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Coupled flexural-torsional vibration band gap in periodic beam including warping effect*

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The propagation of coupled flexural-torsional vibration in the periodic beam including warping effect is investigated with the transfer matrix theory. The band structures of the periodic beam, both including warping effect and ignoring warping effect, are obtained. The frequency response function of the finite periodic beams is simulated with finite element method, which shows large vibration attenuation in the frequency range of the gap as expected. The effect of warping stiffness on the band structure is studied and it is concluded that substantial error can be produced in high frequency range if the effect is ignored. The result including warping effect agrees quite well with the simulated result.

Keywords: phononic crystals, coupled flexural-torsional beam, transfer matrix method, warping stiffness

PACC: 0340K, 4600, 0260, 8160H

1. Introduction

In the last decade, the propagation of elastic or acoustic waves in periodic composites, so-called phononic crystals (PCs), which are phononic materials with elastic wave band gaps where the propagation of vibrations is forbidden, has received much attention.[1−6] The study on PCs materials and structures is driven partly by potential applications such as elastic filters, vibrationless environments for high-precision systems, transducer improvements, as well as pure physical concerns with the Anderson localization of sound and vibration.

The vibration propagation in periodic structure was researched some time ago.[3,7] These theories used to predict that the vibration response of periodic structure could be applied primarily to the analysis of them as pass band and stop band. Recently, with the ideas of PCs, the vibration band gaps including longitudinal vibration, flexural vibration, coupled flexural-torsion vibration and so on, in periodic beams have been studied both theoretically and experimentally.[3,7−9]

Coupled bending-torsional vibration of beam has been investigated in recent years.[10−15] These investigations showed that the coupling between the bending and the torsional displacements in the free vibratory modes of a beam could occur when the mass centre and the shear centre of the beam cross-section are non-coincident. However, the torsion theory used by these authors assumes that the beam twists according to the and there is no allowance for the warping of the beam cross-section. In Saint-Venant theory, the assumption that plane sections remain plane forbids the cross-section to deform into a non-planar surface. Such an assumption can lead to large errors when calculating the complex band structures of such phononic crystals as in Ref.[8]. However, the warping of the cross-section, which is permitted by the flexibility of the reinforcing sheets, is taken into account by introducing a kinematic displacement function that produces no rotation of the section but measures the deviation from plane of the deformed cross-section. It should be noted that the terminology of warping used here is not associated with torsion, and it just specifies the distortion of the cross-section created by moment and shear. Bishop *et al*[16] demonstrated this convincingly when they extended the governing differential equations of a bending-torsion coupled beam to reveal the warping effect. In Ref.[17], the dynamic stiffness matrix was

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provided for the coupled flexural-torsional vibration in a uniform beam including warping effects. Since the dynamic stiffness matrix method uses the exact theory, it gives more accurate results.

In this paper, the coupled flexural-torsional vibration in periodic beam including warping effect is studied by the exact transfer matrix (TM) method. And the frequency response function (FRF) is calculated by the finite element (FE) method to validate our theory.

2. The exact transfer matrix method

Periodic binary straight uniform beam members of channel section are shown in Fig.1. The beam consists of an infinite repetition of alternating member of material A with length L_1 and material B with length L_2. Thus, the cell constant is $L = L_1 + L_2$.

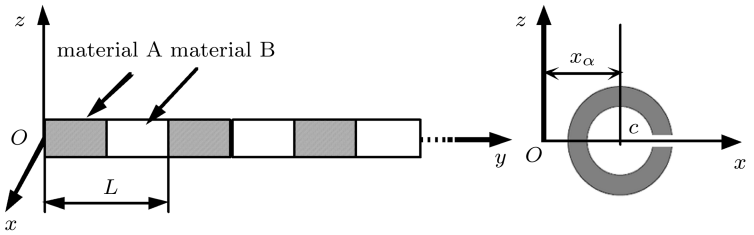


Fig.1. Coordinate system and notations for periodic binary beam. (a) Infinite extended periodic structure; (b) Sample of beam cross-section with shear centre O and centroid c being separated by x_α.

In the right-handed coordinate system of Fig.1, the elastic axis, which is assumed to coincide with the y-axis, is permitted with flexural translation $h(y,t)$ in the z-direction and torsional rotation $\psi(y,t)$ about the y-axis, where y and t denote distance from the origin and time, respectively. The governing differential equations of the coupled bending-torsion vibration of the beam with the inclusion of warping stiffness are given by

$$EIh'''' + m(\ddot{h} - x_\alpha\ddot{\psi}) = 0, \tag{1}$$

$$E\Gamma\psi'''' - GJ\psi'' + I_\alpha\ddot{\psi} - mx_\alpha\ddot{h} = 0, \tag{2}$$

where EI, GJ and $E\Gamma$ are, respectively, the bending, torsion and warping rigidity of the beam cross-section; m is the mass per unit length and I_α is the polar mass moment of inertia per unit length about the y-axis (i.e. an axis through the shear centre). Primes and dots denote differentiation with respect to position y and time t, respectively.

If a sinusoidal variation of h and ψ, with circular frequency ω, is assumed, then

$$h(y,t) = H(y)\sin\omega t, \tag{3}$$

$$\psi(y,t) = \Psi(y)\sin\omega t, \tag{4}$$

where $H(y)$ and $\Psi(y)$ are the amplitudes of the sinusoidal displacement and torsional rotation, respectively.

Substituting Eqs.(3) and (4) into Eqs.(1) and (2) gives

$$EIH'''' - m\omega^2 H + mx_\alpha\omega^2\Psi = 0, \tag{5}$$

$$E\Gamma\Psi'''' - GJ\Psi'' - \mathrm{I}_\alpha\omega^2\Psi + mx_\alpha\omega^2 H = 0. \tag{6}$$

Equations (5) and (6) can be combined into one equation, by eliminating either H or Ψ, to give the eighth-order differential equation

$$[\lambda_{Wi}D_i^8 - D_i^6 - (\lambda_{Wi}\lambda_{Bi} + \lambda_{Ti})D_i^4 + \lambda_{Bi}D_i^2 + c_i\lambda_{Bi}\lambda_{Ti}]W_i = 0, \tag{7}$$

where subscript $i = 1, 2$ represents respectively material A and material B, and

$$W_i = H_i \text{ or } \Psi_i, \tag{8}$$

$$D = \mathrm{d}/\mathrm{d}\xi_i, \tag{9}$$

$$\xi_i = (y - nL)/L_i, \quad nL \le y \le (n+1)L, \tag{10}$$

$$\lambda_{Bi} = m_i\omega^2 L_i^4/E_i\mathrm{I},$$

$$\lambda_{Ti} = I_{\alpha_i}\omega^2 L_i^2/G_i J,$$

$$\lambda_{Wi} = E_i\Gamma/G_iJL_i^2, \tag{11}$$

$$c_i = 1 - m_i x_\alpha / I_{\alpha_i}. \tag{12}$$

The solution of the differential equation (7) can be obtained by substituting the trial solution $W_i = \mathrm{e}^{s_i\xi_i}$ to give the characteristic equation

$$[\lambda_{Wi}s_i^8 - s_i^6 - (\lambda_{Wi}\lambda_{Bi} + \lambda_{Ti})s_i^4 + \lambda_{Bi}s_i^2 + c_i\lambda_{Bi}\lambda_{Ti}]W_i = 0. \tag{13}$$

Let

$$\kappa_i = s_i^2, \tag{14}$$

Substituting Eq.(14) into Eq.(13) gives

$$[\lambda_{Wi}\kappa_i^4 - \kappa_i^3 - (\lambda_{Wi}\lambda_{Bi} + \lambda_{Ti})\kappa_i^2 + \lambda_{Bi}\kappa_i + c_i\lambda_{Bi}\lambda_{Ti}]W_i = 0. \tag{15}$$

It can be shown[16] that all four roots of Eq.(15) are real, and they lie within four distinct intervals, namely $(-\infty, -\sqrt{\lambda_{Bi}})$, $(-\sqrt{\lambda_{Bi}}, 0)$, $(0, \sqrt{\lambda_{Bi}})$ and $(\sqrt{\lambda_{Bi}}, +\infty)$. Let these four roots be κ_{1i}, κ_{2i}, $-\kappa_{3i}$ and $-\kappa_{4i}$, where κ_{mi} (m = 1, 2, 3, 4) are real and positive. Then the eight roots of the characteristic equation (13) are

$$\alpha_i,\ -\alpha_i;\ \beta_i,\ -\beta_i;\ j\gamma_i,\ -j\gamma_i;\ j\delta_i,\ -j\delta_i; \tag{16}$$

$$\alpha_i = \sqrt{\kappa_{1i}}, \quad \beta_i = \sqrt{\kappa_{2i}}, \quad \gamma_i = \sqrt{\kappa_{3i}}, \quad \delta_i = \sqrt{\kappa_{4i}}. \tag{17}$$

Then the solutions of Eqs.(1) and (2) for W_{ni} in the n-th cell are obtained as follows:

$$\begin{aligned} W_{ni}(\xi_i) = {} & C_{ni1}\cosh\alpha_i\xi_i + C_{ni2}\sinh\alpha_i\xi_i \\ & + C_{ni3}\cosh\beta_i\xi_i + C_{ni4}\sinh\beta_i\xi_i \\ & + C_{ni5}\cos\gamma_i\xi_i + C_{ni6}\sin\gamma_i\xi_i \\ & + C_{ni7}\cos\delta_i\xi_i + C_{ni8}\sin\delta_i\xi_i. \end{aligned} \tag{18}$$

Equation (18) represents the solution for both the bending displacement $H_{ni}(\xi_i)$ and the torsional rotation $\Psi_{ni}(\xi_i)$. Thus we have

$$\begin{aligned} H_{ni}(\xi_i) = {} & A_{ni1}\cosh\alpha_i\xi_i + A_{ni2}\sinh\alpha_i\xi_i \\ & + A_{ni3}\cosh\beta_i\xi_i + A_{ni4}\sinh\beta_i\xi_i \\ & + A_{ni5}\cos\gamma_i\xi_i + A_{ni6}\sin\gamma_i\xi_i \\ & + A_{ni7}\cos\delta_i\xi_i + A_{ni8}\sin\delta_i\xi_i, \end{aligned} \tag{19}$$

$$\begin{aligned} \Psi_{ni}(\xi_i) = {} & B_{ni1}\cosh\alpha_i\xi_i + B_{ni2}\sinh\alpha_i\xi_i \\ & + B_{ni3}\cosh\beta_i\xi_i + B_{ni4}\sinh\beta_i\xi_i \\ & + B_{ni5}\cos\gamma_i\xi_i + B_{ni6}\sin\gamma_i\xi_i \\ & + B_{ni7}\cos\delta_i\xi_i + B_{ni8}\sin\delta_i\xi_i, \end{aligned} \tag{20}$$

where A_{nik} and B_{nik} ($k = 1, 2, 3, \ldots\ldots, 8$) are two different sets of constants. They can be related through Eqs.(5) and (6) as

$$\begin{aligned} & B_{ni1} = (k_{\alpha i}/x_\alpha)A_{ni1}, \quad B_{ni2} = (k_{\alpha i}/x_\alpha)A_{ni2}, \\ & B_{ni3} = (k_{\beta i}/x_\alpha)A_{ni3}, \quad B_{ni4} = (k_{\beta i}/x_\alpha)A_{ni4}, \\ & B_{ni5} = (k_{\gamma i}/x_\alpha)A_{ni5}, \quad B_{ni6} = (k_{\gamma i}/x_\alpha)A_{ni6}, \\ & B_{ni7} = (k_{\delta i}/x_\alpha)A_{ni7}, \quad B_{ni8} = (k_{\delta i}/x_\alpha)A_{ni8}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} & k_{\alpha i} = (\lambda_{Bi} - \alpha_i^4)/\lambda_{Bi}, \quad k_{\beta i} = (\lambda_{Bi} - \beta_i^4)/\lambda_{Bi}, \\ & k_{\gamma i} = (\lambda_{Bi} - \gamma_i^4)/\lambda_{Bi}, \quad k_{\delta i} = (\lambda_{Bi} - \delta_i^4)/\lambda_{Bi}. \end{aligned} \tag{22}$$

The expressions for the bending rotation $\Theta_{ni}(\xi_i)$, the bending moment $M_{ni}(\xi_i)$, the shear force $S_{ni}(\xi_i)$, the torque $T_{ni}(\xi_i)$ and the bi-moment $B_{ni}(\xi_i)$ can be obtained from Eqs.(19)–(22) as follows:

$$\Theta_{ni}(\xi_i) = H'_{ni}(\xi_i)/L_i, \tag{23}$$

$$M_{ni}(\xi_i) = -(E_iI/L_i^2)H''_{ni}(\xi_i), \tag{24}$$

$$S_{ni}(\xi_i) = -(1/L_i)M'_{ni}(\xi_i), \tag{25}$$

$$T_{ni}(\xi_i) = (G_iJ/L_i)[\Psi'_{ni}(\xi_i) - \lambda_{Wi}\Psi'''_{ni}(\xi_i)], \tag{26}$$

$$B_{ni}(\xi_i) = (\mathrm{E}_i\Gamma/L_i^2)\Psi''_{ni}(\xi_i), \tag{27}$$

Let

$$\begin{aligned} U_{ni}(\xi_i) = {} & [H_{ni}(\xi_i), \Theta_{ni}(\xi_i), M_{ni}(\xi_i), S_{ni}(\xi_i), \Psi_{ni}(\xi_i), \\ & \Psi'_{ni}(\xi_i), T_{ni}(\xi_i), B_{ni}(\xi_i)]^T, \\ A_{ni} = {} & [A_{ni1}, A_{ni2}, A_{ni3}, A_{ni4}, A_{ni5}, A_{ni6}, \\ & A_{ni7}, A_{ni8}]^T. \end{aligned} \tag{28}$$

Following Eqs.(19)–(28), one can obtain the associated matrix between the constants A_{ni} and U_{ni} as

$$U_{ni}(\xi_i) = V_{ni}(\xi_i)A_{ni}. \tag{29}$$

Since the continuity of $U_{ni}(\xi_i)$ at the interface between cell $n-1$ and n,[18] when $y = nL$ and $y = nL + L_1$, one can obtain the relationship as follows:

$$U_{(n-1)2}(L/L_2) = U_{n1}(0/L_1), \tag{30}$$

$$U_{n1}(L_1/L_1) = U_{n2}(L_1/L_2). \tag{31}$$

Equations (30) and (31) can be expressed in matrix forms

$$KA_{(n-1)2} = HA_{n1}, \tag{32}$$

$$K_1A_{n1} = H_1A_{n2}. \tag{33}$$

Based on Eqs.(32) and (33), the relationship between the n-th and $(n-1)$-th cell can be obtained as follows:

$$A_{n2} = TA_{(n-1)2}, \tag{34}$$

where

$$T = H_1^{-1}K_1H^{-1}K, \tag{35}$$

T is the transfer matrix.

Due to the periodicity of the infinite structure in the y-direction, the vector A_{ni} must satisfy the Bloch theorem[9]

$$A_n = \mathrm{e}^{\mathrm{i}kL}A_{n-1}, \tag{36}$$

where k is the wave vector. For convenience of expression, we write all of the one-dimensional vector in the form of scalar in this paper.

Following Eqs.(35) and (36), the eigenvalues of the infinite periodic structure are determined by the following equation

$$\det(T - \mathrm{e}^{\mathrm{i}kL}I) = 0, \tag{37}$$

where I is the 8×8 unit matrix.

Given ω, the value of k can be obtained from Eq.(37), and the wave can propagate through the infinite beam if k is real, corresponding to a pass band, but it cannot if k is imaginary, corresponding to a band gap.

3. Results and discussion

In order to validate and confirm the accuracy of the theory, an illustrative example, a periodic epoxy-steel binary straight infinite beam with a circular cross section having a crack in it, is chosen to demonstrate the effect of warping on the calculation of band gap, as shown in Fig.1.

Material A stands for epoxy and material B the steel. The length of A is 0.075 m, which is the same as that of B. The geometric properties and the material parameters of the beam are as follows:

$$\begin{aligned}
&S = 6.195 \times 10^{-5}(\mathrm{m}^2),\\
&I = 4.635 \times 10^{-10}(\mathrm{m}^4),^{[19]}\\
&I_{\alpha 1} = 3.505 \times 10^{-6}(\mathrm{kg}\cdot\mathrm{m}),^{[19]}\\
&I_{\alpha 2} = 2.311 \times 10^{-5}(\mathrm{kg}\cdot\mathrm{m}),^{[19]}\\
&J = 1.677 \times 10^{-10}(\mathrm{m}^4),^{[20]}\\
&x_\alpha = 5.799 \times 10^{-3}(\mathrm{m}),^{[21]}\\
&\Gamma = 8.468 \times 10^{-15}(\mathrm{m}^6),^{[20]}\\
&\rho_1 = 1180(\mathrm{kg/m}^3), \qquad \rho_2 = 7780(\mathrm{kg/m}^3),\\
&E_1 = 4.35 \times 10^9(\mathrm{Pa}), \qquad E_2 = 21.06 \times 10^{10}(\mathrm{Pa}),\\
&G_1 = 1.59 \times 10^9(\mathrm{Pa}), \qquad G_2 = 8.10 \times 10^{10}(\mathrm{Pa}).
\end{aligned}$$

Figure 2 illustrates the band structure of the infinite periodic structure epoxy/steel beam, taking into consideration the warping effect. The real wave vector is illustrated in Fig.2(a), and the absolute value of the imaginary part of complex wave vector is illustrated in Fig.2(b). One can find four complete band gaps within the range from 0 to 8000 Hz, shown as the shadowed regions in Fig.2(a). In the band gap frequency range, the coupled flexural-torsional wave cannot propagate through.

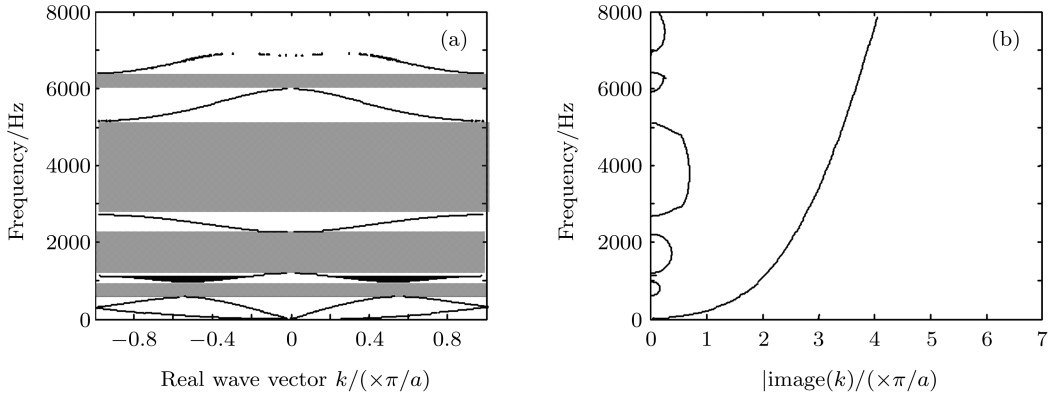


Fig.2. The complex band structure of the coupled vibration of the infinite periodic beam including warping effect; (a) Real wave vector, (b) the absolute value of the imaginary part of complex wave vector.

In Fig.3, the complex band structure of the same beam is given without considering the effect of warping. There are seven band gaps between 0 and 8000 Hz, shown as the shadowed regions in Fig.3(a).

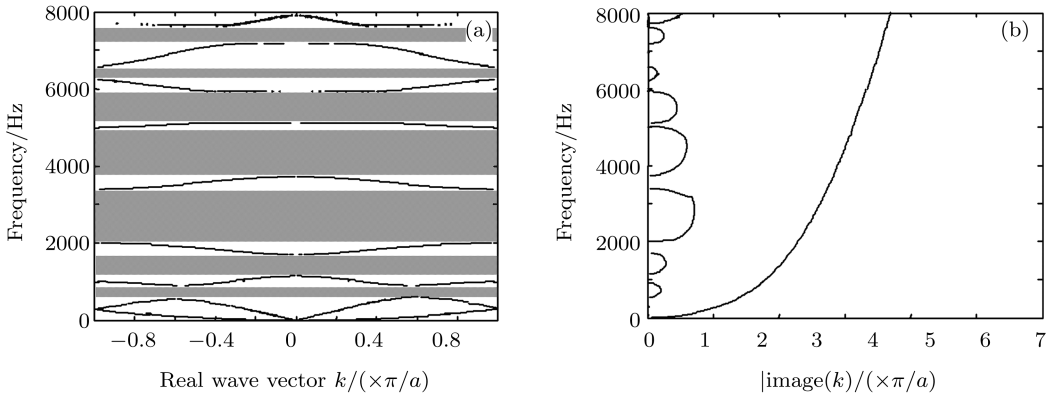


Fig.3. The complex band structure of the same beam when ignoring warping effect; (a) Real wave vector, (b) the absolute value of the imaginary part of complex wave vector.

Comparing Figs.2 and 3, one can observe that both the start and the end frequencies of the first four gaps in coupled vibration become higher, and the frequency range of the first three gaps becomes wider due to the warping effect. It is because warping leads to an additional stress with attenuation character that is described by the warping stiffness in Eqs.(1) and (2). This means that the rigidity of beam increases, therefore the start and the end frequencies become higher. The attenuation speed of the additional stress is concerned with the geometric structure of the beam and the physical parameter of material. As frequency increases, the attenuation in a single vibration period decreases because the length of time decreases. Therefore the effect of warping on the end frequency of a gap is larger than that on the start frequency, which leads to the increase of the frequency range of the gap. And also the effect of the warping on the third gap is apparently higher than that on the first and the second gaps, therefore the change of the third gap is higher than that of the first and the second gaps.

As we know, for the finite beam with N cells, the transmission coefficient can give an intuitive representation of the propagation property, instead of the dispersion relation of an infinite system. Since the frequency response function (FRF) of vibration can be used to describe the gaps effectively, we calculate the FRF of the coupled vibration in the six periodic beams with FE method.

For comparison, we use the same parameters as those of the model shown in Fig.1. In the simulation, the displacement stimulation in z direction of the centroid from 0 to 8000 Hz is applied to one end of the beam. The frequency response of the other end of the beam is shown in Fig.4. There are four sharp drops in the frequency respond curve. Every band gap has a response drop-off over 40 dB compared with the response outside the band gaps.

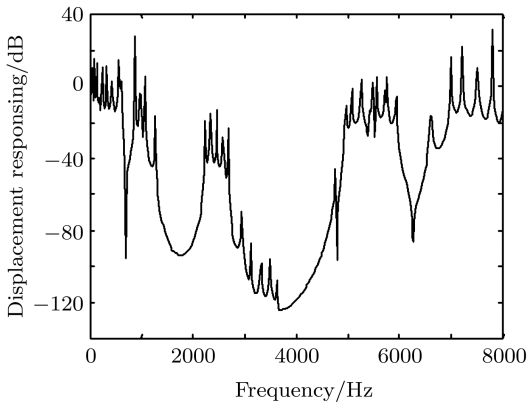


Fig.4. The displacement response at the end of the epoxy/steel beam whose number of periods is six, with the lattice constant $L = 0.15$ m and $L_1 : L_2 = 1 : 1$.

Comparing Figs.2 and 4, one can observe that there is also a little deviation between them. The reason is that figure 2 is plotted based on calculation of an infinite periodic beam while figure 4 is the result of a finite periodic beam.

A set of results for comparing the band gap of such beam, calculated with not only TM method including or excluding warping effect but also FEM are given in Table 1. The theory given in Ref.[12] was used to obtain the results for the case when warping was ignored. It is clear from the results in Table 1 that warping has a profound influence on the calculation

of band structure of the beam. The first four band gaps of the beam agree quite well with the ones which were obtained from simulation experiment, shown in Table 1; but if ignoring the effect of warping stiffness, we find that the difference of result from that with warping is so large that it cannot be accepted, especially for high frequency range although it is in good agreement with the result from the experiment in low frequency range.

Table 1. Frequency (Hz) range of band gap for the structure defined in Fig.1.

band gap number	warping ignored	warping included	simulated by FEM
1	(517, 909)	(577, 970)	(590, 880)
2	(1148, 1695)	(1165, 2222)	(1080, 2240)
3	(1994, 3414)	(2690, 5090)	(2680, 5080)
4	(3728, 5021)	(5910, 6420)	(5940, 6600)

4. Conclusions

The dispersion relation, obtained with the TM method used to calculate the complex band structure, has been developed for a bending-torsion-coupled beam by taking into account the effect of warping stiffness. It provides a more accurate result for complex band structure of a bending-torsion-coupled beam. And it has been shown that large errors may be incurred when calculating the complex band structure of a bending-torsion-coupled beam in high frequency range, if ignoring the effect of warping. The result obtained from application of present theory agrees very well with the result of simulation.

References

[1] Kushwaha M S, Halevi P, Dobrzynski L and Djjafari-Rouhani B 1993 *Phys. Rev. Lett.* **71** 2022

[2] Kushwaha M S, Halevi P, Martinez G, Dobrzynski L and Djjafari-Rouhani B 1994 *Phys. Rev.* B **49** 2313

[3] Mead D J 1970 *J. Sound Vib.* **11** 181

[4] Cai L, Han X Y and Wen X S 2008 *Acta Phys. Sin.* **57** 1746 (in Chinese)

[5] Wu F G and Liu Y Y 2002 *Acta Phys. Sin.* **51** 1434 (in Chinese)

[6] Wang G, Wen J H, Liu Y Z, Yu D L and Wen X S 2005 *Acta Phys. Sin.* **54** 1247 (in Chinese)

[7] Liu Z, Zhang X, Mao Y, Zhu Y, Yang Z, Chan C and Sheng P 2000 *Science* **289** 1734

[8] Yu D L, Liu Y Z, Qiu J, Wen J H and Zhao H G 2006 *Developments in Composites: Advanced, Infrastructural, Natural, and Nano-composites* Proceedings of International Composites Conference (Sydney: Australia)

[9] Yu D L, Liu Y Z, Qiu J and Wen J H 2005 *Chin. Phys.* **14** 1501

[10] Hallauer W L and Liu R Y 1982 *J. Sound Vib.* **85** 105

[11] Friberg P O 1983 *Int. J. Numer. Methods Eng.* **19** 479

[12] Banerjee J R 1989 *Int. J. Numer. Methods Eng.* **28** 1283

[13] Gere J M and Lin Y K 1958 *Journal of Applied Mechanics* **80** 373

[14] Dokumaci E 1987 *J. Sound Vib.* **119** 443

[15] Yaman Y 1997 *J. Sound Vib.* **204** 131

[16] Bishop R E D, Cannon S M and Miao S 1989 *J. Sound Vib.* **131** 457

[17] Banerjee J R, Guo S and Howson W P 1996 *Computer & Structures* **59** 613

[18] Camley R E, Djafari-Rouhani B, Dobrzynski L and Maradudin A A 1983 *Phys. Rev.* B **27** 7318

[19] Fan Q S *et al* 2000 *Mechanics of Materials* (Beijing: Higher Education Press)

[20] Oden J T *et al* 1967 *Mechanics of Elastic Structures* (New York: Hemisphere)

[21] Timoshenko S P *et al* 1974 *Vibration Problems in Engineering* (New York: Wiley)