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Quantum logic gates with two-level trapped ions beyond Lamb–Dicke limit*

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In the system with two two-level ions confined in a linear trap, this paper presents a simple scheme to realize the quantum phase gate (QPG) and the swap gate beyond the Lamb–Dicke (LD) limit. These two-qubit quantum logic gates only involve the internal states of two trapped ions. The scheme does not use the vibrational mode as the data bus and only requires a single resonant interaction of the ions with the lasers. Neither the LD approximation nor the auxiliary atomic level is needed in the proposed scheme. Thus the scheme is simple and the interaction time is very short, which is important in view of decoherence. The experimental feasibility for achieving this scheme is also discussed.

Keywords: trapped ions, quantum phase gate, swap gate, beyond the Lamb–Dicke limit
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Over the past several years, the quantum computer, based on the superposition principle of quantum dynamics, has received much attention. It has been shown that any quantum computation[1] can be decomposed into a sequence of one-qubit and two-qubit logic gates.[2,3] Some of the basic two-qubit logic gates are the conditional quantum phase gate (QPG),[4] the controlled-not (CNOT) gate, which is a universal two-qubit gate,[3] the swap gate, etc. A number of experimental systems have been proposed as candidates for implementing these logic gates, and these systems are trapped ions,[5−7] cavity quantum electrodynamics,[8] nuclear magnetic resonance (NMR) system,[9] etc.

With advances in ion cooling and trapping, the ion trap has been considered to be a promising candidate of the quantum computing device since the original ion-trap proposal of Cirac and Zoller.[6] A lot of schemes for quantum computing with trapped ions by means of lasers have been proposed.[10−13] The scheme of Cirac and Zoller, based on the resonant sideband excitation, uses the vibrational mode as the data bus. In this scheme, the information of an electronic system is stored in the vibrational mode and then is transferred back to this electronic system after the conditional dynamics. Thus three resonant ion–laser interactions are required in order to achieve a QPG between two ions. Sørensen and Mølmer[11] have also proposed a scheme for realizing quantum computation in ion traps via virtual vibrational excitations. The advantage of this scheme is that it does not use the motional mode as the data bus and is insensitive to the vibrational states. But the performance of quantum computation in Ref.[11] includes seven single-qubit operations and two two-qubit operations. While Zou *et al*[13] proposed a scheme to implement a QPG for trapped ions in thermal motion with one standing-wave laser pulse. In the schemes of Refs.[11, 13] quantum information is stored in the spin states of an array of trapped cold ions and manipulated by using laser pulses, and the ions have only two levels, but the logic gates operations work in the Lamb–Dicke (LD) limit. In fact, the quantum motion of the trapped ions is not limited to the LD regime.[14] It has been shown that utilizing the laser-ion interaction beyond the LD limit is helpful for reducing the noise in the trap and improving the cooling rate.[15] Morigi *et al* have proposed a proposal for laser cooling to the ground state for $N \geq 2$ ions outside the LD regime and furthermore they have showed that one can cool a given set of modes to the ground state through the choice of the laser detuning outside the LD regime.[16] In fact, recently, several schemes

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have been proposed to implement the quantum computation and prepare entangled states with trapped ions outside the LD regime.[7,17,18] Recently, several schemes[19–21] have also been proposed for fast quantum computations in an ion trap, but the schemes of Refs.[19–21] require a number of laser-ion interactions. In order to reduce laser-ion interactions, Wen-Xing Yang *et al*[22] and Shi-Biao Zheng[23] proposed a simple scheme to implement a QPG for trapped ions respectively, and these schemes only require a single resonant interaction. But two special conditions are required in the above schemes[22,23]; (a) the ions must have three levels; (b) the trap must operate in the LD limit, i.e. the coupling between the internal spin and external vibrational degrees of freedom of the ions must be sufficiently weak, in which the gate operation time is relatively longer.

In this paper, we present an alternative scheme to realize the QPG and the swap gate beyond the LD limit by using two two-level ions trapped in a linear trap. Like the schemes of Refs.[22, 23], the present scheme does not use the vibrational mode as the data bus and only requires a single resonant interaction. Thus the scheme is simpler than the scheme of Ref.[6] and the required interaction time is shortened. However, unlike the schemes of Refs.[6, 22, 23], in present scheme, neither the LD approximation nor the auxiliary atomic level is needed. The elimination of an auxiliary ground state removes the possible existence of spectator internal atomic levels, which can act as potential 'leaks' from the two levels spanned by the qubits.[24] The elimination of the auxiliary level can dramatically decrease the sensitivity of the gates operation to the fluctuations of external magnetic fields, and increase the speed of quantum computing.[24] Thus our scheme is not only simplified but more practical than that in Refs.[6, 22, 23].

We consider two two-level ions confined in a linear trap. We assume that the first ion is excited by a laser and the other ion is excited by another laser. We take the frequencies of both lasers resonant with the centre of mass vibronic transition in the k-th red sideband, i.e. $\omega_L = \omega_0 - k\nu$. For small k values, we may safely assume that only the centre of mass motion will be excited, which gives that the next eigenfrequency is $\nu_r = \sqrt{3}\nu$, corresponding to the stretch mode.[25] We choose $k = 1$, thus the total Hamiltonian of the system may be written as

$$\begin{aligned} H = {} & \omega_0 \sum_{j=1}^{2} \frac{\sigma_{z,j}}{2} + \nu a^\dagger a \\ & + (-\mathrm{i}\Omega_1 \sigma_1^\dagger \mathrm{e}^{\mathrm{i}\eta(a^\dagger + a)} \mathrm{e}^{-\mathrm{i}(\omega_0 - \nu)t} + \mathrm{H.c.}) \\ & + (-\mathrm{i}\Omega_2 \sigma_2^\dagger \mathrm{e}^{\mathrm{i}\eta(a^\dagger + a)} \mathrm{e}^{-\mathrm{i}(\omega_0 - \nu)t} + \mathrm{H.c.}). \end{aligned} \quad (1)$$

The operators a ($a^\dagger$) is the annihilation (creation) operator associated with the centre of mass mode of frequency ν. The operators $\sigma_{z,j} = |e\rangle_j\langle e| - |g\rangle_j\langle g|$, $\sigma_j^\dagger = |e\rangle_j\langle g|$ and $\sigma_j^- = |g\rangle_j\langle e|$, with $|e\rangle_j$ and $|g\rangle_j$ which are the excited and ground states of the j-th ion respectively. The ω_0 is the frequency of the transition $|e\rangle \to |g\rangle$, Ω_j $(j = 1, 2)$ is the Rabi frequency of the laser and η is the LD parameter.

We consider the resolved sideband regime, where the vibrational frequency ν is much larger than other characteristic frequencies of the problem. In this case the rapidly oscillating terms can be discarded and the interactions of the ions with the laser can be treated by using a nonlinear Jaynes–Cummings model.[26,27] Therefore, the Hamiltonian for such a system is rewritten in the interaction picture as

$$\begin{aligned} H = {} & \left(-\mathrm{i}\Omega_1 \mathrm{e}^{-\frac{\eta^2}{2}} \sigma_1^\dagger \sum_{l=0}^{\infty} \frac{(\mathrm{i}\eta)^{2l+1}}{l!(l+1)!} a^{+l} a^{l+1} + \mathrm{H.c.} \right) \\ & + \left(-\mathrm{i}\Omega_2 \mathrm{e}^{-\frac{\eta^2}{2}} \sigma_2^\dagger \sum_{l=0}^{\infty} \frac{(\mathrm{i}\eta)^{2l+1}}{l!(l+1)!} a^{+l} a^{l+1} + \mathrm{H.c.} \right). \end{aligned} \quad (2)$$

Equation (2) can be written in another form as follows:

$$\begin{aligned} H = {} & g_1[F(a^\dagger, a)\sigma_1^\dagger a + \sigma_1^- a^\dagger F(a^\dagger, a)] \\ & + g_2[F(a^\dagger, a)\sigma_2^\dagger a + \sigma_2^- a^\dagger F(a^\dagger, a)], \end{aligned} \quad (3)$$

where $g_j = \eta\Omega_j \exp(-\eta^2/2)$ and $F(a^\dagger, a) = \sum_{l=0}^{\infty} \frac{(i\eta)^{2l}}{l!(l+1)!} a^{+l} a^l$.

Assume that the vibrational mode is initially in the vacuum state $|0\rangle$. If the two ions are initially in the state $|g_1 g_2\rangle$, $|g_1 e_2\rangle$, $|e_1 g_2\rangle$ or $|e_1 e_2\rangle$, then the evolution of the system is

$$|0\rangle|g_1 g_2\rangle \longrightarrow |0\rangle|g_1 g_2\rangle,$$

$$|0\rangle|g_1 e_2\rangle \longrightarrow \frac{g_2}{E^2} \left\{ \left[g_2 \cos(Et) + \frac{g_1^2}{g_2} \right] |0\rangle|g_1 e_2\rangle + g_1[\cos(Et) - 1]|0\rangle|e_1 g_2\rangle - \mathrm{i}E \sin(Et)|1\rangle|g_1 g_2\rangle \right\},$$

$$
\begin{aligned}
|0\rangle|e_1g_2\rangle &\longrightarrow \frac{g_1}{E^2}\left\{\left[g_1\cos(Et)+\frac{g_2^2}{g_1}\right]|0\rangle|e_1g_2\rangle + g_2[\cos(Et)-1]|0\rangle|g_1e_2\rangle - \mathrm{i}E\sin(Et)|1\rangle|g_1g_2\rangle\right\},\\
|0\rangle|e_1e_2\rangle &\longrightarrow \frac{1}{2}[(|2\rangle|g_1g_2\rangle+|0\rangle|e_1e_2\rangle)\cos(g_1+g_2)t-\mathrm{i}(|1\rangle|g_1e_2\rangle+|1\rangle|e_1g_2\rangle)\sin(g_1+g_2)t\\
&\quad+(|0\rangle|e_1e_2\rangle-|2\rangle|g_1g_2\rangle)\cos(g_1-g_2)t+\mathrm{i}(|1\rangle|e_1g_2\rangle-|1\rangle|g_1e_2\rangle)\sin(g_1-g_2)t]], \qquad (4)
\end{aligned}
$$

where $E=\sqrt{g_1^2+g_2^2}$, the state to the left represents the initial state and the state to the right is the state at the interaction time t. If the following conditions are satisfied

$$
\begin{aligned}
&\cos(Et)=1,\quad \cos(g_1+g_2)t=-1,\\
&\cos(g_1-g_2)t=-1, \qquad (5)
\end{aligned}
$$

then a universal two-qubit logic gate, the controlled-Z logic operation can be realized directly by representing the qubits with the spin states of trapped cold ions, so

$$
\begin{aligned}
&|0\rangle|g_1g_2\rangle \longrightarrow |0\rangle|g_1g_2\rangle,\\
&|0\rangle|g_1e_2\rangle \longrightarrow |0\rangle|g_1e_2\rangle,\\
&|0\rangle|e_1g_2\rangle \longrightarrow |0\rangle|e_1g_2\rangle,\\
&|0\rangle|e_1e_2\rangle \longrightarrow -|0\rangle|e_1e_2\rangle \qquad (6)
\end{aligned}
$$

with the choice $g_1=g_2$, furthermore the following conditions are satisfied

$$
\cos(Et)=-1,\quad \cos(g_1+g_2)t=1, \qquad (7)
$$

we obtain

$$
\begin{aligned}
&|0\rangle|g_1g_2\rangle \longrightarrow |0\rangle|g_1g_2\rangle,\\
&|0\rangle|g_1e_2\rangle \longrightarrow -|0\rangle|e_1g_2\rangle,\\
&|0\rangle|e_1g_2\rangle \longrightarrow -|0\rangle|g_1e_2\rangle,\\
&|0\rangle|e_1e_2\rangle \longrightarrow |0\rangle|e_1e_2\rangle. \qquad (8)
\end{aligned}
$$

Performing the trivial single-qubit rotation $|e_j\rangle \rightarrow -|e_j\rangle$ $(j=1,2)$, we have

$$
\begin{aligned}
&|0\rangle|g_1g_2\rangle \longrightarrow |0\rangle|g_1g_2\rangle,\\
&|0\rangle|g_1e_2\rangle \longrightarrow |0\rangle|e_1g_2\rangle,\\
&|0\rangle|e_1g_2\rangle \longrightarrow |0\rangle|g_1e_2\rangle,\\
&|0\rangle|e_1e_2\rangle \longrightarrow |0\rangle|e_1e_2\rangle. \qquad (9)
\end{aligned}
$$

In this way we obtain a quantum swap gate for the two ions with the vibrational mode left in the vacuum state.

We now show how to set up the experimental parameters to realize the above quantum logic gates. These parameters include the parameter g_1 and the durations of the applied light field. First, for the QPG, requisite parameters are determined by Eq.(5), which implies

$$
\begin{aligned}
&(g_1+g_2)t=(2k+1)\pi,\quad k=0,1,2,3,\ldots,\\
&(g_1-g_2)t=(2n+1)\pi,\quad n=0,1,2,3,\ldots,\\
&\left(\sqrt{g_1^2+g_2^2}\right)t=2m\pi,\qquad m=1,2,3,\ldots \qquad (10)
\end{aligned}
$$

For the quantum swap gate, requisite parameters are determined by Eq.(7), which implies

$$
\begin{aligned}
&g_1=g_2=g,\\
&\sqrt{2}gt=(2i+1)\pi,\quad i=0,1,2,3,\ldots,\\
&gt=j\pi,\quad j=1,2,3,\ldots \qquad (11)
\end{aligned}
$$

It is seen from Eqs.(10) and (11) that if we want to realize determinate quantum logic gates, the parameters k, n, m and the parameters i and j must be integers, but by numerically calculating, if the parameters k, n and the parameter i are integers, the parameters m and j cannot be integers, while they are approximately integers. Thus the fidelity of realizing the quantum logic gates is not 1 and is approximatively 1. Some values of these experimental parameters and the fidelity for realizing the quantum logic gates are given in Table 1 and Table 2.

Table 1. Some experimental parameters and the fidelity for realizing the QPG.

k	n	m	g_1/g_2	g_2t	F
15	1	11.011	17/14	14π	0.9988
12	2	9.0139	3/2	10π	0.9981
22	2	16.008	5/4	20π	0.9994
4	3	4.0311	8	π	0.9905
19	3	14.009	23/16	16π	0.9992
29	3	21.006	33/26	26π	0.9996
5	4	5.0249	10	π	0.9939
26	4	19.007	31/22	22π	0.9995
6	5	6.0208	12	π	0.9957
11	5	9.0139	17/6	6π	0.9981
17	5	12.971	23/12	12π	0.9917

The durations of the applied light field are given by the quantities g_2t in the Table 1 and gt in the Table 2. It is seen from the tables that the switching speed

of the logic gates depends on the ion–laser coupling constant Ω and the LD parameter η. It can be estimated numerically once the experimental parameters are defined. For the QPG and swap gate operation, we show relation between the scale time $\Omega_2 t$ (or Ωt) and the LD parameter η required in performing logic gates operation (see Fig.1).

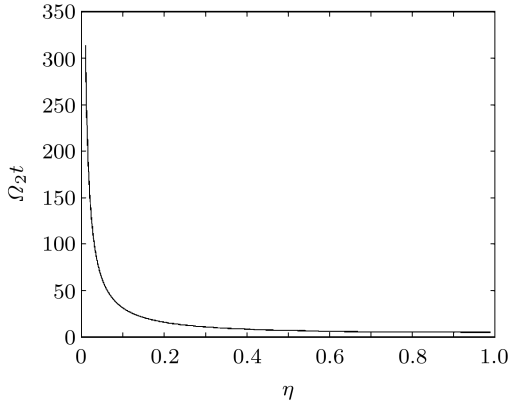


Fig.1. With the choice $g_2 t = \pi$ for the QPG, relation between the scale time $\Omega_2 t$ and the LD parameter η.

Here we will discuss the experimental feasibility of the proposed scheme. Because the logic gates operations only involve the internal states of the trapped ions, while the main detrimental effect on our scheme is the spontaneous emission of the ionic internal state, we have to consider the possible spontaneous emission of the ionic internal state during the implementation of quantum computing. Next we will estimate the operation time of the above logic gates. We consider calcium ions $^{40}Ca^{\dagger}$ with the "ground" (computational) state $|g = S_{1/2}\rangle$ and the "excited"(computational) state $|e = D_{5/2}\rangle$. According to Ref.[28], we choose ion-field coupling constant $\Omega = 4\pi\,\mathrm{MHz}$, then the operation times of the logic gates are at the order of $10^{-5}\,\mathrm{s}$, while the spontaneous radiative time is $T_{\mathrm{a}} = 1.1689\,\mathrm{s}$,[29] much longer than the interaction times needed to implement the present scheme. Thus the losses due to the spontaneous emission can be neglected. According to the recent experiment of trapped $^{40}Ca^{\dagger}$ ions,[30] it is possible to reach the LD regime. So the scheme for the LD regime is feasible in the current technology. Although experimentally it is not reported that laser cooling to the ground state is possible for $N \geq 2$ ions outside the LD regime, theoretically Morigi *et al* have shown that laser cooling to the ground state is possible for $N \geq 2$ ions outside the LD regime.[16] Once experimentally the collective mode of motion of trapped ions is successfully cooled to ground state beyond the LD limit, the present scheme may be realized beyond the LD limit. Therefore we believe that beyond the LD limit our scheme is reachable by the near future trapped-ion techniques.

Table 2. Some experimental parameters and the fidelity for realizing the quantum swap gate.

i	j	gt	F
8	12.0204	$\frac{17\pi}{\sqrt{2}}$	0.9996
20	28.9914	$\frac{41\pi}{\sqrt{2}}$	0.9999
49	70.0036	$\frac{99\pi}{\sqrt{2}}$	0.99999

In conclusion, we have presented a simple scheme to realize the QPG and the swap gate beyond the LD limit by using two two-level ions trapped in a linear trap. Unlike the scheme[6] for quantum computation, the present scheme does not use the vibrational mode as the data bus and only requires a single resonant interaction of the ions with a laser field. Moreover, as in Refs.[7, 18], neither the LD approximation nor the auxiliary atomic level is needed in the present scheme. While in the above Refs.[6, 22, 23], the quantum computing only works in the LD regime, in which the gate operation time is relatively longer, and the ions must have three levels. As involvement of more levels would increase the sensitivity of the gates operation to the fluctuation of external magnetic fields, and decrease the speed of quantum computing,[24] unlike the scheme in Ref.[18], which requires three interactions (three-step red-sideband pulses), in present scheme only one operation is involved in QPG operation. It is known that the more the operation steps, the longer the operational time of the logic-gate operations, because there exists the delay time between any two neighbouring operations in actual experiments. More importantly, there are some uncontrollable factors in each operation that might lead to errors, and an accumulation of errors would probably destroy the correct results of quantum computing. In this sense, the best way to reduce these errors is to decrease the number of the operations. Therefore our scheme is simpler and more efficient than that in Refs.[6, 11, 18–23].

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