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A generalized Mei conserved quantity and Mei symmetry of Birkhoff system

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This paper studies a new conserved quantity which can be called generalized Mei conserved quantity and directly deduced by Mei symmetry of Birkhoff system. The conditions under which the Mei symmetry can directly lead to generalized Mei conserved quantity and the form of generalized Mei conserved quantity are given. An example is given to illustrate the application of the results.

Keywords: Birkhoff system, Mei symmetry, conserved quantity
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1. Introduction

The research on symmetry and conservation laws possesses important significance in mathematics and mechanics and physics. In mechanical systems there are mainly Noether symmetry,[1] Lie symmetry[2] and Mei symmetry,[3] as well as unified symmetry,[4] and many important results[5−15] have achieved. Mei symmetry is a new symmetry which grows up in recent years. Mei symmetry can lead to Mei conserved quantities directly, and lead to Noether conserved quantities or Hojman conserved quantities indirectly.[6] Since Prof. Mei proposed Mei symmetry, a lot of discussions[5,6,9,10,16,17] have done on it. Very newly, Fang[18] *et al* got a new conserved quantity from Mei symmetry of Lagrange system and name it generalized Mei conserved quantity. Fang's studies further promote the development of Mei symmetry.[19−21]

The dynamics of a Birkhoff system is an important branch of classical mechanics in modern time. It can be applied to quantum mechanics, statistical mechanics, biological physics, mechanics of spaceflight and some fields of modern engineering.[22,23] In recent years, people has made systemic research on Birkhoff systems and give the symmetry and conservation laws,[24−28] perturbation to symmetry,[29,30] the stability,[31] geometric picture of Birkhoff systems and construct the theoretical framework of the dynamics of a Birkhoff system, publish monograph of Dynamics of Birkhoff system,[23] and extend the Birkhoff system to relativistic Birkhoff systems.[32−34]

This paper studies Mei symmetry of Birkhoff systems and get generalized Mei conserved quantities directly from Mei symmetry of the Birkhoff system. Firstly, we give the criterion of Mei symmetry for Birkhoff system, secondly, we give the conditions under which the Mei symmetry can lead to generalized Mei conserved quantity and get the generalized Mei conserved quantity of Birkhoff system, last, we give an example to illustrate the application of our results.

2. Definition and criterion of Mei symmetry

The Birkhoff equations can be written as[22,23]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^v}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0 \quad (\mu,v=1,...,2n), \tag{1}$$

where the function $B=B(t,\ \boldsymbol{a})$ is called Birkhoffian, $R_\mu=R_\mu(t,\ \boldsymbol{a})$ are called Birkhoff's functions.

Introduce the infinitesimal transformations

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{a}),\ a^{\mu*}(t^*)=a^\mu+\varepsilon\xi_\mu(t,\boldsymbol{a}), \tag{2}$$

where ε is an infinitesimal parameter, ξ_0, ξ_μ are the infinitesimal generators of transformations. Under the infinitesimal transformations of Eq.(2), $B=$

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$B(t,\ \boldsymbol{a})$ and $R_\mu = R_\mu(t,\ \boldsymbol{a})$ respectively become $B^* = B(t^*,\ \boldsymbol{a}^*)$ and $R^*_\mu = R_\mu(t^*,\ \boldsymbol{a}^*)$. Expanding B^* and R^*_μ into Taylor series at $(t,\ \boldsymbol{a})$, we have

$$\begin{aligned} B^* &= B(t^*, \boldsymbol{a}^*) = B(t, \boldsymbol{a}) + \varepsilon X^{(0)}(B) + O(\varepsilon^2), \\ R^*_\mu &= R_\mu(t^*, \boldsymbol{a}^*) = R_\mu(t, \boldsymbol{a}) + \varepsilon X^{(0)}(R_\mu) \\ &\quad + O(\varepsilon^2), \end{aligned} \tag{3}$$

where

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_\mu \frac{\partial}{\partial a^\mu} \tag{4}$$

is the operator of the infinitesimal generator.

Definition[25] Under the infinitesimal transformations (2), if the Birkhoff equations (1) keep their form invariance, that make the following equations hold

$$\begin{aligned} &\left(\frac{\partial R^*_v}{\partial a^\mu} - \frac{\partial R^*_\mu}{\partial a^v}\right)\dot{a}^v - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial R^*_\mu}{\partial t} = 0 \\ &(\mu, v = 1, ..., 2n), \end{aligned} \tag{5}$$

then this kind of invariance is called Mei symmetry of Birkhoff system.

Substituting Eq.(3) into Eq.(5), neglecting the second and higher order terms of ε, and considering Eq. (1), we can get the criterion of Mei symmetry.

Criterion For the Birkhoffian system (1), if the infinitesimal generators (2) satisfy

$$\begin{aligned} &\left\{\frac{\partial X^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X^{(0)}(R_\mu)}{\partial a^v}\right\}\dot{a}^v \\ &- \frac{\partial X^{(0)}(B)}{\partial a^\mu} - \frac{\partial X^{(0)}(R_\mu)}{\partial t} = 0, \\ &(\mu, v = 1, ..., 2n), \end{aligned} \tag{6}$$

then the system (1) has Mei symmetry.

3. The generalized Mei conserved quantity

The condition under which the Mei symmetry will lead to a generalized Mei conserved quantity and the form of the generalized Mei conserved quantity are given in the following theorem.

Theorem If the generators of the infinitesimal transformations $\xi_0,\ \xi_\mu$ of Mei symmetry and gauge function $G_F = G_F(t,\ \boldsymbol{a})$ satisfy the following conditions

$$\begin{aligned} &\left[f\frac{\partial X^{(0)}(R_\mu)}{\partial t} + \xi_\nu \frac{\partial X^{(0)}(R_\mu)}{\partial a^\nu}\right]\dot{a}^\mu + X^{(0)}(R_\mu)\dot{\xi}_\mu \\ &\left[f\frac{\partial X^{(0)}(B)}{\partial t} + \xi_\nu \frac{\partial X^{(0)}(B)}{\partial a^\nu}\right] \\ &- X^{(0)}(B)\dot{f} + \dot{G}_F = 0, \end{aligned} \tag{7}$$

then the Mei symmetry can lead to a conserved quantity as follows:

$$I_F = X^{(0)}(R_\mu)\xi_\mu - X^{(0)}(B)f + G_F = \text{const}, \tag{8}$$

where $f = f(t,\ \boldsymbol{a})$ is an arbitrary function whose duty is make the gauge function exist, we named f coordination function.[18]

Proof Calculate the derivative of Eq.(8) with respect to time t, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I_F =& \left[\frac{\partial X^{(0)}(R_\mu)}{\partial t} + \frac{\partial X^{(0)}(R_\mu)}{\partial a^v}\dot{a}^v\right]\xi_\mu + X^{(0)}(R_\mu)\dot{\xi}_\mu \\ &- \left[\frac{\partial X^{(0)}(B)}{\partial t} + \frac{\partial X^{(0)}(B)}{\partial a^v}\dot{a}^v\right]f \\ &- X^{(0)}(B)\dot{f} + \dot{G}_F, \end{aligned} \tag{9}$$

substituting Eq.(7) into Eq.(9), and considering the criterion Eq.(6) as well as the following conditions

$$\left[\frac{\partial X^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X^{(0)}(R_\nu)}{\partial a^v}\right]\dot{a}^v\dot{a}^\mu = 0, \tag{10}$$

we can get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I_F =& \left\{\left[\frac{\partial X^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X^{(0)}(R_\mu)}{\partial a^v}\right]\dot{a}^\nu \right. \\ &\left. - \frac{\partial X^{(0)}(B)}{\partial a^\mu} - \frac{\partial X^{(0)}(R_\mu)}{\partial t}\right\}(f\dot{a}^\mu - \xi_\mu) \\ =& f\left[\frac{\partial X^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X^{(0)}(R_\mu)}{\partial a^v}\right]\dot{a}^\nu\dot{a}^\mu = 0. \end{aligned} \tag{11}$$

So, equation (8) is a conserved quantity.

If we let $f = \xi_0$, then equations (7) and (8) become respectively the results in Refs.[27, 28]. So our results in this paper are more common, and we call the conserved quantity (8) as generalized Mei conserved quantity for Birkhoff system.

4. An illustrative example

Suppose the fourth order Birkhoff's functions and Birkhoffian are

$$R_1 = a^3,\ R_2 = a^4,\ R_3 = R_4 = 0, \tag{12}$$

$$B = [(a^3)^2 + (a^4)^2] + a^2. \tag{13}$$

Let us study its Mei symmetry and generalized Mei conserved quantity. The Birkhoff equation (1) gives out

$$-\dot{a}^3 = 0, \quad -\dot{a}^4 - 1 = 0,$$

$$\dot{a}^1 - a^3 = 0, \ \dot{a}^2 - a^4 = 0. \tag{14}$$

Take the criterion Eq.(6) into account, we have

$$\left(\frac{\partial \xi_4}{\partial a^1} - \frac{\partial \xi_3}{\partial a^2}\right)\dot{a}^2 - \frac{\partial \xi_3}{\partial a^3}\dot{a}^3 - \frac{\partial \xi_3}{\partial a^4}\dot{a}^4 - \frac{\partial \xi_3}{\partial a^1}a^3 - \frac{\partial \xi_4}{\partial a^1}a^4 - \frac{\partial \xi_2}{\partial a^1} - \frac{\partial \xi_3}{\partial t} = 0, \tag{15}$$

$$\left(\frac{\partial \xi_3}{\partial a^2} - \frac{\partial \xi_4}{\partial a^1}\right)\dot{a}^1 - \frac{\partial \xi_4}{\partial a^3}\dot{a}^3 - \frac{\partial \xi_4}{\partial a^4}\dot{a}^4 - \frac{\partial \xi_3}{\partial a^2}a^3 - \frac{\partial \xi_4}{\partial a^2}a^4 - \frac{\partial \xi_2}{\partial a^2} - \frac{\partial \xi_4}{\partial t} = 0, \tag{16}$$

$$\frac{\partial \xi_3}{\partial a^3}\dot{a}^1 + \frac{\partial \xi_4}{\partial a^3}\dot{a}^2 - \frac{\partial \xi_3}{\partial a^3}a^3 - \frac{\partial \xi_4}{\partial a^3}a^4 - \frac{\partial \xi_2}{\partial a^3} - \xi_3 = 0, \tag{17}$$

$$\frac{\partial \xi_3}{\partial a^4}\dot{a}^1 + \frac{\partial \xi_4}{\partial a^4}\dot{a}^2 - \frac{\partial \xi_3}{\partial a^4}a^3 - \frac{\partial \xi_4}{\partial a^4}a^4 - \frac{\partial \xi_2}{\partial a^4} - \xi_4 = 0. \tag{18}$$

The above equations have the following solutions

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = t, \quad \xi_3 = \xi_4 = 0. \tag{19}$$

The structure equation (7) gives

$$\dot{G}_F = f + \dot{f}. \tag{20}$$

Let $f = t$, we have

$$G_F = \frac{1}{2}t^2 + \frac{1}{3}t^3. \tag{21}$$

From Eq.(8), we can get

$$I_F = \frac{1}{3}t^3 - \frac{1}{2}t^2. \tag{22}$$

Let $f = a^3$, we have

$$G_F = a^1. \tag{23}$$

From Eq.(8), we can get

$$I_F = a^1 - a^3 t. \tag{24}$$

Let $f = a^4$, we have

$$G_F = a^2 - \frac{1}{2}t^2. \tag{25}$$

From Eq.(8), we can get

$$I_F = a^2 - a^3 t - a^4 t. \tag{26}$$

Let $f = \xi_0 = 0$, we have

$$G_F = a^3. \tag{27}$$

From Eq.(8), we can get

$$I_F = -a^3. \tag{28}$$

5. Conclusion

In this paper, we get generalized Mei conserved quantity which is a new conserved quantity of Birkhoffian system. The Mei conserved quantity is a special case of generalized Mei conserved quantity. The coordination function broadens the area of Mei conserved quantity and our results are of more general significance. We hope it can be applied to more other systems.

References

[1] Noether A E 1918 *Nachr. Akad. Wiss. Göttingen. Math. Phys.* K **I II** 235

[2] Lutzky M 1979 *J. Phys.* A: *Math. Gen.* **12** 973

[3] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120

[4] Mei F X, Xu X J and Zhang Y F 2004 *Acta Mech. Sin.* **20** 668 (in Chinese)

[5] Mei F X 2001 *Chin. Phys.* **10** 177

[6] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[7] Fang J H and Zhang P Y 2004 *Acta Phys. Sin.* **53** 4041 (in Chinese)
[8] Zhang H B, Chen L Q, Liu R W and Gu S L 2005 *Acta Phys. Sin.* **54** 2489 (in Chinese)
[9] Qiao Y F, Li R J and Ma Y S 2005 *Chin. Phys.* **14** 12
[10] Wu H B and Mei F X 2006 *Acta Phys. Sin.* **55** 3285 (in Chinese)
[11] Xu X J, Qin M C and Mei F X 2005 *Chin. Phys.* **14** 1287
[12] Wu H B 2005 *Chin. Phys.* **14** 452
[13] Fang J H, Wang P and Ding N 2006 *Commun. Theor. Phys.* **45** 882
[14] Wang P, Fang J H, Zhang P Y and Ding N 2006 *Commun. Theor. Phys.* **46** 385
[15] Li Y C, Jing H X, Xia L L, Wang J and Hou Q B 2007 *Chin. Phys.* **16** 2154
[16] Wang S Y and Mei F X 2002 *Chin. Phys.* **11** 5
[17] Fang J H, Liao Y P and Peng Y 2005 *Acta Phys. Sin.* **54** 496 (in Chinese)
Jia L Q, Luo S K and Zhang Y Y 2008 *Acta. Phys. Sin.* **57** 2006 (in Chinese)
[18] Fang J H, Ding N and Wang P 2007 *Chin. Phys.* **16** 887
[19] Wang P, Fang J H and Ding N 2007 *Commun. Theor. Phys.* **49** 993
[20] Ding N, Fang J H, Wang P and Zhang X N 2007 *Commun. Theor. Phys.* **48** 799
[21] Fang J H, Liu Y K and Zhang X N 2008 *Chin. Phys.* B **17** 1962
[22] Santilli R M 1983 *Foundations of Theoretical Mechanics* (New York: Springer)
[23] Mei F X, Shi R C, Zhang Y F and Wu H B 1996 *Dynamics of Birkhoffian Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)
[24] Mei F X 1993 *Sci. Chin.* A **23** 709
[25] Mei F X and Chen X W 2001 *J. Beijing Inst. Technol.* **10** 138
[26] Zhang Y 2002 *Acta Phys. Sin.* **51** 461 (in Chinese)
[27] Xu X J, Mei F X and Qin M C 2004 *Chin. Phys.* **13** 1999
[28] Zhang Y, Fan C X and Ge W K 2004 *Acta Phys. Sin.* **53** 3644 (in Chinese)
Zhang Y 2008 *Acta Phys. Sin.* **57** 2643 (in Chinese)
[29] Chen X W, Zhang R C and Mei F X 2000 *Acta Mech. Sin.* **16** 282 (in Chinese)
[30] Zhang Y 2006 *Chin. Phys.* **15** 2489
[31] Mei F X 1993 *Chin. Sci. Bull.* **38** 311
[32] Fu J L, Luo S K, Chen X W and Wang X M 2001 *Acta Phys. Sin.* **50** 2289 (in Chinese)
[33] Luo S K, Chen X W and Fu J L 2001 *Chin. Phys.* **10** 271
[34] Luo S K and Cai J L 2003 *Chin. Phys.* **12** 357