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Properties on the distant distribution of entanglement for arbitrary two-qubit pure states via noisy quantum channels*

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This paper investigates the change of entanglement for transmitting an arbitrarily entangled two-qubit pure state via one of three typical kinds of noisy quantum channels: amplitude damping quantum channel, phase damping quantum channel and depolarizing quantum channel. It finds, in all these three cases, that the output distant entanglement (measured by concurrence) reduces proportionately with respect to its initial amount, and the decaying ratio is determined only by the noisy characteristics of quantum channels and independent of the form of initial input state.

Keywords: entanglement distribution, quantum noisy channel, concurrence
PACC: 0367, 0365

A real physical system inevitably interacts with its surrounding environment and thus two basic effects arise. The first effect is that the coherence of the system reduces, a phenomenon that is usually called decoherence in quantum physics. It is well known that different states may decohere at drastically different rates, and a small fraction of them may be particularly stable under interaction with the environment.[1−6] Such states have intrinsic advantages for quantum computation and quantum information processing. It is also important in understanding what states will naturally emerge from a quantum system subjected to decoherence. Another effect for a quantum system under the influence of environment is the change of entanglement. For a compound quantum system that is initially entangled between subsystems, when it interacts with the environment, the entanglement generally also changes. By making an analogy to decoherence, one may wonder what are the rules for change of different entangled states under the influence of environment? Does also there exist a set of particular entangled states whose entanglement is not affected by the environment?

In addition, for many tasks of quantum information processing, such as teleportation,[7−10] dense coding,[11−13] quantum cryptography,[14,15] remote-state preparation,[16] and distributed quantum computation,[17,18] distant entanglement is the essential ingredient. In order to establish distant entanglement, one may at first produce a local entangled state, and then let one component of the entangled state transmit to a distant receiver via a quantum channel. This is the so-called process of entanglement distribution. Owing to the effect of noises, the obtained distant entanglement is inevitably different from its initial amount. In most practical cases, the initial states to be transmitted are maximally entangled pure ones. One thus naturally ask questions such as: 1) Why does people transmit entanglement usually by using maximally entangled states as input? Is there any other state with which as the input, the obtained distant entanglement is contrarily larger? 2) Maximally entangled stats have manifold forms, for example, there are four maximally entangled Bell states in two-qubit system. When we use them as inputs, the obtained entanglements are all the same?

In order to answer the questions presented in the previous two paragraphs. We thus, in this paper, deal with such a problem: let an arbitrarily entangled two-qubit pure state be input and transmit one of the qubits through a noisy quantum channel to a distant receiver, we then see how the quantum entan-

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glement of the output distant entangled state changes. The noisy quantum channels considered here include three basic kinds of type: amplitude damping quantum channel, phase damping quantum channel and depolarizing quantum channel. We show that for all these types of noisy channels and for an initially randomly entangled two-qubit pure state, after transmitting one of the qubits to a distant receiver, the output distant entanglement (measured by concurrence) always reduces proportionately with respect to its initial amount, and the decaying ratio is determined only by the noisy characteristics of quantum channels, independent of the form of initial input state. From the viewpoint of theoretical research, this result is interesting on its own. It also has some practical meanings for distant entanglement distribution and study of disentanglement process.

An arbitrary two-qubit pure state can be written as

$$|\psi_{\rm in}\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle\,, \tag{1}$$

where the superposition coefficients are complex and satisfy normalization condition $|a|^2+|b|^2+|c|^2+|d|^2 = 1$. In base set $\{|00\rangle\,,|01\rangle\,,|10\rangle\,,|11\rangle\}$, the corresponding density matrix is

$$\rho_{\rm in} = \begin{bmatrix} |a|^2 & ab^* & ac^* & ad^* \\ ba^* & |b|^2 & bc^* & bd^* \\ ca^* & cb^* & |c|^2 & cd^* \\ da^* & db^* & dc^* & |d|^2 \end{bmatrix}. \tag{2}$$

In general, this pure state is entangled with respect to its two qubits and the degrees of entanglement may be described by the so-called concurrence[19]

$$C(|\psi_{\rm in}\rangle) = 2|ad-bc|. \tag{3}$$

Now, we use this entangled two-qubit pure state as input, transmit one of the two qubits via a quantum noisy channel to a distant receiver, and thus establish distant entanglement between the qubits held by the sender and receiver.

Amplitude damping is a process in which a quantum system dissipates energy to its environment. For an amplitude damping quantum channel, the effect of noise can be described by the following Kraus operators:[17,20]

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} \tag{4}$$

with real parameter p characterizing the properties of amplitude damping quantum channel. For ideal noise-free quantum channels, we have $p = 0$, otherwise $0 < p \leq 1$. Assume that the sender now transmits the second qubit in the input state of Eq.(1) via the amplitude damping quantum channel to a distant receiver, then the output state of the two-qubit system can be obtained according to $\rho^A_{\rm out} = (I\otimes M_0)\rho_{\rm in}(I\otimes M_0^+) + (I\otimes M_1)\rho_{\rm in}(I\otimes M_1^+)$, which may be written as

$$\rho^A_{\rm out} = \begin{bmatrix} |a|^2+p|b|^2 & \sqrt{1-p}ab^* & ac^*+pbd^* & \sqrt{1-p}ad^* \\ \sqrt{1-p}a^*b & (1-p)|b|^2 & \sqrt{1-p}bc^* & (1-p)bd^* \\ a^*c+pb^*d & \sqrt{1-p}b^*c & |c|^2+p|d|^2 & \sqrt{1-p}cd^* \\ \sqrt{1-p}a^*d & (1-p)b^*d & \sqrt{1-p}c^*d & (1-p)|d|^2 \end{bmatrix}. \tag{5}$$

In general, this is an entangled mixed state. According to the definition of concurrence and after a very tedious deduction, we find that the concurrence for this output state has a very simple expression,

$$C(\rho^A_{\rm out}) = 2\sqrt{1-p}|ad-bc|. \tag{6}$$

This expression shows a very interesting result: when the input state has no entanglement ($C(|\psi_{\rm in}\rangle) = 0$), then the output state also has no entanglement ($C(\rho^A_{\rm out}) = 0$). But if the input state possesses any entanglement, then the output state also takes on entanglement. The decaying ratio for the output state is $1 - C(\rho^A_{\rm out})/C(|\psi_{\rm in}\rangle) = 1 - \sqrt{1-p}$ which is just determined by the characteristics of amplitude damping quantum channel, and does not depend on the form of input state. Therefore, if one wants to obtain a distant entangled state with maximally possible entanglement, a local maximally-entangled input state is needed. This is the reason why people usually transmits entanglement by using Bell states as input.

Phase damping is uniquely a quantum mechanical process which describes the loss of quantum information without loss of energy. When a quantum superposition state suffers from phase damping, each superposition component accumulates a random relative phase so as to the partial information about the phase loses. For a phase damping quantum channel, the corresponding effect of noise may be described by three Kraus operators,[20]

$$M_0 = \sqrt{1-p}\begin{bmatrix}1&0\\0&1\end{bmatrix}, \quad M_1 = \sqrt{p}\begin{bmatrix}1&0\\0&0\end{bmatrix},$$
$$M_2 = \sqrt{p}\begin{bmatrix}0&0\\0&1\end{bmatrix}, \tag{7}$$

where the real parameter p characterizes the properties of phase damping quantum channel. For ideal noise-free quantum channels, $p = 0$, otherwise $0 < p \leq 1$. For the input state of Eq.(1), if let the second qubit transmit via the phase damping quantum channel to a distant receiver, then the output state of the two-qubit system becomes

$$\rho_{\text{out}}^{p} = \begin{bmatrix} |a|^2 & (1-p)ab^* & ac^* & (1-p)ad^* \\ (1-p)ba^* & |b|^2 & (1-p)bc^* & bd^* \\ ca^* & (1-p)cb^* & |c|^2 & (1-p)cd^* \\ (1-p)da^* & db^* & (1-p)dc^* & |d|^2 \end{bmatrix}. \tag{8}$$

After a very tedious deduction, we find that the concurrence for this output state is

$$C(\rho_{\text{out}}^{p}) = 2(1-p)|ad-bc|. \tag{9}$$

This expression shows a similar result: the entanglement of the output state is proportional to the entanglement of the input state, and the decaying ratio $1 - C(\rho_{\text{out}}^{p})/C(|\psi_{\text{in}}\rangle) = p$ is just determined by the characteristics of phase damping quantum channel, and does not depend on the form of input state.

The depolarizing channel is also a basic type of quantum noise which may be described in terms of the following four Kraus operators:[17,20]

$$M_0 = \sqrt{1-p}\begin{bmatrix}1&0\\0&1\end{bmatrix}, \quad M_1 = \sqrt{\frac{p}{3}}\begin{bmatrix}0&1\\1&0\end{bmatrix}, \quad M_2 = \sqrt{\frac{p}{3}}\begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix}, \quad M_3 = \sqrt{\frac{p}{3}}\begin{bmatrix}1&0\\0&-1\end{bmatrix}, \tag{10}$$

where $p = 0$ corresponds to ideal noise-free quantum channel, and $0 < p \leq 1$ corresponds to noisy depolarizing quantum channels. For the input state of Eq.(1), after the second qubit transmits via the depolarizing quantum channel to a distant receiver, then the output state of the two-qubit system becomes

$$\rho_{\text{out}}^{d} = \begin{bmatrix} \left(1-\frac{2}{3}p\right)|a|^2+\frac{2}{3}p|b|^2 & \left(1-\frac{4}{3}p\right)ab^* & \left(1-\frac{2}{3}p\right)ac^*+\frac{2}{3}pbd^* & \left(1-\frac{4}{3}p\right)ad^* \\ \left(1-\frac{4}{3}p\right)a^*b & \left(1-\frac{2}{3}p\right)|b|^2+\frac{2}{3}p|a|^2 & \left(1-\frac{4}{3}p\right)bc^* & \left(1-\frac{2}{3}p\right)bd^*+\frac{2}{3}pac^* \\ \left(1-\frac{2}{3}p\right)a^*c+\frac{2}{3}pb^*d & \left(1-\frac{4}{3}p\right)b^*c & \left(1-\frac{2}{3}p\right)|c|^2+\frac{2}{3}p|d|^2 & \left(1-\frac{4}{3}p\right)cd^* \\ \left(1-\frac{4}{3}p\right)a^*d & \left(1-\frac{2}{3}p\right)b^*d+\frac{2}{3}pa^*c & \left(1-\frac{4}{3}p\right)cd^* & \left(1-\frac{2}{3}p\right)|d|^2+\frac{2}{3}p|c|^2 \end{bmatrix}. \tag{11}$$

The concurrence for this output state can be calculated directly and for $0 \leq p < 1/2$, it can be written as

$$C(\rho_{\text{out}}^{d}) = (2-4p)|ad-bc|, \tag{12}$$

thus the ratio of concurrence between output and input states is $C(\rho_{\text{out}}^{d})/C(|\psi_{\text{in}}\rangle) = 1-2p$. For $1/2 \leq p \leq 1$, the concurrence for the output state is zero and the ratio of concurrence between output and input states is also zero. Of course, in practical quantum information processing, the noise usually is not very large. Thus one may rationally assume that $0 \leq p < 1/2$ so that the concurrence of the output state is presented by Eq.(12). Synthetically, we can write the concur-

rence of the output state as

$$C(\rho_{\rm out}^d) = \max\{0, (2-4p)|ad-bc|\}, \qquad (13)$$

and the decaying ratio is $1 - C(\rho_{\rm out}^d)/C(|\psi_{\rm in}\rangle) = \min\{1, 2p\}$. Therefore, we once again conclude that the entanglement of the output state is proportional to that of the input state, and the decaying ratio is just determined by the characteristics of depolarizing quantum channels, and does not depend on the form of initial input state.

In summary, we have studied the properties of distant entanglement distribution for an arbitrarily entangled two-qubit pure state as input via three typical kinds of noisy quantum channels. It has been shown that, in all these three cases, the output distant entanglement reduces proportionately with respect to its initial amount, and the decaying ratio is determined only by the noisy characteristics of quantum channels and independent of the form of the input state. This result has important meanings in quantum information processing. Firstly, it theoretically demonstrates such a fact that maximally entangled two-qubit pure states are favourable to be considered as inputs for transmitting entanglement, because they permit us to obtain distant entangled states with maximally possible entanglement. This is the reason why people usually transmits entanglement by using Bell states as inputs. Secondly, it also tells us that, if only from the viewpoint of preserving entanglement, all two-qubit entangled pure states are equivalent, no one has any more advantage than the others. Suppose we have two sets of entangled pure-state qubit pairs. One set consists of Bell states, the other consists of non-maximally entangled states. We also assume that the total amount of entanglement for the two sets are same, i.e., the sums of concurrence for the two sets are equal. After these two sets of states are made entanglement distributing via a same noisy quantum channel, the output total amount of entanglement for the two sets are also equal. If we have enough ability to distill entanglement, we will obtain two new sets for which each with equal number of singlets. Finally, the result still tells us that there is no specific entangled two-qubit pure state whose entanglement is not affected by the above three types of noisy environment, which makes a contrast to decoherence.

Of course, in the above discussions we only assume that one component of the two-qubit system suffers from the effect of a given noisy environment (one-qubit transmitting via a noisy quantum channel). There are many related problems still worthy of research. For example, if the two qubits are both affected by the noisy environment (two-qubit transmitting via noisy quantum channels), what is the rule for the change of entanglement? Or, if the initial state to be distributed is not a pure but an entangled two-qubit mixed state, how about it is? We have done some numerical research on these problems and find that in these cases the result of proportionate reduction of entanglement is generally not true. Actually, recent research shows that for some specific entangled two-qubit states, when both of them simultaneously suffer from the effect of noisy environment, an interesting phenomenon of entanglement sudden death has been observed.[21−24]

In the above calculations, we only consider the effect of a single basic type of quantum noise at each time. Though these basic types of quantum noise have practical physical meaning (for example, amplitude damping quantum noise corresponds to a process of spontaneous emission), in most practical cases, we will encounter simultaneously more than two basic types of quantum noise (complex quantum noise). To deal with the cases of complex quantum noise, we firstly require to know the interaction forms through which the system interacts with its environment. By using the method of master equation, we can deduce the Kraus representation. Then following the steps in this paper, we can study the change of entanglement over entanglement distribution.

Unfortunately, it is not straightforward to generalize our idea for studying multi-qubit entanglement distribution. The main handicap lies in the quantification of entanglement for multi-qubit states. Though the concept of concurrence has been generalized to the case of multi-partite systems,[25] but its intrinsical meaning is not well understood. The evaluation of the generalized concurrence is also rather cumbersome. In fact, there is not yet an effective way to evaluate the concurrence for general multi-partite mixed states. It has a very long distance to run in the classification of entanglement and its quantification for multi-partite systems.

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