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Painlevé properties and exact solutions for the high-dimensional Schwartz Boussinesq equation*

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The usual (1+1)-dimensional Schwartz Boussinesq equation is extended to the (1+1)-dimensional space–time symmetric form and the general (n+1)-dimensional space–time symmetric form. These extensions are Painlevé integrable in the sense that they possess the Painlevé property. The single soliton solutions and the periodic travelling wave solutions for arbitrary dimensional space–time symmetric form are obtained by the Painlevé–Bäcklund transformation.

Keywords: high-dimensional integrable model, Schwartz Boussinesq equation, Painlevé integrable
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1. Introduction

Modern soliton theory is widely applied in almost all the physics fields such as field theory, condensed matter physics, plasma physics, optics, particle, nuclear physics, etc.[1,2] It is significant in soliton theory to find more new high-dimensional integrable systems. There are several ways to obtain high-dimensional equations.[3−10] It is said that the known (1+1)- and (2+1)-dimensional integrable models both possess a Schwartz form which is conformal invariant, and the conformal invariant plays a very important role in finding other integrable properties.[4] In Refs.[7, 8], Lou has pointed out that starting from a conformal invariant form to find a high-dimensional integrable model may be one of the best convenient ways and has obtained the $(n+1)$-dimensional extensions by using the Schwartz Korteweg–de Vries (KdV) and the Kadomtsev–Petviashvili (KP) equations which possess Painlevé property. Naturally, we hope that the more Schwartz equations can be extended to high-dimensional Painlevé integrable systems.

In the present paper, we extend the Schwartz Boussinesq equation to high dimension case, and the usual Boussinesq equation is[11]

$$U_{tt} + \frac{1}{3}U_{xxxx} + UU_{xx} + U_x^2 = 0. \tag{1}$$

The above equation was introduced by Boussinesq to describe the propagation of long waves in shallow water. In recent years, a great deal of research work has been carried out for the study of Boussinesq equation.[12−15]

The remainder of the present paper is organized as follow. In Section 2, a (1+1)-dimensional Painlevé integrable space–time symmetric form is obtained. In Section 3 the arbitrary dimensional Painlevé integrable space–time symmetric form is discussed. In Section 4, the single soliton solution and the travelling wave solution with an arbitrary function are obtained by the Painlevé–Bäcklund transformation for an (n+1)-dimensional space–time symmetric form. Finally a brief summary and discussion are given in Section 5.

2. (1+1)-dimensional space–time symmetric Boussinesq extension

For the Boussinesq equation, the Schwartz form is[16]

$$\left(\frac{\Phi_t}{\Phi_x}\right)_t + \frac{1}{2}\left(\frac{\Phi_t^2}{\Phi_x^2}\right)_x + \frac{1}{3}\{\Phi; x\}_x = 0, \tag{2}$$

where

$$\{\Phi; x\} = \left(\frac{\Phi_{xx}}{\Phi_x}\right)_x - \frac{1}{2}\left(\frac{\Phi_{xx}}{\Phi_x}\right)^2$$

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is Schwartz derivative, while Eq.(1) and its Schwartz form Eq.(2) are related to each other by

$$U = \frac{2\Phi_{xx}}{\Phi} - \frac{2\Phi_x^2}{\Phi^2} + \frac{\Phi_t^2}{\Phi_x^2} - \frac{\Phi_{xx}^2}{\Phi_x^2} + \frac{4}{3}\frac{\Phi_{xxx}}{\Phi_x}. \quad (3)$$

Equation (2) can be extended into several forms. We extend the Schwartz form (2) into a space–time symmetric form expressed as

$$\left(\frac{\Phi_t}{\Phi_x}\right)_t + \frac{1}{2}\left(\frac{\Phi_t^2}{\Phi_x^2}\right)_x + \frac{1}{3}\{\Phi; x\}_x + c_1\Bigg\{\left(\frac{\Phi_x}{\Phi_t}\right)_x + \frac{1}{2}\left(\frac{\Phi_x^2}{\Phi_t^2}\right)_t + \frac{1}{3}\{\Phi; t\}_t\Bigg\} = 0, \quad (4)$$

where c_1 is constant. When $c_1 = 0$, Eq.(4) turns into the usual Schwartz form Eq.(2), and Eq.(4) is symmetric by exchanging the space–time $x \leftrightarrow t$ for $c_1 = 1$. Meanwhile, Eq.(4) is invariant under the Möbious transformation

$$\Phi \to \frac{a + b\Phi}{c + d\Phi}, \quad ad \neq bc. \quad (5)$$

In order to make use of the Weiss–Tabor–Carnevale (WTC) approach,[16] we make the following transformation:

$$\Phi = \exp F, \quad (6)$$

$$F_x = u \ \text{ and } \ F_t = v. \quad (7)$$

Substituting expressions (6) and (7) into Eq.(4) yields the following system:

$$\begin{aligned} &3u^2v^2(vv_t + c_1uu_x) + u^2v^2(vu_{xxx} + c_1uv_{ttt}) \\ &- u^3v^3(uu_x + c_1vv_t) \\ &- 4uv(v^2u_{xx}u_x + c_1u^2v_{tt}v_t) - 3(v^3u_x^3 + c_1u^3v_t^3) \\ &- 3(v^5u_x + c_1u^5v_t) = 0, \end{aligned} \quad (8)$$

$$u_t = u_x, \quad (9)$$

where Eq.(9) is the compatibility condition of transformations (7).

We use the standard WTC approach to prove the Painlevé property of Eqs.(8) and (9). According to the standard WTC method, we can represent u and v as

$$u = \sum_{j=0}^{\infty} u_j \Phi^{j+\alpha} \ \text{ and } \ v = \sum_{j=0}^{\infty} v_j \Phi^{j+\beta} \quad (10)$$

with u_j and v_j being arbitrary functions of (x, t); $u_0 \neq 0$ and $v_0 \neq 0$. By using the leading order analysis, we obtain

$$\alpha = -1, \quad \beta = -1,$$

$$u_0^2 = \Phi_x^2 \ \text{ and } \ v_0^2 = \Phi_t^2. \quad (11)$$

Now substituting expressions (10) and (11) into Eqs.(8) and (9) and comparing the coefficients of the $(\Phi^{j-9}, \ \Phi^{j-2})$, we can obtain the recursion relations between functions u_j and v_j as follows:

$$(j+1)(j-1)(j-2)(u_0^2v_0\Phi_x u_j + c_1u_0v_0^2\Phi_t v_j) = 0, \quad (12)$$

$$(j-1)(v_0u_j + u_0v_j) = 0. \quad (13)$$

Thus, the resonance values of j are given by

$$\begin{vmatrix} (j^2-1)(j-2)u_0^2v_0\phi_x, & c_1(j^2-1)(j-2)u_0v_0^2\phi_t \\ (j-1)v_0, & (j-1)u_0 \end{vmatrix} = 0, \quad (14)$$

i.e.

$$j = -1, 1, 1, 2. \quad (15)$$

The resonance at $j = -1$ corresponds to an arbitrary singularity manifold $\Phi(x, t) = 0$. In order to prove the existence of arbitrary function at two other resonances, $j = 1$, we obtain the following equations by collecting the coefficients of $(\Phi^{j-8}, \ \Phi^{j-1})$:

$$\begin{aligned} &u_0^2v_0^2(-v_0u_{0x}\Phi_x^2 + 2u_0v_0\Phi_x\Phi_{xx} \\ &- v_0u_0^2u_{0x} - c_1u_0v_{0t}\Phi_t^2 + 2c_1v_0u_0\Phi_t\Phi_{tt} \\ &+ c_1u_0v_{0t}v_0^2) = 0, \end{aligned} \quad (16)$$

$$u_{0t} - v_{0x} = 0. \quad (17)$$

Substituting expression (11) into Eqs.(16) and (17), we can easily verify that Eqs.(16) and (17) are fully satisfied, and both u_1 and v_1 must be arbitrary functions. Similarly, gathering up the coefficients of $(\Phi^{j-7}, \ \Phi^j)$ yields the following equations:

$$\begin{aligned} &u_0v_0\{v_0^2u_0u_{0xx}\Phi_x + v_0u_0u_{0x}\Phi_{xx} - v_0^2\Phi_xu_{0x}^2 \\ &- v_0^2u_0^2\Phi_{xxx} + c_1(v_0u_0^2v_{0tt}\Phi_t \\ &+ v_0u_0^2v_{0t}\Phi_{tt} - v_0^2u_0^2\Phi_{ttt})\} = 0, \end{aligned} \quad (18)$$

$$u_{1t} + u_2 \Phi_t - v_{1x} - v_2 \Phi_x = 0. \tag{19}$$

We find that Eq.(18) is fully satisfied by expression (11) and only Eq.(19) contains u_2 and v_2. So Eqs.(8) and (9) pass the Painlevé test in the sense of WTC approach.

In the same way, we can obtain (2+1)-dimensional and (3+1)-dimensional Painlevé integrable space–time symmetric forms. We can find a rule to construct an arbitrary dimensional Painlevé integrable space–time symmetric form in the next section.

3. $(n+1)$-dimensional Boussinesq extension

Following the above method, we can take an $(n+1)$-dimensional Boussinesq extension as

$$\sum_{i=1}^{n}\left[\left(\frac{\Phi_t}{\Phi_{x_i}}\right)_t + \frac{1}{2}\left(\frac{\Phi_t}{\Phi_{x_i}}\right)^2 + \frac{1}{3}\{\Phi; x_i\}_{x_i}\right] + c_1\sum_{i=1}^{n}\left\{\left(\frac{\Phi_{x_i}}{\Phi_t}\right)_t + \frac{1}{2}\left(\frac{\Phi_{x_i}^2}{\Phi_t^2}\right)_t + \frac{1}{3}\{\Phi; t\}_t\right\} = 0, \tag{20}$$

where c_1 is an arbitrary constant. In order to check whether Eq.(20) is Painlevé integrable, we still take the transformation

$$\Phi = \exp F, \quad u_0 = F_t \ \text{ and}$$

$$u_i = F_{x_i}, \quad (i = 1, 2, \ldots, n). \tag{21}$$

Substituting expression (21) into Eq.(20), $(n+1)$-dimensional Boussinesq extension (20) becomes

$$\prod_{i=1}^{n} u_i^3 \sum_{i=1}^{n}\left(3u_0^3 u_{0,t} + \frac{u_0^3 u_{i,x_i x_i x_i}}{u_i} - u_0^3 u_i u_{i,x_i} - \frac{4u_0^3 u_{i,x_i} u_{i,x_i x_i}}{u_i^2} + \frac{3u_0^3 u_{i,x_i}^3}{u_i^3} - \frac{3u_0^5 u_{i,x_i}}{u_i^3} + 3c_1 u_0^2 u_{i,x_i} - 3c_1 u_i^2 u_{0,t}\right) + nc_1(u_0^2 u_{0,ttt} - u_0^4 u_{0,t} - 4u_0 u_{0,t} u_{0,tt} + 3u_{0,t}^3) = 0, \tag{22}$$

$$u_{i,t} - u_{0,x_i} = 0, \quad (i = 1, 2, \ldots, n), \tag{23}$$

where Eq.(23) comes from the compatibility condition transformation (21). Then, we use the standard WTC method to prove that Eqs.(22) and (23) possess Painlevé property. We expand u_i as

$$u_i = \sum_{j=0}^{\infty} u_{ij}\, \Phi^{j+\alpha_i}, \quad (i = 0, 1, 2, \ldots, n). \tag{24}$$

By using the leading analysis, we can obtain the following result:

$$\alpha_i = -1, \quad u_{i0}^2 = \Phi_{x_i}^2 \ \text{ and}$$

$$u_{00}^2 = \Phi_t^2, \quad (i = 1, 2, \ldots, n). \tag{25}$$

Substituting expressions (24) and (25) into Eqs.(22) and (23), we have

$$(j^2-1)(j-2)\prod_{i=1}^{n}(u_{0,t}^3 u_{i0,x_i}^3) \times \sum_{i=1}^{n}(u_{i0,x_i}^2 u_{ij} + nc_1 u_{0,t}^2 u_{0,j}) = f(u_{ik}, i = 0, 1, 2, \ldots, n, k \le j-1), \tag{26}$$

$$(j-1)(u_{0j}u_{i0} - u_{ij}u_{00}) = u_{i(j-1),t} - u_{0(j-1),x_i}, \quad i = 0, 1, 2, \ldots, n, \tag{27}$$

with $n+3$ resonances

$$j = -1, \underbrace{1, 1 \ldots 1}_{n+1}, 2, \tag{28}$$

where the function f is a very complicated function of u_{ik}, $i \le j-1$. The resonance at $j = -1$ corresponds to an arbitrary singularity manifold. For the other $n+1$ resonances, $j = 1$, there exist $n+1$ compatibility conditions

$$\prod_{i=0}^{n} u_{i0}^2 \prod_{i=1}^{n} u_{i0} \sum_{i=1}^{n}[(2u_{i0}u_{i0,x_i}u_{i0,x_i x_i} - u_{00}u_{i0}^2 u_{i0,x_i} - u_{00}u_{i0}u_{i0,x_i}) + (nc_1 u_{00,t}) \times (2u_{00}u_{00,tt} - u_{00,t}^2 - u_{00}^2)] = 0, \tag{29}$$

$$u_{i0,t} - u_{00,x_i} = 0, \quad i = 1, 2, \ldots, n. \tag{30}$$

Equations (29) and (30) are fully satisfied by expression (25). Similarly, collecting the coefficients of the next order, we can find that compatibility condition is fully satisfied. So Eqs.(22) and (23) are invariant under the Möbius transformation (5) and possess the Painlevé property. It demonstrates the new systems are integrable in the Lou's sense.[17] This confirms the integrable nature of Eqs.(22) and (23).

4. Single soliton and travelling wave solution for $(n+1)$-dimensional Boussinesq extension

In order to obtain the solution of the $(n+1)$-dimensional Boussinesq extension, we research the solutions of the (1+1)-dimensional equations (8) and (9)

and the $(n+1)$-dimensional equation in sequence. By using the Painlevé–Bäcklund transformation

$$u = \frac{\phi_x}{\phi} \quad \text{and} \quad v = \frac{\phi_t}{\phi}. \tag{31}$$

Equations (8) and (9) can be simplified into the following form:

$$\begin{aligned} &3\phi_x^2\phi_t^2\phi_{tt} + \phi_x^2\phi_t^2\phi_{xxxx} + 3\phi_t^3\phi_{zz}^3 - 3\phi_t^5\phi_{xx} \\ &- 4\phi_x\phi_t^3\phi_{xxx}\phi_{xx} + 3c_1\phi_x^3\phi_{tt}^3 - 3c_1\phi_x^5\phi_{tt} \\ &+ 3c_1\phi_x^3\phi_t^2\phi_{xx} + c_1\phi_x^3\phi_t^2\phi_{tttt} - 4c_1\phi_x^3\phi_t\phi_{ttt}\phi_{tt} = 0. \end{aligned} \tag{32}$$

It is straightforward to see Eq.(32) possesses a single soliton solution

$$\phi = 1 + \exp(kx + wt), \tag{33}$$

where k and w need not satisfy any relation like dispersion relation. Substituting expression (33) into expression (31), we can obtain a single soliton solution of Eqs.(8) and (9)

$$u = \frac{k\exp(kx + wt)}{1 + \exp(kx + wt)}$$

and

$$v = \frac{w\exp(kx + wt)}{1 + \exp(kx + wt)}. \tag{34}$$

In the same way, we can obtain a single soliton solution of $(n+1)$-dimensional equations (22) and (23) as

$$u_i = \frac{k_i\exp\left(\sum\limits_{i=1}^{n} k_i x_i + wt\right)}{1 + \exp(\sum\limits_{i=1}^{n} k_i x_i + wt)}$$

and

$$u_0 = \frac{w\exp(\sum\limits_{i=1}^{n} k_i x_i + wt)}{1 + \exp\left(\sum\limits_{i=1}^{n} k_i x_i + wt\right)}, \quad i = 1, 2, \ldots, n, \tag{35}$$

where k_i and w are arbitrary constants.

In addition, we can obtain the travelling wave solution by taking the function ϕ as

$$\phi(x, t) = \phi(\xi) \quad \text{and} \quad \xi = k(x - ct), \tag{36}$$

where k and c are arbitrary constants to be determined. Substituting expression (36) into Eq.(32) and using the software Maple, we can easily find that the equation is fully satisfied when $c_1 = \dfrac{1}{c^3}$. We obtain the solution for Eqs.(8) and (9) by substituting expression (36) into expression (31)

$$u(x, t) = \frac{k\phi'(\xi)}{\phi(\xi)} \quad \text{and} \quad v(x, t) = -\frac{kc\phi'(\xi)}{\phi(\xi)}. \tag{37}$$

Following the above way, we take the function ϕ as the following form for the arbitrary dimensional equation:

$$\phi = \phi(\xi) \quad \text{and} \quad \xi = k\left(\sum_{i=1}^{n} x_i - ct\right). \tag{38}$$

The solution with an arbitrary function for arbitrary dimensional equations (22) and (23) can be obtained as

$$u_i = \frac{k_i\phi'(\xi)}{\phi(\xi)} \quad \text{and} \quad u_0 = -\frac{ck\phi'(\xi)}{\phi(\xi)}, \quad i = 1, 2, \ldots, n, \tag{39}$$

where $c_1 = \dfrac{1}{c^3}$.

Therefore, we can obtain different forms of solutions via the different selections of the arbitrary function ϕ and give out the five exact solutions for the (1+1)-dimensional time–space symmetric forms (8) and (9) as an example in the following.

When the arbitrary function $\phi(\xi)$ is considered to be the Jacobi elliptic function similar to that in Ref.[18], then we can derive the Jacobi periodic wave solutions. The expansion has the following form by randomly choosing $cn(\xi, m)$:

$$\phi(\xi) = \sum_{i=0}^{n} a_i cn^i(\xi), \tag{40}$$

where n, a_i, k and c are arbitrary constants, m is the modulus of the Jacobi elliptic function. We can obtain the solution for the (1+1)-dimensional time–space symmetric forms (8) and (9) by substituting expression (40) into expression (37)

$$u(x, t) = \frac{k\sum\limits_{i=0}^{n} (a_i cn^i(\xi))'}{\sum\limits_{i=0}^{n} a_i cn^i(\xi)}. \tag{41}$$

Figure 1 shows the solution u of expression (41) for the parameters $n = 2$, $a_0 = -1$, $a_1 = 0.05$, $a_2 = 0.6$, $k = 1$, and $c = 1$.

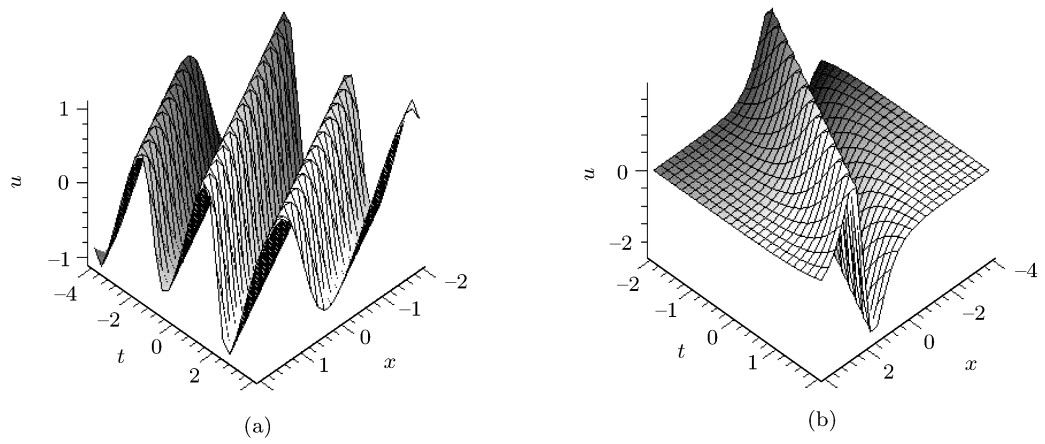


Fig.1. (a) Evolution of the Jacobi periodic elliptic solution u of expression (41) with $m = 0.1$, and (b) evolution of the soliton solution u of expression (41) with $m = 1$.

Furthermore, the solution of Eqs.(8) and (9) is obtained as

$$u = \frac{k(\cosh(\xi)\mathrm{sech}^2(\xi) + 2\tanh(\xi) - 2\tanh^3(\xi) - 2\sinh(\xi)\tanh(\xi)\mathrm{sech}^2(\xi) - 2\mathrm{cn}(\xi)\mathrm{dn}(\xi)\mathrm{sn}(\xi))}{1 + \tanh^2(\xi) + \mathrm{cn}^2(\xi) + \sinh(\xi)\mathrm{sech}(\xi)} \tag{42}$$

by selecting

$$\phi(\xi) = 1 + \tanh^2(\xi) + \mathrm{cn}^2(\xi) + \sinh(\xi)\mathrm{sech}^2(\xi). \tag{43}$$

The corresponding solution u of expression (42) is plotted with the parameters $k = 1$ and $c = 1$ in Fig.2.

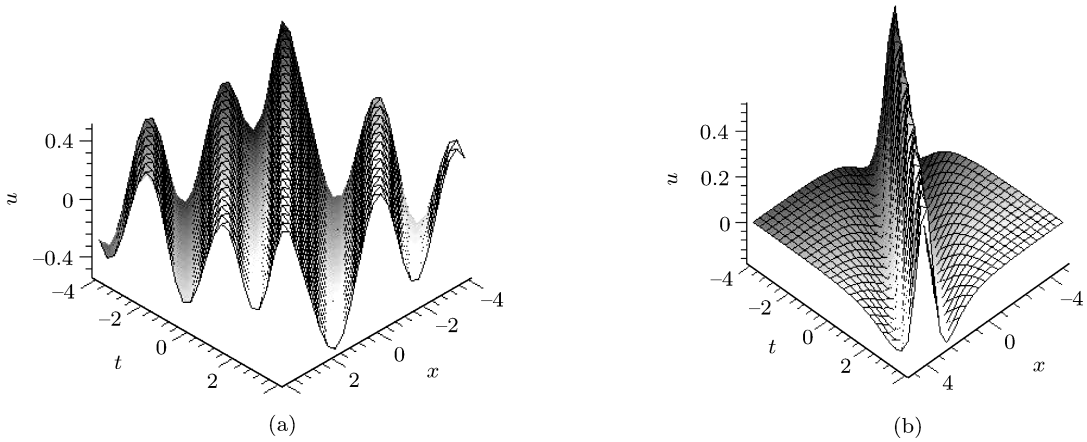


Fig.2. (a) Evolution of the quasi-periodic wave solution u of expression (42) with $m = 0.5$, and (b) evolution of the soliton solution u of expression (42) with $m = 1$.

Figure 3 shows the exact solution for Eqs.(8) and (9)

$$u = \frac{k(1 + \sinh(\xi)\mathrm{dn}(\xi) + \mathrm{cn}(\xi)\mathrm{dn}(\xi) - \tanh(\xi) - m^2\cosh(\xi)\mathrm{cn}(\xi)\mathrm{sn}(\xi))}{1 + \mathrm{sn}(\xi) + \tanh(\xi) + \cosh(\xi)\mathrm{dn}(\xi)}, \tag{44}$$

which is obtained by selecting

$$\phi = 1 + \mathrm{sn}(\xi) + \tanh(\xi) + \cosh(\xi)\mathrm{dn}(\xi), \tag{45}$$

and choosing the parameters $k = 1$ and $c = 1$. Solution (44) describes a kind of periodic-kink interaction

solitary wave which has been obtained in Refs.[19–21].

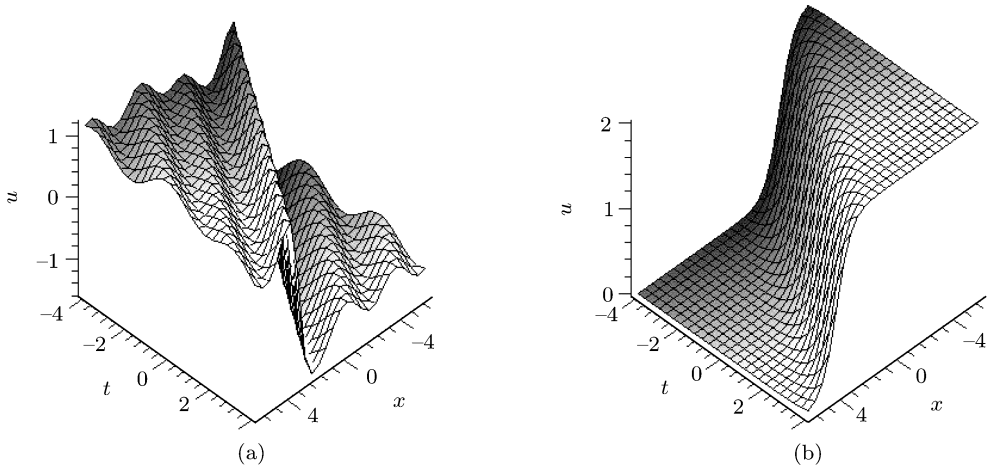


Fig.3. (a) Evolution of the typical periodic-kink wave interaction solution u of expression (44) with $m = 0.6$, and (b) evolution of the kink soliton solution u of expression (44) with $m = 1$.

If ϕ is fixed to be in the following form:

$$\phi(\xi) = 1 + \cosh(\xi) + \mathrm{cn}(\xi), \tag{46}$$

we can obtain

$$u(x,t) = \frac{k(\sinh(\xi) - \mathrm{dn}(\xi)\mathrm{sn}(\xi))}{1 + \cosh(\xi) + \mathrm{cn}(\xi)}. \tag{47}$$

Solution (47) is plotted with the parameters $k = 1$ and $c = 1$ in Fig.4.

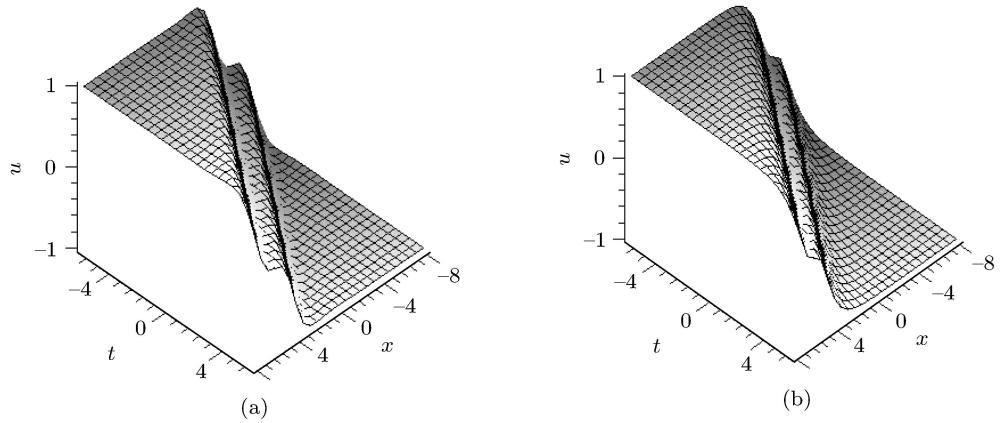


Fig.4. Evolution of the similar kink solution u of expression (47) with $m = 0.1$ (a) and 1 (b).

Figure 5 is the plot of the solution for Eqs.(8) and (9)

$$u(x,t) = \frac{-k\mathrm{sech}(\xi)(1 + \tanh(\xi) + 3\mathrm{sech}^2(\xi)\tanh(\xi) + \mathrm{sech}^2(\xi))}{(2\exp(\xi) + \mathrm{sech}(\xi) + \mathrm{sech}^3(\xi))} \tag{48}$$

with the selection of

$$\phi = 1 + \frac{\mathrm{sech}(\xi) + \mathrm{sech}^3(\xi)}{2\exp(\xi)} \tag{49}$$

and the parameters $k = 1$ and $c = 1$.

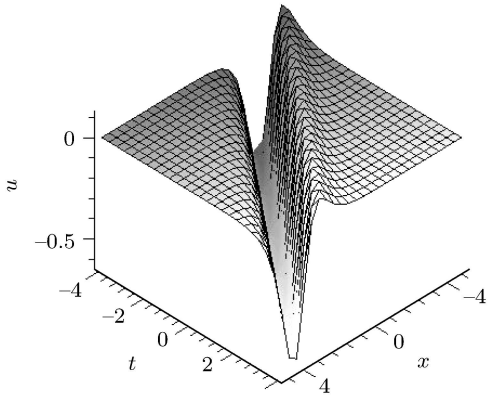


Fig.5. Evolution of the soliton solution u of expression (48).

5. Conclusion

In this paper, according to the idea that the conformal-invariant form may be one of the best convenient ways to obtain high-dimensional integrable model, we acquire (1+1)-dimensional and $(n + 1)$-dimensional space–time symmetric forms. Then, by using the standard WTC method, we prove that the new systems possess the Painlevé property and invariant under the Möbius transformation. Therefore, the new systems are integrable in the Lou sense. By the Painlevé–Bäcklund transform, the single soliton solutions and the periodic travelling wave solutions are obtained for the $(n+1)$-dimensional space–time symmetric form. More properties of the $(n + 1)$-dimensional space–time symmetric form such as multi-soliton solutions, infinitely many conservation laws and symmetries are worth further studying.

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