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New approach for deriving density operator for describing continuum photodetection process*

Fan Hong-Yi(范洪义)[a)b)] and Hu Li-Yun(胡利云)[a)†]

[a)] *Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China*

[b)] *Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, China*

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By introducing the two-mode entangled state representation $\langle\eta|$ whose one mode is a fictitious one accompanying the system mode, this paper presents a new approach for deriving density operator for describing continuum photodetection process.

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In continuum photodetection process photons of a light field are absorbed by detectors, so this is a nonunitary time evolution of the field, we may tackle this process from the point of view of field amplitude damping. In another word, we can introduce the following master equation of density operator[1]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=\kappa\left(2a\rho a^{\dagger}-a^{\dagger}a\rho-\rho a^{\dagger}a\right) \tag{1}$$

to describe a photodetection process, where κ is the rate of decay. Equation (1) is actually an amplitude decay model (dissipation in a lossy cavity). The purpose of this paper is to explore how the density operator evolves in continuous photodetection process by virtue of the entangled state representation. This paper will present a clear picture in this aspect, i.e., the readers will see the origin of the density operator's time-evolution. To solve Eq.(1) we introduce the following entangled state in the doubled Fock space:[2,3]

$$|\eta\rangle=\exp\left(-\frac{1}{2}|\eta|^{2}+\eta a^{\dagger}-\eta^{*}\tilde{a}^{\dagger}+a^{\dagger}\tilde{a}^{\dagger}\right)\left|0,\tilde{0}\right\rangle, \tag{2}$$

where $\tilde{a}^{\dagger}$ is a fictitious mode accompanying the real photon creation operator $a^{\dagger}$, $\left|0,\tilde{0}\right\rangle=|0\rangle\left|\tilde{0}\right\rangle$, $\left|\tilde{0}\right\rangle$ is annihilated by $\tilde{a}$, $\left[\tilde{a},\tilde{a}^{\dagger}\right]=1$. The new entangled state representation is different from all the previous known representations: Q-function, P-representation, positive P-representation, complex P-representation, and the Wigner function derived through characteristic functions. Equation (1) provides a new approach to obtaining density operators as we demonstrate now.

Operating a and $\tilde{a}$ on $|\eta\rangle$ in Eq.(2) we obtain the eigen-equations of $|\eta\rangle$,

$$\begin{aligned}
&(a-\tilde{a}^{\dagger})\,|\eta\rangle=\eta\,|\eta\rangle\,,\\
&(a^{\dagger}-\tilde{a})\,|\eta\rangle=\eta^{*}\,|\eta\rangle\,,\\
&\langle\eta|\,(a^{\dagger}-\tilde{a})=\eta^{*}\,\langle\eta|\,,\\
&\langle\eta|\,(a-\tilde{a}^{\dagger})=\eta\,\langle\eta|\,.
\end{aligned} \tag{3}$$

Note that $\left[(a-\tilde{a}^{\dagger}),(a^{\dagger}-\tilde{a})\right]=0$, thus $|\eta\rangle$ is the common eigenvector of $(a-\tilde{a}^{\dagger})$ and $(\tilde{a}-a^{\dagger})$.

Using the normally ordered form of vacuum projector $\left|0,\tilde{0}\right\rangle\left\langle 0,\tilde{0}\right|=:\exp\left(-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}\right):$, and the technique of integration within an ordered product (IWOP) of operators,[4−6] we can easily prove that $|\eta\rangle$ is complete,

$$\begin{aligned}
\int\frac{\mathrm{d}^{2}\eta}{\pi}\,|\eta\rangle\,\langle\eta| &= \int\frac{\mathrm{d}^{2}\eta}{\pi}{:}\exp(-|\eta|^{2}+\eta a^{\dagger}-\eta^{*}\tilde{a}^{\dagger}\\
&\quad+\eta^{*}a-\eta\tilde{a}+a^{\dagger}\tilde{a}^{\dagger}+\tilde{a}a\\
&\quad-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}){:}\;=1,
\end{aligned} \tag{4}$$

this merit provides us with a possibility to express density operators in terms of $|\eta\rangle$. From Eq.(3) it is easily seen that $|\eta\rangle$ is orthogonal

$$\langle\eta'\,|\eta\rangle=\pi\delta\left(\eta'-\eta\right)\delta\left(\eta'^{*}-\eta^{*}\right). \tag{5}$$

It is worth mentioning that the two-mode squeezing

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†Corresponding author. E-mail: hlyun2008@126.com

operator $S_2 = \exp\left[\lambda\left(a\tilde{a} - \tilde{a}^\dagger a^\dagger\right)\right]$[7−10] has its natural expression in $\langle\eta|$ representation,[2,3]

$$S_2 \equiv \mathrm{e}^\lambda \int \frac{\mathrm{d}^2\eta}{\pi} \left|\mathrm{e}^\lambda\eta\right\rangle \langle\eta| . \tag{6}$$

In fact, using the IWOP technique to perform the integration in Eq.(6) we have

$$\begin{aligned}
&\mathrm{e}^\lambda \int \frac{\mathrm{d}^2\eta}{\pi} \left|\mathrm{e}^\lambda\eta\right\rangle \langle\eta| \\
=&\mathrm{e}^\lambda \int \frac{\mathrm{d}^2\eta}{\pi} : \exp\Big[-\frac{|\eta|^2}{2}\left(1+\mathrm{e}^{2\lambda}\right) + \eta\left(\mathrm{e}^\lambda a^\dagger - \tilde{a}\right) \\
&- \eta^*\left(\tilde{a}^\dagger \mathrm{e}^\lambda - a\right) + a^\dagger\tilde{a}^\dagger + \tilde{a}a - a^\dagger a - \tilde{a}^\dagger\tilde{a}\Big] : \\
=&\mathrm{sech}\lambda \exp(-a^\dagger\tilde{a}^\dagger \tanh\lambda) \exp[\left(a^\dagger a + \tilde{a}^\dagger\tilde{a}\right) \\
&\times \ln(\mathrm{sech}\lambda)] \exp[\tilde{a}a \tanh\lambda] \\
=&\exp\left[\lambda\left(a\tilde{a} - \tilde{a}^\dagger a^\dagger\right)\right] ,
\end{aligned} \tag{7}$$

so

$$S_2 \left|\eta\right\rangle = \mathrm{e}^\lambda \left|\mathrm{e}^\lambda\eta\right\rangle . \tag{8}$$

When $\eta = 0$, Eq.(2) becomes

$$\left|\eta = 0\right\rangle = \mathrm{e}^{a^\dagger\tilde{a}^\dagger}\left|0,\tilde{0}\right\rangle = \sum_{n=0}^{\infty} \left|n,\tilde{n}\right\rangle \equiv \left|I\right\rangle , \tag{9}$$

so

$$\left|\eta\right\rangle = D\left(\eta\right)\left|I\right\rangle , \tag{10}$$

where $\left|n,\tilde{n}\right\rangle = \frac{1}{n!}\left(a^\dagger\tilde{a}^\dagger\right)^n\left|0,\tilde{0}\right\rangle$, and $D\left(\eta\right) = \exp\left(\eta a^\dagger - \eta^* a\right)$ is the displacement operator. It is easily seen that $\left|I\right\rangle$ has the properties

$$a\left|I\right\rangle = \tilde{a}^\dagger\left|I\right\rangle ,\ a^\dagger\left|I\right\rangle = \tilde{a}\left|I\right\rangle ,\ \left(a^\dagger a\right)^n\left|I\right\rangle = \left(\tilde{a}^\dagger\tilde{a}\right)^n\left|I\right\rangle . \tag{11}$$

For general density operators expressed in Fock space as

$$\rho = \sum_{m,n=0} \rho_{n,m} a^{\dagger m} a^n , \tag{12}$$

one can see that

$$\left|\rho\right\rangle \equiv \rho\left|I\right\rangle = \sum_{m,n=0} \left(\rho^*_{n,m}\tilde{a}^{\dagger m}\tilde{a}^n\right)^\dagger \left|I\right\rangle . \tag{13}$$

After the classical quantity $\langle\eta|\,\rho\rangle$ are solved from the c-number master equations, the relationships between $\langle\eta|\,\rho\rangle$ and the characteristic function, positive P-representation are also deduced. The entangled state representation has been recently used in discussing quantum optical version of classical circular harmonic correlation.[11]

In order to resolve the above master equation, acting the both sides of Eq.(1) on the state $\left|I\right\rangle$, and noticing $\left|\rho\right\rangle = \rho\left|I\right\rangle$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\rho\right\rangle = \kappa\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right)\left|I\right\rangle . \tag{14}$$

Using Eq.(11) and noting that density operators defined in the original space are commutative with operators in the tilde space, we find that Eq.(14) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\rho\right\rangle = \kappa\left(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a}\right)\left|\rho\right\rangle , \tag{15}$$

so its formal solution is

$$\left|\rho\right\rangle = \exp\{\kappa t\left(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a}\right)\}\left|\rho_0\right\rangle , \tag{16}$$

where $\left|\rho_0\right\rangle \equiv \rho_0\left|I\right\rangle$, ρ_0 is the initial density operator. Thus we see that by introducing a fictitious mode being a counterpart mode of the system mode under review we can arrange some density operators' master equations in quantum statistics as state-vector evolution equations.

Noticing that the operators in Eq.(16) obey the following commutative relation:

$$\left[a\tilde{a}, a^\dagger a\right] = \left[a\tilde{a}, \tilde{a}^\dagger\tilde{a}\right] = \tilde{a}a,\ \left[\frac{a^\dagger a + \tilde{a}^\dagger\tilde{a}}{2}, a\tilde{a}\right] = -\tilde{a}a \tag{17}$$

and using the operator identity[12]

$$\mathrm{e}^{\lambda(A+\sigma B)} = \mathrm{e}^{\lambda A}\exp\left[\sigma\left(1-\mathrm{e}^{-\lambda\tau}\right)B/\tau\right] , \tag{18}$$

(valid for $[A,B] = \tau B$), we have

$$\begin{aligned}
&\exp\left[-2\kappa t\left(\frac{a^\dagger a + \tilde{a}^\dagger\tilde{a}}{2} - a\tilde{a}\right)\right] \\
=&\exp\left[-\kappa t\left(a^\dagger a + \tilde{a}^\dagger\tilde{a}\right)\right]\exp\left[Ta\tilde{a}\right] ,
\end{aligned} \tag{19}$$

where

$$T = 1 - \mathrm{e}^{-2\kappa t} . \tag{20}$$

Then substituting Eq.(19) into Eq.(16) we obtain

$$\begin{aligned}
\left|\rho\right\rangle =&\exp\left[-\kappa t\left(a^\dagger a + \tilde{a}^\dagger\tilde{a}\right)\right]\sum_{n=0}^{\infty}\frac{T^n}{n!}a^n\tilde{a}^n\left|\rho_0\right\rangle \\
=&\mathrm{e}^{-\kappa t a^\dagger a}\sum_{n=0}^{\infty}\frac{T^n}{n!}a^n\rho_0 a^{\dagger n}\mathrm{e}^{-\kappa t\tilde{a}^\dagger\tilde{a}}\left|I\right\rangle \\
=&\sum_{n=0}^{\infty}\frac{T^n}{n!}\mathrm{e}^{-\kappa t a^\dagger a}a^n\rho_0 a^{\dagger n}\mathrm{e}^{-\kappa t a^\dagger a}\left|I\right\rangle ,
\end{aligned} \tag{21}$$

which leads to

$$\rho = \sum_{n=0}^{\infty}\frac{T^n}{n!}\mathrm{e}^{-\kappa t a^\dagger a}a^n\rho_0 a^{\dagger n}\mathrm{e}^{-\kappa t a^\dagger a} . \tag{22}$$

This is an operator-sum representation of ρ. So we can rewrite $\rho = \sum_{n=0}^{\infty}\frac{T^n}{n!}\rho_n$,

$$\rho_n = \mathrm{e}^{-\kappa t a^\dagger a}a^n\rho_0 a^{\dagger n}\mathrm{e}^{-\kappa t a^\dagger a} , \tag{23}$$

and ρ_n itself is a statistical operator, we can normalize it by dividing a factor $\mathrm{Tr}\rho_n$, i.e.,

$$\begin{aligned}\rho_n(t) \equiv & \mathrm{e}^{-\kappa t a^\dagger a} a^n \rho_0 a^{\dagger n} \mathrm{e}^{-\kappa t a^\dagger a} / \mathrm{Tr}\left[a^{\dagger n} \mathrm{e}^{-2\kappa t a^\dagger a} a^n \rho_0\right] \\ = & \rho_n / \mathrm{Tr}\rho_n, \end{aligned} \tag{24}$$

so if ρ_0 is the photon statistical operator at initial time, then $\rho_n(t)$ is the photon statistical operator at time t. In the case of n-photon detection, $\rho_n(t)$ represents the density operator for continuum photodetection process.

For example, when the initial state is a thermal state with average photon number $\bar{n}_0$,

$$\rho_{0,\mathrm{th}} \equiv \sum_{m=0}^{\infty} \frac{\bar{n}_0^m}{(1+\bar{n}_0)^{m+1}} |m\rangle \langle m|, \tag{25}$$

from Eq.(24) and $a^n |m\rangle = \sqrt{\frac{m!}{(m-n)!}} |m-n\rangle$ we know that the density operator is

$$\rho_n(t) = \sum_{m=0}^{\infty} \frac{\bar{n}_0^m}{(1+\bar{n}_0)^{m+1}} \mathrm{e}^{-\kappa t a^\dagger a} a^n |m\rangle \langle m| a^{\dagger n} \mathrm{e}^{-\kappa t a^\dagger a} / \mathrm{Tr}\left[a^{\dagger n} \mathrm{e}^{-2\kappa t a^\dagger a} a^n \rho_0\right], \tag{26}$$

where the numerator is

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{\bar{n}_0^m}{(1+\bar{n}_0)^{m+1}} \mathrm{e}^{-\kappa t a^\dagger a} a^n |m\rangle \langle m| a^{\dagger n} \mathrm{e}^{-\kappa t a^\dagger a} \\ = & \sum_{m=n}^{\infty} \frac{\bar{n}_0^m}{(1+\bar{n}_0)^{m+1}} \frac{m!}{(m-n)!} \mathrm{e}^{-2\kappa t(m-n)} |m-n\rangle \langle m-n| \\ = & \frac{\bar{n}_0^n}{(1+\bar{n}_0)^{n+1}} \sum_{m'=0}^{\infty} \left(\frac{\bar{n}_0 \mathrm{e}^{-2\kappa t}}{1+\bar{n}_0}\right)^{m'} \frac{(m'+n)!}{m'!} |m'\rangle \langle m'|, \end{aligned} \tag{27}$$

and the denominator is

$$\begin{aligned} \mathrm{Tr}\left[a^{\dagger n} \mathrm{e}^{-2\kappa t a^\dagger a} a^n \rho_0\right] = & \sum_{m=0}^{\infty} \frac{\bar{n}_0^m}{(1+\bar{n}_0)^{m+1}} \langle m| a^{\dagger n} \mathrm{e}^{-2\kappa t a^\dagger a} a^n |m\rangle \\ = & \sum_{m=n}^{\infty} \frac{\bar{n}_0^m}{(1+\bar{n}_0)^{m+1}} \frac{m!}{(m-n)!} \mathrm{e}^{-2\kappa t(m-n)} \\ = & \frac{\bar{n}_0^n n!}{(1+\bar{n}_0)^{n+1}} \sum_{j=0}^{\infty} \left(\frac{\bar{n}_0 \mathrm{e}^{-2\kappa t}}{1+\bar{n}_0}\right)^{j} \frac{(j+n)!}{n! j!}. \end{aligned} \tag{28}$$

Using the binomial formula

$$\sum_{m=0}^{\infty} \binom{m+n}{m} (-x)^m = (1+x)^{-n-1}, \tag{29}$$

we can further write Eq.(28) as

$$\mathrm{Tr}\left[a^{\dagger n} \mathrm{e}^{-2\kappa t a^\dagger a} a^n \rho_0\right] = \frac{\bar{n}_0^n n!}{(1+\bar{n}_0 - \bar{n}_0 \mathrm{e}^{-2\kappa t})^{n+1}}. \tag{30}$$

Substituting Eqs.(27) and (30) into Eq.(24) we have

$$\begin{aligned} \rho_n(t) = & \sum_{m'=0}^{\infty} \left(\frac{\bar{n}_0 \mathrm{e}^{-2\kappa t}}{1+\bar{n}_0}\right)^{m'} \frac{(m'+n)!}{m'! n!} \\ & \times \left(1 - \frac{\bar{n}_0 \mathrm{e}^{-2\kappa t}}{1+\bar{n}_0}\right)^{n+1} |m'\rangle \langle m'| \\ = & \sum_{m'=0}^{\infty} \frac{(m'+n)!}{m'! n!} \gamma^{n+1} (1-\gamma)^{m'} |m'\rangle \langle m'|, \end{aligned} \tag{31}$$

where

$$\gamma = 1 - \frac{\bar{n}_0 \mathrm{e}^{-2\kappa t}}{1+\bar{n}_0} = \frac{1+\bar{n}_0\left(1-\mathrm{e}^{-2\kappa t}\right)}{1+\bar{n}_0}. \tag{32}$$

Equation (31) indicates that after detecting n photons, the light field exhibits negative-binomial distribution.

In summary, we have explored how a light field evolves after its n photons are continuously detected,

namely, we have obtained the photon statistical operator for describing continuum photodetection process. Our approach uses the newly developed entangled state representation[2,3,13] which seems convenient. We have not employed so-called superoperator method since those superoperators are imaginary operators, though their operation role is assumed, one never knows what are their concrete forms. For more applications about entangled state representation, we refer to Refs.[14–16].

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