Vol 18 No 4, April 2009
1674-1056/2009/18(04)/1301-05

Chinese Physics B

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Topological structure of Gauss–Bonnet–Chern theorem and $\tilde{p}$-branes*

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(Received 8 June 2008; revised manuscript received 4 July 2008)

By making use of the ϕ-mapping topological current theory, this paper shows that the Gauss–Bonnet–Chern density (the Euler–Poincaré characteristic $\chi(M)$ density) can be expressed in terms of a smooth vector field $\boldsymbol{\phi}$ and take the form of $\delta(\boldsymbol{\phi})$, which means that only the zeros of $\boldsymbol{\phi}$ contribute to $\chi(M)$. This is the elementary fact of the Hopf theorem. Furthermore, it presents that a new topological tensor current of $\tilde{p}$-branes can be derived from the Gauss–Bonnet–Chern density. Using this topological current, it obtains the generalized Nambu action for multi $\tilde{p}$-branes.

Keywords: topological structure, branes
PACC: 0230

1. Introduction

The Gauss–Bonnent–Chern (GBC) theorem is one of the most significant results in differential geometry, which relates the curvature of the compact and oriented even-dimensional Riemannian manifold M with an important topological invariant, the Euler–Poincaré characteristic $\chi(M)$ of M. An elegantly intrinsic proof of the theorem was proposed by Chern.[1] A useful summary and some historical comments on GBC theorem were given, respectively, by Kabayashi, Nomizu,[2] and Spivak.[3] Reference [4] presented a detailed review of Chern's proof of the GBC theorem. A great advance in this field is the discovery of the relationship between supersymmetry and the index theorem, which includes the derivation of the GBC theorem via supersymmetry and the path integral techniques as presented by Alvarez–Gaumé.[5] On the physics side, the optical Berry phase is a direct result of the Gauss–Bonnent theorem, the black hole entropy emerges as the Euler class through dimensional continuation of the Gauss–Bonnent theorem and the topological current of GBC is important to study the topological defects in Ginzburg–Landau theory.

Extended objects with p spatial dimension, known as "branes", play an essential role in revealing the nonperturbative structure of the superstring theories and M-theory.[6−8] Antisymmetric tensor gauge fields have been widely studied in the theories of $\tilde{p}$-branes.[9] In the context of the effective $D = 10$ or $D = 11$ supergravity theory a $\tilde{p}$-brane is a p-dimensional extended source for a $(p+2)$-form gauge field strength F. It is well known that the $(p+2)$-form strength F satisfies the field equation $\nabla_\mu F^{\mu\mu_1\cdots\mu_{p+1}} = j^{\mu_1\cdots\mu_{p+1}}$, where $j^{\mu_1\cdots\mu_{p+1}}$ is a $(p+1)$-form tensor current, corresponding to the electric source and the dual field strength ${}^*F$ satisfies $\nabla^*_\mu F^{\mu\mu_1\cdots\mu_{p+1}} = \tilde{j}^{\mu_1\cdots\mu_{p+1}}$, in which $\tilde{j}^{\mu_1\cdots\mu_{p+1}}$ is a $(p+1)$-form tensor current, corresponding to the magnetic source.[10]

It must be pointed out that the essence of GBC theorem is to relate the curvature tensor to the Euler characteristic $\chi(M)$. Moreover, the Hopf–Poincaré theorem[11] shows that the $\chi(M)$ is the sum of the indices of the zeros of the vector field on M. In this paper, based on the ϕ-mapping topological current theory,[12,13] we obtain the inner relationship between the GBC theorem and the Hopf–Poincaré theorem. Furthermore, we present a topological tensor current of $\tilde{p}$-branes derived from the GBC density and use this current to get the generalized Nambu action for multi-$\tilde{p}$-branes. This paper is arranged as follows. In Section 2, we discuss the Chern form and express it completely in terms of a unit vector. In Section 3, we study the $\tilde{p}$-branes derived from the GBC density. The

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topological concepts of the Hopf index and Brouwer degree play an important role in the investigation.

2. Topological structure of Gauss–Bonnet–Chern density

Let M be a compact and oriented N (even)-dimensional Riemannian manifold and $P(\pi, M, G)$ be a principal bundle with the structure group $G = \mathrm{SO}(N)$. A smooth vector field ϕ^a $(a = 1, 2 \ldots N)$ can be found on the base manifold M, which is a section of a vector bundle over M. Defining a unit vector $\hat{n}$ as

$$n^a = \frac{\phi^a}{\|\phi\|}, \quad n^a n^a = 1, \tag{1}$$

one can see that the zeros of $\boldsymbol{\phi}$ are just the singularities of $\hat{n}$ and the unit vector field $\hat{n}$ is identified as a section of the sphere bundle over M. The GBC form is an N (even)-form Λ over the manifold M, such that when pulled back to $S(M)$, it is exact

$$\pi^* \Lambda = \mathrm{d}\Omega. \tag{2}$$

The famous GBC theorem can thus be expressed as

$$\chi(M) = \int_M \Lambda = \int_M n^* \pi^* \Lambda, \tag{3}$$

where

$$\Lambda = \frac{(-1)^{N/2}}{2^N \pi^{N/2} (N/2)!} \epsilon_{a_1 a_2 \ldots a_{N-1} a_N} F^{a_1 a_2} \wedge F^{a_3 a_4} \wedge \ldots \times \wedge F^{a_{N-1} a_N}. \tag{4}$$

Using a recursion method, reference [1] has proved that the $(N-1)$-form Ω on $S(M)$ can be expressed as

$$\Omega = \frac{1}{(2\pi)^{N/2}} \sum_{k=0}^{\frac{N}{2}-1} (-1)^k \frac{2^{-k}}{(N-2k-1)!!k!} \Theta_k, \tag{5}$$

which is called the Chern form with

$$(N-2k-1)!! = (N-2k-1)(N-2k-3)(N-2k-5)\ldots 1, \tag{6}$$

$$\Theta_k = \epsilon_{a_1 a_2 \ldots a_{N-2k} a_{N-2k+1} a_{N-2k+2} \ldots a_{N-1} a_N} n^{a_1} Dn^{a_2} \wedge \ldots \wedge \mathrm{D}n^{a_{N-2k}} \wedge F^{a_{N-2k+1} a_{N-2k+2}} \wedge \ldots \wedge F^{a_{N-1} a_N}. \tag{7}$$

Here $\mathrm{D}n^a = \mathrm{d}n^a - \omega^{ab} n^b$ is the covariant derivative 1-form of n^a, $F^{ab} = \mathrm{d}\omega^{ab} - \omega^{ac} \wedge \omega^{cb}$ is the $\mathrm{SO}(N)$ gauge field tensor and ω^{ab} corresponds to the $\mathrm{SO}(N)$ gauge potential. In fact, let P be an arbitrary but fixed point of M. In a neighbourhood of P, we can choose a family of frames such that at P

$$\omega^{ab} = 0, \quad F^{ab} = 0, \quad \mathrm{D}n^a = \mathrm{d}n^a. \tag{8}$$

Then the $(N-1)$-form expressed by Eq.(7) can be written as

$$\Theta_0 = \epsilon_{a_1 \ldots a_N} n^{a_1} \mathrm{d}n^{a_2} \wedge \ldots \wedge \mathrm{d}n^{a_N}. \tag{9}$$

Let ρ be the GBC density

$$\rho = dd\Omega = \frac{1}{(2\pi)^{N/2}(N-1)!!} \mathrm{d}\Theta_0 = \frac{1}{(2\pi)^{N/2}(N-1)!!} \epsilon_{a_1 \ldots a_N} \epsilon^{\mu_1 \cdots \mu_N} \partial_{\mu_1} n^{a_1} \ldots \times \partial_{\mu_N} n^{a_N} \mathrm{d}^N x. \tag{10}$$

Based on the ϕ-mapping topological current theory, substituting Eq.(1) into Eq.(10) and considering that

$$\partial_\mu n^a = \frac{1}{\|\phi\|} \partial_\mu \phi^a + \phi^a \partial_\mu \frac{1}{\|\phi\|}, \quad \frac{\partial}{\partial \phi^a} \left(\frac{1}{\|\phi\|} \right) = -\frac{\phi^a}{\|\phi\|^3}, \tag{11}$$

we can find that the GBC density can be expressed as

$$\rho = C_N \epsilon^{\mu_1 \mu_2 \cdots \mu_N} \epsilon_{a_1 a_2 \ldots a_N} \partial_{\mu_1} \phi^{a_1} \ldots \partial_{\mu_N} \phi^{a_N} \times \sum_a \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^a} (G_N(\|\phi\|)) \mathrm{d}^N x, \tag{12}$$

where C_N is a constant

$$C_N = \begin{cases} -\dfrac{1}{A(S^{N-1}) N! (N-2)} & \text{for } N > 2, \\ \dfrac{1}{4\pi} & \text{for } N = 2, \end{cases} \tag{13}$$

and $G_N(\|\phi\|)$ is a generalized function

$$G_N(\|\phi\|) = \begin{cases} \dfrac{1}{\|\phi\|^{N-2}} & \text{for } N > 2, \\ \ln(\|\phi\|) & \text{for } N = 2, \end{cases} \tag{14}$$

where $A(S^{N-1})$ is the area of S^{N-1}: $A(S^{N-1}) = \dfrac{(2\pi)^{N/2}}{\Gamma(\frac{N}{2})}$. If we define a generalized Jacobian tensor as

$$\begin{aligned} &\epsilon_{\mu_1\mu_2\ldots\mu_N} D\left(\frac{\phi}{x}\right) \\ &= \epsilon_{a_1a_2\ldots a_N}\partial_{\mu_1}\phi^{a_1}\partial_{\mu_2}\phi^{a_2}\ldots\partial_{\mu_N}\phi^{a_N}, \end{aligned} \tag{15}$$

and make use of the generalized Laplacian–Green function relation[14] in ϕ space

$$\begin{aligned} &\sum_a \frac{\partial}{\partial\phi^a}\frac{\partial}{\partial\phi^a}G_N(\|\phi\|) \\ &= \begin{cases} -\dfrac{4\pi^{N/2}}{\Gamma(N/2-1)}\delta(\boldsymbol{\phi}) & \text{for} \quad N > 2, \\ 2\pi\delta(\boldsymbol{\phi}) & \text{for} \quad N = 2, \end{cases} \end{aligned} \tag{16}$$

we obtain a δ-function like current density

$$\rho = \delta(\boldsymbol{\phi})D\left(\frac{\phi}{x}\right)\mathrm{d}^N x. \tag{17}$$

The GBC theorem can now be written as

$$\chi(M) = \int_M \delta(\boldsymbol{\phi})D\left(\frac{\phi}{x}\right)\mathrm{d}^N x, \tag{18}$$

which shows that only at the zero points of the vector field $\boldsymbol{\phi}$, we can get the nontrivial GBC density. Suppose that the vector field $\boldsymbol{\phi}$ possesses l isolated zeros, according to the implicit function theorem,[15] when the Jacobian $D(\phi/x) \neq 0$, the solutions of $\boldsymbol{\phi}(x) = 0$ are generally expressed as $\boldsymbol{x}_i = \boldsymbol{z}_i$, $i = 1, 2, ..., l$. In terms of the δ-function theory,[16] one can rigorously prove that the $\delta(\boldsymbol{\phi})$ can be expressed as

$$\delta(\boldsymbol{\phi}) = \sum_{i=1}^{l}\frac{W_i\delta(\boldsymbol{x}-\boldsymbol{z}_i)}{D(\phi/x)|_{\boldsymbol{x}=\boldsymbol{z}_i}}, \quad W_i = \beta_i\eta_i, \tag{19}$$

where W_i is the winding number of the vector field, $\beta_i = |W_i|$ is the Hopf index and η_i is the Brouwer degree of ϕ-mapping

$$\eta_i = \mathrm{sgn}\ D\left(\frac{\phi}{x}\right)\Big|_{\boldsymbol{x}=\boldsymbol{z}_i} = \pm 1. \tag{20}$$

Then equation (18) can be written as

$$\chi(M) = \int_M \beta_i\eta_i\delta(\boldsymbol{x}-\boldsymbol{z}_i)\mathrm{d}^N x = \sum_{i=1}^{l} W_i, \tag{21}$$

which states that the sum of indices of the zeros of vector $\boldsymbol{\phi}$ is the Euler characteristic. Therefore, the topological structure of GBC form reveals the expected result of the Hopf–Poincaré theorem. The above discussion, especially, the expression (17) is very valuable to establish the theory of the $\tilde{p}$-branes derived from GBC density.

3. $\tilde{p}$-branes derived from GBC density

Let us consider the D-dimensional smooth manifold X with metric tensor $g_{\mu\nu}$ and local coordinate x^μ $(\mu, \nu = 0, ..., D-1)$ with $x^0 = t$ as time, and let R^N be an Euclidean space of dimension N ($N < D$ and N is even number). We consider a smooth map $\phi : X \mapsto R^N$, which gives an N-dimensional smooth vector field on X,

$$\phi^a = \phi^a(x), \quad a = 1, 2, ..., N. \tag{22}$$

The direction unit field of $\boldsymbol{\phi}(x)$ can be expressed as

$$n^a = \frac{\phi^a}{\|\phi\|}, \quad \|\phi\| = \sqrt{\phi^a\phi^a}. \tag{23}$$

A generally covariant topological current is defined as

$$\begin{aligned} j^{\mu_1\mu_2\ldots\mu_{D-N}} &= \frac{1}{A(S^{N-1})(N-1)!}\left(\frac{1}{\sqrt{g}}\right)\epsilon^{\mu_1\ldots\mu_{D-N}\mu_{D-N+1}\ldots\mu_D} \\ &\times \epsilon_{a_1\ldots a_N}\partial_{\mu_{D-N+1}}n^{a_1}\partial_{\mu_{D-N+2}}n^{a_2}\ldots\partial_{\mu_D}n^{a_N}, \end{aligned} \tag{24}$$

just corresponding to the GBC density Eq.(10). Repeating the same derivation as before, we get

$$\begin{aligned} j^{\mu_1\mu_2\ldots\mu_{D-N}} &= C_N\left(\frac{1}{\sqrt{g}}\right)\epsilon^{\mu_1\ldots\mu_{D-N}\mu_{D-N+1}\ldots\mu_D} \\ &\times \epsilon_{a_1\ldots a_N}\partial_{\mu_{D-N+1}}\phi^{a_1}\ldots\partial_{\mu_D}\phi^{a_N}\sum_a\frac{\partial}{\partial\phi^a}\frac{\partial}{\partial\phi^a}(G_N(\|\phi\|)). \end{aligned} \tag{25}$$

Let us define Jacobian as

$$\epsilon^{a_1\cdots a_N} D^{\mu_1\cdots\mu_{D-N}}\left(\frac{\phi}{x}\right) = \epsilon^{\mu_1\cdots\mu_{D-N}\mu_{D-N+1}\cdots\mu_D}\partial_{\partial_\mu(D-N+1)}\phi^{a_1}\cdots\partial_{\mu_D}\phi^{a_N}, \tag{26}$$

in which $D^0\left(\frac{\phi}{x}\right)$ is the usual N-dimensional Jacobian

$$D^0\left(\frac{\phi}{x}\right) = D\left(\frac{\phi}{x}\right) = \frac{\partial\left(\phi^1,\ldots,\phi^N\right)}{\partial(x^1,\ldots,x^N)}. \tag{27}$$

Making use of the Laplacian relation,[14] we obtain the δ-function like current

$$j^{\mu_1\cdots\mu_{D-N}} = \left(\frac{1}{\sqrt{g}}\right)\delta(\boldsymbol{\phi})D^{\mu_1\cdots\mu_{D-N}}\left(\frac{\phi}{x}\right). \tag{28}$$

One can see that $j^{\mu_1\cdots\mu_{D-N}} \neq 0$ only when $\boldsymbol{\phi} = 0$, so it is essential to discuss the solutions of the equations

$$\phi^a(x) = 0, \quad a = 1,\ldots,N. \tag{29}$$

Suppose that the vector field $\boldsymbol{\phi}(x)$ possesses l isolated zeros, according to the implicit function theorem,[15] when the zeros are regular points of ϕ-mapping, i.e., $D^{\mu_1\cdots\mu_{D-N}}(\frac{\phi}{x}) \neq 0$, the solutions of $\boldsymbol{\phi}(x) = 0$ can be parametrized by

$$\begin{aligned} &x^\mu = Z_i^\mu(u^1,u^2,\ldots u^{D-N}),\\ &i = 1,\ldots,l,\ \ \mu = 1,\ldots,N, \end{aligned} \tag{30}$$

where the subscript i represents the i-th solution and the parameters $u = u(u^1,u^2,\ldots,u^{D-N})$ span a $(D-N)$-dimensional submanifold of X, denoted by N_i, each of which corresponds to a $\tilde{p}$-brane and N_i is its world volume. One sees that the tensor current $j^{\mu_1\cdots\mu_{D-N}}$ is not vanished only on the world volume manifolds N_i $(i = 1,\ldots,l)$. Therefore, every isolated zero of $\boldsymbol{\phi}(x)$ on X corresponds to a GBC $\tilde{p}$-brane.

In the following, we will discuss the inner structure of the topological tensor current $j^{\mu_1\cdots\mu_{D-N}}$. It can be proved that there exists an N-dimensional submanifold M in X with the parameter equation

$$x^\mu = x^\mu(v^1,\ldots,v^N) \quad \mu = 1,\ldots,D, \tag{31}$$

which is transversal to every N_i at the point p_i with

$$\begin{aligned} &g_{\mu\nu}\frac{\partial x^\mu}{\partial u^I}\frac{\partial x^\nu}{\partial v^A} = 0,\\ &I = 1,\ldots,D-N, \quad A = 1,\ldots,N. \end{aligned} \tag{32}$$

This is to say that the equations $\boldsymbol{\phi}(x) = 0$ have isolated zero points on M. In order to investigate the property of the topological tensor current $j^{\mu_1\cdots\mu_{D-N}}$, we must expand the N-dimensional δ-function of the singular point in terms of the δ-function on the singular submanifold N_i, which have been given in Ref.[16]

$$\begin{aligned} &\delta(N_i) = \int_{N_i}\delta^N(x - Z_i(u))\sqrt{g_u}\mathrm{d}^{(D-N)}u,\\ &i = 1,\ldots,l, \end{aligned} \tag{33}$$

in which

$$\begin{aligned} &g_u = \det\left(g_{\mu\nu}\frac{\partial x^\mu}{\partial u^I}\frac{\partial x^\nu}{\partial u^J}\right),\\ &I,J = 1,\ldots.,(D-N). \end{aligned} \tag{34}$$

Then, from Eq.(19), and by considering the property of the δ-function, one will obtain

$$\delta(\boldsymbol{\phi}(x)) = \sum_{i=1}^{l}\beta_i\eta_i\int_{N_i}\delta^N(x - Z_i(u))\sqrt{g_u}\mathrm{d}^{(D-N)}u. \tag{35}$$

Therefore, the general topological current of the $\tilde{p}$-branes can be expressed directly as

$$j^{\mu_1\cdots\mu_{D-N}} = \left(\frac{1}{\sqrt{g}}\right)D^{\mu_1,\cdots,\mu_{D-N}}\left(\frac{\phi}{x}\right)\sum_{i=1}^{l}\beta_i\eta_i\int_{N_i}\delta^N(x - Z_i(u))\sqrt{g_u}\mathrm{d}^{(D-N)}u. \tag{36}$$

If we define a Lagrangian as

$$L = \sqrt{\frac{1}{(D-N)!}g_{\mu_1\nu_1}\cdots g_{\mu_{D-N}\nu_{D-N}}j^{\mu_1\cdots\mu_{D-N}}j^{\nu_1\cdots\nu_{D-N}}}, \tag{37}$$

which is just the generalization of Nielsen's Lagrangian.[17] then we can prove that

$$L = \left(\frac{1}{\sqrt{g}}\right)\delta(\boldsymbol{\phi}(x)). \tag{38}$$

Then, the action takes the form as

$$S = \int_X L\sqrt{g}\mathrm{d}^D x = \int_X \delta(\boldsymbol{\phi}(x))\mathrm{d}^D x. \tag{39}$$

By substituting Eq.(35) into Eq.(39), we obtain an important result

$$\begin{aligned} S &= \int_X \sum_{i=1}^{l} \beta_i \eta_i \int_{N_i} \delta^N(x - Z_i(u))\sqrt{g_u}\mathrm{d}^{(D-N)}u\mathrm{d}^D x \\ &= \sum_{i=1}^{l} \beta_i \eta_i \int_{N_i} \sqrt{g_u}\mathrm{d}^{(D-N)}u, \end{aligned} \tag{40}$$

i.e.,

$$S = \sum_{i=1}^{l} \eta_i S_i, \tag{41}$$

where $S_i = \beta_i \int_{N_i} \sqrt{g_u}\mathrm{d}^{(D-N)}u$. This is just the generalized Nambu action for multi-$\tilde{p}$-branes ($p = D - N - 1$), which is the straightforward generalization of Nambu action for the string world-sheet action. Here this action for multi-$\tilde{p}$-branes is obtained directly by ϕ-mapping theory, and it is easy to see that this action is just Nambu action for multistring when $D - N = 2$.

4. Conclusion

In conclusion, based on the ϕ-mapping topological current method, we obtain the inner relationship between the GBC theorem and Hopf–Poincaré theorem: only the zeros of $\boldsymbol{\phi}$ contribute to the GBC density (the Euler–Poincaré characteristic $\chi(M)$ density). Furthermore, we present a topological current of $\tilde{p}$-branes derived from the GBC density. Using this current, we obtain the generalized Nambu action for multi-$\tilde{p}$-branes.

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