\varepsilon\zeta, \tag{29}$$

where $c\equiv-2+\ln2+\gamma\approx-0.73$ and $\gamma\approx0.577$ is Euler's constant. Clearly n_{s} depends only on three parameters ε, η and ζ, which are defined by

$$\varepsilon=\frac{1}{4\pi G_N}\left[\frac{\hat{H}'(\phi)}{\hat{H}(\phi)}\right]^2, \tag{30}$$

$$\eta=\frac{1}{4\pi G_N}\left[\frac{\hat{H}''(\phi)}{\hat{H}(\phi)}\right], \tag{31}$$

$$\zeta=\frac{1}{4\pi G_N}\left[\frac{\hat{H}'(\phi)\hat{H}'''(\phi)}{\hat{H}^2(\phi)}\right]^{1/2}, \tag{32}$$

where $\hat{H}=\mathrm{d}(\ln\hat{a})/\mathrm{d}\hat{t}$, and the primes denote the differentiation with respect to ϕ. From Eqs.(20), (21), (22) and (25), we have

$$\varepsilon = \frac{1}{k+3}\left[\frac{2\left(\frac{k+3}{k+1}\beta+km\right)\varphi^2+4kn}{\alpha\varphi^4-\left(\frac{k+3}{k+1}\beta+km\right)\varphi^2-kn}\right]^2, \tag{33}$$

$$\eta = \frac{1}{k+3}\frac{-4\left(\frac{k+3}{k+1}\beta+km\right)\varphi^2-16kn}{\alpha\varphi^4-\left(\frac{k+3}{k+1}\beta+km\right)\varphi^2-kn}, \tag{34}$$

$$\zeta = \frac{1}{k+3}\left\{\frac{16\left(\frac{k+3}{k+1}\beta+km\right)^2\varphi^4+160kn\beta\varphi^2+(16kn)^2}{\left[\alpha\varphi^4-\left(\frac{k+3}{k+1}\beta+km\right)\varphi^2-kn\right]^2}\right\}^{1/2}. \tag{35}$$

When inflation ends, the parameter $\varepsilon(\phi_{\rm end})$ is taken to the unity,

$$\varepsilon(\phi_{\rm end}) = 1. \tag{36}$$

Moreover, to solve the puzzles of big bang successfully, the scale factor should achieve enough expansion during the inflation. Usually, the time of the last horizon crossing t_{Hc} is difficult to solve exactly, but should occur around 60 e-folds before the end of inflation, namely

$$\frac{\hat{a}(\hat{t}_{\rm end})}{\hat{a}(\hat{t}_{Hc})} = \frac{\Omega(\varphi_{\rm end})a(t_{\rm end})}{\Omega(\varphi_{Hc})a(t_{Hc})} \approx {\rm e}^{60}. \tag{37}$$

From Eq.(36), we can get the value of $\varphi_{\rm end}$, and from Eq.(37) we can find the relationship between φ_{Hc} and $\varphi_{\rm end}$. Taking $8\pi G = 1$, $\alpha/\beta = 1$ and $\xi = 10^3$,[14] we get

$$\varepsilon(\phi_{Hc}) \approx 1.907\times10^{-4}, \tag{38}$$

$$\eta(\phi_{Hc}) \approx -0.0159, \tag{39}$$

$$\zeta(\phi_{Hc}) \approx 0.0159. \tag{40}$$

Substituting these parameters into Eq.(29), we get

$$n_s \approx 0.967. \tag{41}$$

Using WMAP (Wilkinson Microwave Anisotropy Probe) data only, the best fit value for $n_{\rm s}$ for the power-law flat ΛCDM (Lambda-Cold Dark Matter) model is $n_{\rm s} = 0.958 \pm 0.016$.[19] Our result is consistent with it.

Quantum fluctuations of the scalar field produce density fluctuations on current astronomical scales when $\varphi \approx \varphi_{Hc}$. The fluctuation of scalar density is amplified during inflation, which satisfies

$$\delta = \frac{\delta\rho}{\rho} = C\frac{\hat{H}^2(\phi)}{{\rm d}\phi/{\rm d}\hat{t}}, \tag{42}$$

where C is a model free constant. The result of COBE (Cosmic Background Explorer) is $\delta \approx 10^{-5}$. But the value is only an upper bound of Eq.(42) because the perturbations responsible for a large-scale structure can be formed by some other mechanisms. According to the result of COBE and Eq.(42) we can obtain the quartic coupling constant

$$\lambda \le 4.76\times10^{-7}, \tag{43}$$

where we have used $C = 1$, $\alpha/\beta = 1$ and $\xi = 10^3$.[14] Obviously, from Eq.(1) we know, when $\xi \gg 1$ the result of non-minimal coupling will tend to that of minimal coupling,[12] but when ξ is very small the results should be distinctly different. For example, when $\xi = 1$ the result becomes

$$\lambda \le 6.29\times10^{-13}, \tag{44}$$

which is close to that reported previously.[20] Thus the non-minimal coupling does have better applicability from this view.

Another important problem for inflation scenario is the requirement that the inflation stage should end opportunely and enter smoothly into a radiation-dominated stage. The values of the scalar field at different time can be calculated from Eqs.(33), (36) and (37). Using $\xi = 10^3$, we get $\varphi_{Hc} \approx 15.93$ and $\varphi_{\rm end} \approx 2.54$, using $\xi = 1$, we get $\varphi_{Hc} \approx 15.37$ and $\varphi_{\rm end} \approx 2.22$. It is clear that

$$\varphi_{Hc} > \varphi_{\rm end}. \tag{45}$$

From Eq.(13) we know that the value of scalar field φ reduces with time t monotonically. Thus the value in radiation-dominated stage φ_r should smaller than

$\varphi_{\rm end}$. The effective energy density and effective pressure are given by

$$\rho_{\rm eff} = (1 + 8\pi G\xi\varphi^2)^{-1}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi) - 6\xi H\varphi\dot{\varphi}\right], \tag{46}$$

$$P_{\rm eff} = (1 + 8\pi G\xi\varphi^2)^{-1}\left[\left(\frac{1}{2} + 2\xi\right)\dot{\varphi}^2 - V(\varphi) + 4\xi H\varphi\dot{\varphi} + 2\xi\varphi\ddot{\varphi}\right]. \tag{47}$$

In the radiation-dominated stage, the equation of state is

$$3P_r = \rho_r. \tag{48}$$

Making use of the results of Section 2 and the above three equations, when use $\xi = 10^3$, we get $\varphi_r \approx 1.76$, and when use $\xi = 1$, we get $\varphi_r \approx 1.67$. It is clear that

$$\varphi_{\rm end} > \varphi_r, \tag{49}$$

which shows that in our model it is possible for an inflation-like stage to enter smoothly into a radiation-dominated stage without any discontinuities in $\rho_{\rm eff}$.

4. Discussion

In this paper, we obtained an exact inflationary solution in the chaotic model to non-minimally coupled scalar field from a special $H(\varphi)$. The solution has a new inflation potential, $V(\varphi) = (\lambda/4)\varphi^4 + b_1\varphi^2 + b_2 + b_3\varphi^{-2} + b_4\varphi^{-4}$. The scalar field φ has no zero point in the potential, which is self-consistent with Eq.(13). In addition, the inflation potential $V(\varphi)$ may be meaningful because of the existence of negative index terms. It may imply the accelerative expansion of universe and more work can be done on this topic.

In the course of calculating the spectral index $n_{\rm s}$ of density fluctuations, when we use any values of $\xi \gg 1$ and $\alpha/\beta \approx 1$, the results are consistent with the observed data and seldom vary. But for the quartic coupling constant, the case is not the same, the coupling constant ξ has strong effect to λ. When $\xi \gg 1$, the non-minimal coupling has a relaxed constraint to quartic coupling constant which can lead to inflation with large λ, but when $\xi = 1$, λ has a smaller value which is close to that reported previously. This agrees with the conclusion of Ref.[14]. Finally, the calculations show that in our model it is possible for an inflation-like stage to enter smoothly into a radiation-dominated stage without any discontinuities in $\rho_{\rm eff}$.

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