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Four-mode continuous-variables entangled state: generation from beam splitter, a parametric down-conversion and a polarizer*

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We introduce a new kind of four-mode continuous variable entangled state in Fock space. The completeness relation and the partly nonorthonormal property of such a state are proven. The scheme to generate this state is presented by combining a symmetrical beamsplitter, a parametric down-conversion and a polarizer. After making a single-mode quadrature amplitude measurement, the remaining three modes are kept in entanglement. And its applications are also discussed.

Keywords: four-mode entangled state, integration within ordered product (IWOP) technique, quadrature amplitude measurement

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1. Introduction

The conception of quantum entanglement is increasingly of interest and importance in the study of quantum information and quantum communication. It was originated by Einstein, Podolsky and Rosen (EPR) in a paper arguing the incompleteness of quantum mechanics[1] and has played a key role in understanding some fundamental problems in quantum mechanics and quantum optics.[2−7]

Multipartite entanglement in a pure state of many systems is a fundamental property and a crucial resource for quantum information science. Its complete theoretical understanding can be deepen by virtue of continuous-variables (CV) representation theory. On the other hand, it is still challenging to construct an entangled state representation. On the basis of EPR's idea, a two-mode entangled state and a parametrized entangled state in Fock space were constructed.[8,9] Late a kind of three-mode CV entangled state was proposed[8,10−12] and prepared by using an asymmetric beam splitter (BS)[13] and a parametric down-conversion (PDC) instrument,[14−17] and then Hu and Fan[18] introduced a four-mode CV entangled state generated by BS and PDCs. In addition, Fan used a BS and a polarizer to make a three-mode CV entangled state[19] and introduced a four-mode CV entangled state[20] generated by BS and polarizers. It has been proved that these entangled states make up new quantum mechanical representations. All these studies above exhibit the role of BS in generating entangled states and indicate also that these physical instruments, such as PDC, BS and polarizer, can be used to implement entangled states.

An interesting question thus naturally arises: is there any kind of entangled state which can be generated by combining these three instruments? As far as our knowledge goes, there has been found no such a study in the literature up to now. In the present paper, we present a new four-mode CV entangled state in Fock space, denoted by $|\gamma, s, \tau\rangle_{\theta,\lambda}$, which can compose a new quantum mechanical representation. By using a symmetrical BS, a PDC and a polarizer, this entangled state can be generated experimentally. The remainder of the present paper is arranged as follows. In Section 2 we introduce the new entangled state $|\gamma, s, \tau\rangle_{\theta,\lambda}$. In Section 3, the main properties of this state are analysed, such as the completeness property and partly non-orthogonal property. In Section 4, a feasible scheme is proposed to implement this state

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experimentally. Section 5 is devoted to quadrature-amplitude measurement for $|\gamma,s,\tau\rangle_{\theta,\lambda}$, which leads it to collapse into a three- or two-mode entangled state. Finally, the applications of $|\gamma,s,\tau\rangle_{\theta,\lambda}$ are presented in Section 6.

2. New four-mode entangled state

First, we present an explicit expression of $|\gamma,s,\tau\rangle_{\theta,\lambda}$ in four-mode Fock space as

$$\begin{aligned}|\gamma,s,\tau\rangle_{\theta,\lambda} =&\mathrm{sech}\lambda\exp\left[-\frac{1}{2}\left(|\gamma|^2+|s|^2+|\tau|^2\right)+\left(a_1^+-\gamma^*\right)\left(a_3^+-\tau^*\right)\cos\theta\mathrm{sech}\lambda\right.\\&+\left(a_1^+-\gamma^*\right)a_4^+\sin\theta\mathrm{sech}\lambda+\left(a_1^+-\gamma^*\right)\left(a_2^+-s^*\right)\tanh\lambda\\&\left.+\gamma a_1^++sa_2^++\tau a_3^+\right]|0000\rangle\,,\end{aligned}\tag{1}$$

where $|0000\rangle$represents the vacuum state, annihilated by $a_i\,(i=1,2,3,4)$, θ and λ are two real numbers (λ is the squeezing parameter characterizing a PDC, and a polarizer transmits the component of the electric field in the direction that makes an angle θ with respect to the horizontal direction). Particularly, when $\theta=\dfrac{\pi}{2}$, $\lambda=0$, thus expression (1) reduces to

$$\begin{aligned}|\gamma,s,\tau\rangle_{\frac{\pi}{2},0} =&\exp\left[-\frac{1}{2}\left(|\gamma|^2+|s|^2+|\tau|^2\right)+a_1^+a_4^+-a_4^+\gamma^*+\gamma a_1^++sa_2^++\tau a_3^+\right]|0000\rangle\\=&|\eta=\gamma\rangle_{14}\otimes|s\rangle_2\otimes|\tau\rangle_3\,,\end{aligned}\tag{2}$$

where $|\eta=\gamma\rangle_{14}$ is the bipartite entangled state in $a_1^+-a_4^+$ mode,[10−12] while $|s\rangle_2$ and $|\tau\rangle_3$ are the coherent states in a_2^+ and a_3^+ modes, respectively. Expression (1) will become a tripartite entangled state with $\theta=\dfrac{\pi}{2}$ and $\lambda=0$. Operating a_i on $|\gamma,s,\tau\rangle_{\theta,\lambda}$ yields four independent eigenvector equations,

$$\begin{aligned}&\left(a_1-a_2^+\tanh\lambda-a_3^+\cos\theta\mathrm{sech}\lambda-a_4^+\sin\theta\mathrm{sech}\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}\\=&\left(\gamma-\tau^*\cos\theta\mathrm{sech}\lambda-s^*\tanh\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}\,,\end{aligned}\tag{3}$$

$$\left(a_2-a_1^+\tanh\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}=\left(s-\gamma^*\tanh\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}\,,\tag{4}$$

$$\left(a_4-a_1^+\sin\theta\mathrm{sech}\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}=\left(-\gamma^*\sin\theta\mathrm{sech}\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}\,,\tag{5}$$

$$\left(a_3-a_1^+\cos\theta\mathrm{sech}\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}=\left(\tau-\gamma^*\cos\theta\mathrm{sech}\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}\,.\tag{6}$$

Combining Eqs.(4)–(6) yields

$$\begin{aligned}&\left(a_2\tanh\lambda+a_3\cos\theta\mathrm{sech}\lambda+a_4\sin\theta\mathrm{sech}\lambda-a_1^+\right)|\gamma,s,\tau\rangle_{\theta,\lambda}\\=&\left(\tau\cos\theta\mathrm{sech}\lambda+s\tanh\lambda-\gamma^*\right)|\gamma,s,\tau\rangle_{\theta,\lambda}\,.\end{aligned}\tag{7}$$

By introducing $X_i=\dfrac{a_i+a_i^+}{\sqrt{2}}$ and $P_i=\dfrac{a_i-a_i^+}{\sqrt{2}\mathrm{i}}$ and combining Eqs.(3)–(6) we have

$$\begin{aligned}&\left(X_1-X_2\tanh\lambda-X_3\cos\theta\mathrm{sech}\lambda-X_4\sin\theta\mathrm{sech}\lambda\right)|\gamma,s,\tau\rangle_{\theta,\lambda}\\=&\sqrt{2}(\gamma_1-\tau_1\cos\theta\mathrm{sech}\lambda-s_1\tanh\lambda)\,|\gamma,s,\tau\rangle_{\theta,\lambda}\,.\end{aligned}\tag{8}$$

Equations (4)–(8) indicate that $|\gamma,s,\tau\rangle_{\theta,\lambda}$ is a common eigenvector of the above four operators $(a_2-a_1^+\tanh\lambda)$, $(a_3-a_1^+\cos\theta\mathrm{sech}\lambda)$, $(a_4-a_1^+\sin\theta\mathrm{sech}\lambda)$ and $(X_1-X_2\tanh\lambda-X_3\cos\theta\mathrm{sech}\lambda-X_4\sin\theta\mathrm{sech}\lambda)$, which constitute a commutable operator set.

3. Main properties of $|\gamma, s, \tau\rangle_{\theta,\lambda}$

We now investigate the major properties of $|\gamma, s, \tau\rangle_{\theta,\lambda}$. Using the normally ordered form of the vacuum projection operator

$$|0000\rangle\langle 0000| =: \exp\left[-\sum_{i=1}^{4} a_i^+ a_i\right] :\equiv: \mathrm{e}^H :, \qquad (9)$$

(where $H = -a_1^+ a_1 - a_2^+ a_2 - a_3^+ a_3 - a_4^+ a_4$ and the symbol : : denotes normal ordering) and the following integral formula:

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp\left[\xi |z|^2 + \varsigma z + \eta z^*\right] = -\frac{1}{\xi}\exp\left[-\frac{\varsigma\eta}{\xi}\right], \quad \mathrm{Re}\xi < 0, \qquad (10)$$

we can integrate by using the IWOP technique[21−23] to prove the completeness relation of $|\gamma, s, \tau\rangle_{\theta,\lambda}$,

$$\sin^2\theta \int \frac{\mathrm{d}^2\gamma \mathrm{d}^2 s \mathrm{d}^2\tau}{\pi^3} |\gamma, s, \tau\rangle_{\theta,\lambda\theta,\lambda}\ \langle\gamma, s, \tau| =: \exp\left[EE^+ + \frac{F^+F}{1-\cos^2\theta \mathrm{sech}^2\lambda} + GG^+ + A + H\right] : = : \exp(0) := 1, \qquad (11)$$

where

$$\begin{aligned} A &= (a_1^+ a_2^+ + a_1 a_2)\tanh\lambda + (a_1^+ a_3^+ + a_1 a_3)\cos\theta\mathrm{sech}\lambda + (a_1^+ a_4^+ + a_1 a_4)\sin\theta\mathrm{sech}\lambda, \\ E &= a_3^+ - a_1\cos\theta\mathrm{sech}\lambda, \\ F &= a_1^+(1-\cos^2\theta\mathrm{sech}^2\lambda) - a_4\sin\theta\mathrm{sech}\lambda - a_2\tanh\lambda, \\ G &= \sqrt{1-\cos^2\theta\mathrm{sech}^2\lambda}\left[a_2^+ - a_4^+\tanh\lambda/(\sin\theta\mathrm{sech}\lambda)\right]. \end{aligned} \qquad (12)$$

In Eq.(11), the factor $\sin^2\theta$ is integration measurement needed for the completeness relation. It is worth pointing out here that the integration is involved with three-mode, not with four-mode. This is because the state $|\gamma, s, \tau\rangle_{\theta,\lambda}$ is entangled among four modes, which reduces the folds of integration. Thus we notice that although $|\gamma, s, \tau\rangle_{\theta,\lambda}$ is defined in four-mode Fock space, due to its four modes are mutually entangled, it spans a completeness relation with three-fold complex integral measurement. To derive the exact expression of ${}_{\theta',\lambda'}\langle\gamma, s, \tau|\gamma, s, \tau\rangle_{\theta,\lambda}$, we use the overcompleteness relation of four-mode coherent state[24−26]

$$\int \prod_{i=1}^{4} \frac{\mathrm{d}^2 z_i}{\pi} |z_1, z_2, z_3, z_4\rangle\langle z_1, z_2, z_3, z_4| = 1, \qquad (13)$$

where

$$|z_1, z_2, z_3, z_4\rangle = \exp\left[\sum_{i=1}^{4}(-\frac{1}{2}|z_i|^2 + z_i a_i^+)\right]|0000\rangle, \qquad (14)$$

and calculate

$$\begin{aligned} \langle z_1, z_2, z_3, z_4|\gamma, s, \tau\rangle_{\theta,\lambda} =& \mathrm{sech}\lambda \exp\Bigg[-\frac{1}{2}\left(\sum_{i=1}^{4}|z_i|^2 + |\gamma|^2 + |s|^2 + |\tau|^2\right) \\ &+ (z_1^* - \gamma^*)(z_2^* - s^*)\tanh\lambda + (z_1^* - \gamma^*)(z_3^* - \tau^*)\cos\theta\mathrm{sech}\lambda \\ &+ (z_1^* - \gamma^*) z_4^* \sin\theta\mathrm{sech}\lambda + \gamma z_1^* + s z_2^* + \tau z_3^*\Bigg]. \end{aligned} \qquad (15)$$

Using expressions (13) and (15) we have

$$\begin{aligned} {}_{\theta',\lambda'}\langle\gamma', s', \tau'|\gamma, s, \tau\rangle_{\theta,\lambda} &= \int \sum_{i=1}^{4}\left(\frac{\mathrm{d}^2 z_i}{\pi}\right) {}_{\theta',\lambda'}\langle\gamma', s', \tau'|z_1, z_2, z_3, z_4\rangle\langle z_1, z_2, z_3, z_4|\gamma, s, \tau\rangle_{\theta,\lambda} \\ &= \frac{\mathrm{sech}\lambda\mathrm{sech}\lambda'}{f}\exp\left[K + B + \frac{1}{f}B_1\right], \end{aligned} \qquad (16)$$

where

$$K=\exp\left[-\frac{1}{2}\left[\gamma\right]^2+\left[s\right]^2+\left[\tau\right]^2+\left[\gamma'\right]^2+\left[s'\right]^2+\left[\tau'\right]^2\right],$$

$$f=1-\sin\theta\sin\theta'\mathrm{sech}\lambda\mathrm{sech}\lambda'-\cos\theta\cos\theta'\mathrm{sech}\lambda\mathrm{sech}\lambda'-\tanh\lambda\tanh\lambda',$$

$$\begin{aligned}B=&(\gamma^*s^*\tanh\lambda+\gamma's'\tanh\lambda')+(\gamma^*\tau^*\cos\theta\mathrm{sech}\lambda+\gamma'\tau'\cos\theta'\mathrm{sech}\lambda')\\&+\gamma'\gamma^*\sin\theta\sin\theta'\mathrm{sech}\lambda\mathrm{sech}\lambda'+(\tau'^*-\gamma'\cos\theta'\mathrm{sech}\lambda')\\&\times(\tau-\gamma^*\cos\theta\mathrm{sech}\lambda)+(s'^*-\gamma'\tanh\lambda')(s-\gamma^*\tanh\lambda),\\B_1=&[(\gamma'^*-\gamma^*f)+(\tau-\tau')\cos\theta'\mathrm{sech}\lambda'+(s-s')\tanh\lambda']\\&\times[(\gamma-\gamma'f)+(\tau'^*-\tau^*)\cos\theta\mathrm{sech}\lambda+(s'^*-s^*)\tanh\lambda].\end{aligned}\tag{17}$$

Expression (16) shows that $|\gamma,s,\tau\rangle_{\theta,\lambda}$ state vectors are not orthogonal to each other. For simplicity, we further assume $\lambda=\lambda'$ and $\theta=\theta'$ then $f\to1$, the orthogonal relation in expression (16) becomes

$$\begin{aligned}{}_{\theta,\lambda}\langle\gamma',s',\tau'|\,\gamma,s,\tau\rangle_{\theta,\lambda}=&\mathrm{sech}^2\lambda\exp\left[K+C\right]\\&\times\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\exp\left[-\frac{1}{\varepsilon}\left|(\gamma'^*-\gamma^*)+(\tau-\tau')\cos\theta\mathrm{sech}\lambda+(s-s')\tanh\lambda\right|^2\right],\end{aligned}\tag{18}$$

where

$$\begin{aligned}C=&(\gamma^*s^*+\gamma's')\tanh\lambda+(\gamma^*\tau^*+\gamma'\tau')\cos\theta\mathrm{sech}\lambda+\gamma'\gamma^*\sin^2\theta\mathrm{sech}^2\lambda\\&+(\tau'^*-\gamma'\cos\theta\mathrm{sech}\lambda)(\tau-\gamma^*\cos\theta\mathrm{sech}\lambda)+(s'^*-\gamma'\tanh\lambda)(s-\gamma^*\tanh\lambda).\end{aligned}\tag{19}$$

Using the following limiting formula of δ-function:

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\exp\left[-\frac{1}{\varepsilon}\left|\alpha\right|^2\right]=\pi\delta(\alpha)\,\delta(\alpha^*),\tag{20}$$

we finally obtain

$$\begin{aligned}{}_{\theta,\lambda}\langle\gamma',s',\tau'|\,\gamma,s,\tau\rangle_{\theta,\lambda}=&\pi\mathrm{sech}^2\lambda\exp\left[K+C\right]\delta\left(\gamma^*-\gamma'^*+(\tau'-\tau)\cos\theta\mathrm{sech}\lambda+(s'-s)\tanh\lambda\right)\\&\times\delta\left(\gamma-\gamma'+\left(\tau'^*-\tau^*\right)\cos\theta\mathrm{sech}\lambda+\left(s'^*-s^*\right)\tanh\lambda\right).\end{aligned}\tag{21}$$

From expression (21) we see that $|\gamma,s,\tau\rangle_{\theta,\lambda}$ is partly orthonormal.

4. Implementation of $|\gamma,s,\tau\rangle_{\theta,\lambda}$

We now discuss an experimental setup to produce the ideal four-mode entangled state $|\gamma,s,\tau\rangle_{\theta,\lambda}$. Recall that the simultaneous eigenstate of relative coordinate X_1-X_2 and total momentum P_1-P_2 of two particles in two-mode Fock space was constructed as[8,9]

$$\begin{aligned}|\eta\rangle=&\exp\left[-\frac{1}{2}\left|\eta\right|^2+\eta a_1^+-\eta^*b_2^+ +a_1^+b_2^+\right]|00\rangle_{12},\\\eta=&\frac{\eta_1+\mathrm{i}\eta_2}{\sqrt{2}}.\end{aligned}\tag{22}$$

By introducing the displacement operator $D_1(\eta)=\exp\left[\eta a_1^+-\eta^*a_1\right]$, $|\eta\rangle$ can be expressed as

$$|\eta\rangle=D_1(\eta)\exp\left[a_1^+b_2^+\right]|00\rangle_{12}.\tag{23}$$

In particular, $|\eta=0\rangle=\exp\left[a_1^+b_2^+\right]|00\rangle_{12}$, which can be express as

$$\begin{aligned}&\exp\left[a_1^+b_2^+\right]|00\rangle_{12}\\=&B_{12}\left(\frac{\pi}{4}\right)|p=0\rangle_1\otimes|x=0\rangle_2,\end{aligned}\tag{24}$$

where $|p=0\rangle_1\approx\exp\left[\frac{1}{2}a_1^{+2}\right]|0\rangle_1$ is the zero-momentum eigenstate and $|x=0\rangle_2\approx\exp\left[\frac{1}{2}b_2^{+2}\right]|0\rangle_2$ is the zero-position eigenstate. Expression (24) shows that the state $|\eta=0\rangle$ can be produced when a pair of incident squeeze rays falls on a symmetric 50:50 BS without loss and phase shift, and the operation of BS is represented by the operator $B_{12}\left(\frac{\pi}{4}\right)=\exp\left[\frac{\pi}{4}\left(a_1^+b_2-a_1b_2^+\right)\right]$.[20−22]

In a similar way to deriving expression (24), based on the state $\exp\left[a_1^+ b_3^+\right]|00\rangle_{13}$ generated by a beam-splitter, we further make a squeezing transformation for a_1^+ through a PDC,[13]

$$S_{12} a_1^+ S_{12}^{-1} = a_1^+ \cosh\lambda - a_2 \sinh\lambda, \tag{25}$$

where $S_{12} = \exp[\lambda(a_1^+ a_2^+ - a_1 a_2)]$, which is the two-mode squeezing operator, and simultaneously make a mathematical transformation for b_3^+ through a polarizer, i.e.

$$b_3^+ \to a_3^+ \cos\theta + a_4^+ \sin\theta. \tag{26}$$

Physically, the squeezing transformation corresponds to the following nonlinear optical process: the a_1^+-mode (signal mode) and a_2^+-mode (idler mode) become coupled in a PDC where squeezing happens in a frequency domain; The mathematical transformation corresponds to the following situation: if the b_3-mode light falls on a polarizer and resolves into two modes $(a_3,\ a_4)$ corresponding to two orthogonal polarizers and satisfying the same commutation relation as b_3 on the input side (a similar situation may also occur when the b_3-mode light falls on a beam splitter and decomposes into a_3 and a_4). The polarizer transmits the component of the electric field in the direction that makes an angle θ with respect to the horizontal direction.

Combing Eqs.(25) and (26) we have

$$\begin{aligned} S_{12}\exp\left[a_1^+ b_3^+\right]|00\rangle \to & S_{12}\exp\left[a_1^+\left(a_3^+\cos\theta + a_4^+\sin\theta\right)\right]|0000\rangle \\ = & \,\mathrm{sech}\lambda \exp\left[a_1^+ a_3^+ \cosh\lambda\cos\theta + a_1^+ a_4^+ \cosh\lambda\sin\theta\right] \\ & \times \exp\left[-a_2\left(a_3^+\cos\theta + a_4^+\sin\theta\right)\sinh\lambda\right] \times \exp\left[a_1^+ a_2^+ \tanh\lambda\right]|0000\rangle . \end{aligned} \tag{27}$$

Employing the Baker–Hausdorff formula $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^B\mathrm{e}^A\mathrm{e}^{[A,B]}$, which holds for $[A,[A,B]] = [B,[A,B]] = 0$, we have

$$\begin{aligned} & \exp\left[-a_2\left(a_3^+\cos\theta + a_4^+\sin\theta\right)\sin\lambda\right]\exp\left[a_1^+ a_2^+\tanh\lambda\right] \\ = & \exp\left[a_1^+ a_2^+\tanh\lambda\right]\exp\left[-\left(a_3^+\cos\theta + a_4^+\sin\theta\right)a_1^+\sinh\lambda\tanh\lambda\right] \\ & \times \exp\left[-a_2\left(a_3^+\cos\theta + a_4^+\sin\theta\right)\sinh\lambda\right], \end{aligned} \tag{28}$$

and substituting Eq.(28) into Eq.(27) we have

$$\begin{aligned} & S_{12}\exp\left[a_1^+\left(a_3^+\cos\theta + a_4^+\sin\theta\right)\right]|0000\rangle \\ = & \,\mathrm{sech}\lambda\exp\left[a_1^+ a_3^+ \mathrm{sech}\lambda\cos\theta + a_1^+ a_4^+ \mathrm{sech}\lambda\sin\theta + a_1^+ a_2^+\tanh\lambda\right]|0000\rangle \\ = & \,|\,\rangle_{\theta,\lambda} . \end{aligned} \tag{29}$$

Thus we know that an ideal BS, a PDC and a polarizer are the fundamental devices for preparing the ideal state $|\gamma = 0, s = 0, \tau = 0\rangle_{\theta,\lambda}$. Then making a three-mode displacement $D_1(\gamma) D_2(s) D_3(\tau)$ for $|\,\rangle_{\theta,\lambda}$, where $D_1(\gamma) = \exp\left[\gamma a_1^+ - \gamma^* a_1\right]$, we obtain the new entangled state

$$D_1(\gamma) D_2(s) D_3(\tau)\,|\,\rangle_{\theta,\lambda} = |\gamma, s, \tau\rangle_{\theta,\lambda} . \tag{30}$$

Here we should point out that one can achieve the displacement $D_1(\gamma)$ experimentally by reflecting the light field from an almost perfect reflecting mirror and adding through the mirror a field (a strong coherent local oscillator) that has been phase and amplitude-modulated according to the value γ, and a similar analysis holds true for $D_2(s)$ and $D_3(\tau)$.

5. Quadrature amplitude measurement on $|\gamma, s, \tau\rangle_{\theta,\lambda}$

In order to see clearly the entanglement property of this state $|\gamma, s, \tau\rangle_{\theta,\lambda}$, we make the following Fourier transform:

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathrm{d}\gamma_2 \mathrm{d}s_2 \mathrm{d}\tau_2 \mathrm{e}^{\mathrm{i}(u\gamma_2 + v s_2 + \sigma\tau_2)}\,|\gamma, s, \tau\rangle_{\theta,\lambda} \\ = & \, W(u, v, \sigma)\,|x_1\rangle_1 \otimes |x_2\rangle_2 \otimes |x_3\rangle_3 \otimes |x_4\rangle_4 , \end{aligned} \tag{31}$$

where $|x_i\rangle_i$ all belong to the set of coordinate eigen-

vectors

$$|x_i\rangle_i = \pi^{-\frac{1}{4}}\exp\left[-\frac{x_i^2}{2}+\sqrt{2}x_i a_i^+ - \frac{1}{2}a_i^{+2}\right]|0\rangle_i, \quad (32)$$

$W(u,v,\sigma)$ is a less important factor, so we need not to write it down explicitly,

$$x_1 = \frac{(\gamma_1-u)}{\sqrt{2}}, \quad x_2 = \frac{(s_1-v)}{\sqrt{2}}, \quad x_3 = \frac{(\tau_1-\sigma)}{\sqrt{2}},$$

and

$$x_4 = \frac{1}{\sqrt{2}\sin\theta}[(\tau_1+\sigma)\cos\theta + (s_1+v) \times \sinh\lambda - (\gamma_1+u)\cosh\lambda]. \quad (33)$$

Then making the inverse Fourier transform of Eq.(31) yields

$$|\gamma,s,\tau\rangle_{\theta,\lambda} = \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma \mathrm{e}^{-\mathrm{i}(u\gamma_2+vs_2+\sigma\tau_2)}W(u,v,\sigma)|x_1\rangle_1\otimes|x_2\rangle_2\otimes|x_3\rangle_3\otimes|x_4\rangle_4, \quad (34)$$

which confirms that $|\gamma,s,\tau\rangle_{\theta,\lambda}$ itself is an entangled state according to the standard theory of Schmidt decomposition.[27]

Next, we consider the quadrature amplitude measurement for $|\gamma,s,\tau\rangle_{\theta,\lambda}$. When making a single-mode quadrature amplitude measurement, say, $|x\rangle_{11}\langle x|$, on $|\gamma,s,\tau\rangle_{\theta,\lambda}$, where $|x\rangle$ is the coordinate of X operator, using expression (34) we obtain

$${}_1\langle x|\gamma,s,\tau\rangle_{\theta,\lambda} = \mathrm{e}^{-\mathrm{i}u\gamma_2}\int_{-\infty}^{\infty}\mathrm{d}v\mathrm{d}\sigma\mathrm{e}^{-\mathrm{i}(u\gamma_2+vs_2+\sigma\tau_2)}W(v,\sigma)\otimes|x_2\rangle_2\otimes|x_3\rangle_3\otimes|x_4\rangle_4\Big|_{u=\gamma_1-\sqrt{2}x}, \quad (35)$$

which shows that the remaining three modes are kept in entanglement.

If a joint Bell measurement, say $|\eta'\rangle_{2,32,3}\langle\eta'|$, is performed on $|\gamma,s,\tau\rangle_{\theta,\lambda}$, the result of measurement is

$$\begin{aligned}{}_{2,3}\langle\eta|\gamma,s,\tau\rangle_{\theta,\lambda} =& \mathrm{sech}\lambda\mathrm{e}^M\exp\left\{a_1^{+2}\mathrm{sech}\lambda\cos\theta\tanh\lambda + a_1^+[(s-\tau^*-2\gamma^*\tanh\lambda)\,\mathrm{sech}\lambda\cos\theta\right.\\ &\left.+(\eta^*-s^*+\tau)\tanh\lambda+\gamma] + \left(a_1^+a_4^+ - a_4^+\gamma^*\right)\sin\theta\mathrm{sech}\lambda\right\}, \end{aligned} \quad (36)$$

where e^M is an unimportant coefficient. Expression (36) indicates that after the Bell joint measurement, the remaining two-mode is still in entanglement. Thus we conclude that $|\gamma,s,\tau\rangle_{\theta,\lambda}$ possesses the properties of two- and three-mode entangled state.

6. Applications of $|\gamma,s,\tau\rangle_{\theta,\lambda}$

We now present an application of the new state $|\gamma,s,\tau\rangle_{\theta,\lambda}$ in quantum squeezing theory. For simplicity, let $v=\mathrm{e}^{\mu}$, then we will have $\mathrm{sech}\mu = \dfrac{2v}{1+v^2}$, and $\tanh\mu = \dfrac{v^2-1}{v^2+1}$. Using Eq.(10) we construct the following ket-bra integration:

$$\begin{aligned}S =& \sin^2\theta\int\frac{\mathrm{d}^2\gamma\mathrm{d}^2s\mathrm{d}^2\tau}{\pi^3\nu}\left|\frac{\gamma}{\nu},s,\tau\right\rangle_{\theta,\lambda}{}_{\theta,\lambda}\left\langle\frac{\gamma}{\nu},s,\tau\right| = \frac{I}{D}\exp\left[\frac{I}{D}a_1^+a_4\right]\\ &\times\exp\left[\ln\left(\frac{\mathrm{sech}\mu-\sin^2\theta\mathrm{sech}^2\lambda-1}{D}\right)a_1^+a_1+\ln\left(\frac{I}{D}\right)a_4^+a_4\right]\exp\left[-\frac{J}{D}a_1a_4\right], \end{aligned} \quad (37)$$

where S is a two-mode squeezing operator with three feasible parameters,

$$\begin{aligned}&D = 1-\mathrm{sech}\mu(\cos^2\theta\mathrm{sech}^2\lambda+\tanh^2\lambda),\\ &I = \mathrm{sech}\mu\sin^2\theta\mathrm{sech}^2\lambda, \quad \text{and} \quad J = \tanh\mu\sin\theta\mathrm{sech}\lambda.\end{aligned}$$

Operating expression (37) can yield a two-mode squeezed state

$$S|00\rangle_{14} = \frac{\mathrm{sech}\mu\sin^2\theta\mathrm{sech}^2\lambda}{D}\exp\left[\frac{\tanh\mu\sin\theta\mathrm{sech}\lambda}{D}a_1^+a_4^+\right]|00\rangle_{14}. \quad (38)$$

Second, we consider an application of $|\gamma, s, \tau\rangle_{\theta,\lambda}$ in the optical phase operator theory.[28,29] We extend the two-mode optical phase operator $\sqrt{\frac{a^+ - b}{a - b^+}}$, which was proposed by Noh *et al* [28] on the basis of eight-port homodyne scheme, to the four-mode case

$$\sqrt{\frac{a^+ - b}{a - b^+}} \to \mathrm{e}^{\mathrm{i}\hat{\Phi}} \equiv \sqrt{\frac{a_1^+ - a_2 \tanh\lambda - a_3 \cos\theta \mathrm{sech}\lambda - a_4 \sin\theta \mathrm{sech}\lambda}{a_1 - a_2^+ \tanh\lambda - a_3^+ \cos\theta \mathrm{sech}\lambda - a_4^+ \sin\theta \mathrm{sech}\lambda}}. \tag{39}$$

Thus the completeness relation (10) of the state $|\gamma, s, \tau\rangle_{\theta,\lambda}$ and Eqs.(3) and (7) provide a suitable representation for the phase behaviour of $\mathrm{e}^{\mathrm{i}\hat{\Phi}}$,

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\hat{\Phi}} &= \sin^2\theta \int \frac{\mathrm{d}^2\gamma \mathrm{d}^2 s \mathrm{d}^2\tau}{\pi^3} \sqrt{\frac{\gamma^* - \tau\cos\theta\mathrm{sech}\lambda - s\tanh\lambda}{\gamma - \tau^*\cos\theta\mathrm{sech}\lambda - s^*\tanh\lambda}} \, |\gamma, s, \tau\rangle_{\theta,\lambda\theta,\lambda} \, \langle\gamma, s, \tau| \\ &= \sin^2\theta \int \frac{\mathrm{d}^2\gamma \mathrm{d}^2 s \mathrm{d}^2\tau}{\pi^3} \exp\left[\mathrm{i}\arg\left(\gamma^* - \tau\cos\theta\mathrm{sech}\lambda - s\tanh\lambda\right)\right] |\gamma, s, \tau\rangle_{\theta,\lambda\theta,\lambda} \, \langle\gamma, s, \tau| \,, \end{aligned} \tag{40}$$

which exhibits its classical phase image — $\exp\left[\mathrm{i}\arg\left(\gamma^* - \tau\cos\theta\mathrm{sech}\lambda - s\tanh\lambda\right)\right]$.

7. Conclusion

We introduce the four-mode entangled state $|\gamma, s, \tau\rangle_{\theta,\lambda}$ of CV both theoretically and experimentally. On the basis of the IWOP technique, we prove the completeness relation of $|\gamma, s, \tau\rangle_{\theta,\lambda}$ and show that $|\gamma, s, \tau\rangle_{\theta,\lambda}$ is partly orthogonal. We present its Schmidt decomposition and a quadrature amplitude measurement on it. A feasible experimental setup composed of a symmetrical BS, a PDC and a polarizer is proposed to generate such a four-mode entangled state.

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