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Generation of Schrödinger cat state of a single trapped cold ion

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The fidelity of the generated Schrödinger Cat state (SCS) of a single trapped ion in the Lamb–Dicke approximation is discussed. The results show that the fidelity significantly decreases with the values of Lamb–Dicke parameter η and coherent state amplitude α increasing. For $\eta = 0.20$ and $\alpha = 3$, the typical values of experimental parameters, the fidelity is rather low (30%). A scheme for generating the SCS is proposed without making the Lamb–Dike approximation in laser–ion interaction, and the fidelity of the generated SCS is about 99% for the typical values of experimental Lamb–Dicke parameters.

Keywords: Schrödinger cat state, trapped cold ion, Lamb–Dicke approximation, fidelity
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Schrödinger Cat states (SCSs), embodying the famous paradox by Schrödinger, are typical quantum states.[1] SCSs are important not only because of the fundamental research in physics, but also due to many potential applications, such as accurate quantum measurement or quantum computing.[2−8] A laser-cooled ion confined in an electromagnetic trap is one of the best candidates for the generation of SCSs, because it possesses a relatively long decoherence time.[9,10] Thus, the generation of SCSs with the trapped ions has received much attention in recent years.[11−16] For example, Monroe *et al*[14] generated the ion's SCSs first in experiment by sequentially applying five pulses of the laser beams to drive a trapped cold $^9Be^+$ ion. Gerry proposed a scheme to generate one- or two-mode SCS via the Raman coupled models, which required fewer laser manipulation steps.[15] Up to now, generating SCSs with six ions has been realized experimentally.[17] However, most of previous approaches to the generation of SCSs are operated in the Lamb–Dicke (LD) approximation, wherein the effects of higher-order terms of LD parameter are neglected,[9] and the feasibilities such as the fidelities[18,19] of generated SCSs are not reported yet. For the accurate controlling and generating of quantum states,[20−23] the fidelities of generated SCSs in the LD approximation should be considered. Following the way presented by Gerry[15] and Zheng,[16] we discuss the fidelity of the generated SCS in the LD approximation. By choosing the laser driving duration appropriately, we can generate the SCS around the experimental LD parameter η and coherent state amplitudes α with sufficiently high fidelity.

Suppose that a single two-level ion is trapped in a two-dimensional harmonic potential, and resonantly driven by two dispersive laser beams which propagate along the x and the y axes, respectively. Choosing the phases $\theta_x = 0$, and $\theta_y = \pi$ of the applied laser beams so that the interaction Hamiltonian (in the interaction picture and the rotating-wave approximation)[24] for such a system is expressed as[15,16]

$$\hat{H}_{\rm I} = \hbar\left[\Omega_x \mathrm{e}^{-\eta_x^2/2}\sum_{j=0}^{\infty}\frac{(\mathrm{i}\eta_x)^{2j}}{j!^2}(\hat{a}^+)^j\hat{a}^j - \Omega_y \mathrm{e}^{-\eta_y^2/2}\sum_{j=0}^{\infty}\frac{(\mathrm{i}\eta_y)^{2j}}{j!^2}(\hat{b}^+)^j\hat{b}^j\right](\hat{\sigma}_- + \hat{\sigma}_+). \tag{1}$$

Here, $\hat{a}^+$ and $\hat{b}^+$ are boson creation operators in the x and the y directions respectively, $\hat{a}$ and $\hat{b}$ are boson annihilation operators in the x and the y directions respectively. The Pauli operators $\hat{\sigma}_-$ and $\hat{\sigma}_+$ are defined by the internal ground state $|g\rangle$ and the excited state $|e\rangle$ of the two-level ion as $\hat{\sigma}_- = |g\rangle\langle e|$ and $\hat{\sigma}_+ = |e\rangle\langle g|$. $\Omega_i = E_i\lambda/\hbar$ ($i = x, y$) that depends on the amplitude E_i, and dipole-transition matrix element λ of the ap-

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plied laser beams corresponds to the coupling-degree of the laser–ion interaction. η_i is so-called LD parameter which describes the strength of coupling between the internal and the external degrees of freedom of the ion. The above Hamiltonian can be reduced to a single-mode model in two ways.[15,16] One is to assume that there is no vibrational excitation (i.e., initially in vacuum state) in the y direction. The other is that if the oscillator is anisotropic with $\eta_x \gg \eta_y$, then the interaction Hamiltonian can be reduced to

$$\hat{H}_{\rm I}' = \hbar\Omega_0 \left[\sum_{j=1}^{\infty} \frac{({\rm i}\eta)^{2j}}{j!^2} (\hat{a}^+)^j \hat{a}^j\right] (\hat{\sigma}_- + \hat{\sigma}_+), \qquad (2)$$

where $\eta = \eta_x$, and $\Omega_0 = \Omega_x \exp(-\eta_x^2/2) = \Omega_y \exp(-\eta_y^2/2)$[15,16] that could be achieved by adjusting the intensities of the applied laser beams in experiment. If the LD parameter η is small enough, then one can retain terms only to the second order of η and reduce the Hamiltonian (2) to the single-mode Raman coupled model of Knight.[25,26]

The dynamics for the trapped ion governed by the non-linear Hamiltonian (2) is exactly solvable. For example, it is easy to check the dynamical evolution of the initial state $|m\rangle|g\rangle$, where m is the occupation number of the Fock state of external vibrational motion of the ion, by using the evolution operator $\hat{U}(t) = \exp[-({\rm i}/\hbar)t\hat{H}_{\rm I}']$. Indeed, with the help of the relation

$$\langle m|\langle e|\hat{H}_I'|m\rangle|g\rangle = \hbar\Omega_m,$$

where

$$\Omega_m = \Omega_0 \sum_{j=0}^{m} \frac{({\rm i}\eta)^{2j} m!}{(j!)^2 (m-j)!} \delta_{j,0},$$

$$\delta_{j,0} = \begin{cases} 1 & j \neq 0, \\ 0 & j = 0, \end{cases} \qquad (3)$$

the time evolution of the initial state $|m\rangle|g\rangle$ can be expressed as

$$\begin{aligned} |\psi(0)\rangle &= m\rangle|g\rangle \to |\psi(t)\rangle \\ &= \cos(\Omega_m t)|m\rangle|g\rangle - {\rm i}\sin(\Omega_m t)|m\rangle|e\rangle. \end{aligned} \qquad (4)$$

If the external vibrational state of the ion is initially in a coherent state prepared by using one of several methods[14,27–29]

$$|\alpha\rangle = {\rm e}^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle, \qquad (5)$$

where α is the coherent state amplitude, and the internal state is initially in ground state $|g\rangle$, then the dynamical evolution of this system is

$$\begin{aligned} |\psi(0)\rangle &= |\alpha\rangle|g\rangle \to |\psi(t)\rangle \\ &= N_1(t)|\varphi_1(t)\rangle|g\rangle + N_2(t)|\varphi_2(t)\rangle|e\rangle, \end{aligned} \qquad (6)$$

where

$$|\varphi_1(t)\rangle = \frac{{\rm e}^{-|\alpha|^2/2}}{N_1(t)} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \cos(\Omega_m t)|m\rangle, \qquad (7)$$

$$|\varphi_2(t)\rangle = \frac{{\rm e}^{-|\alpha|^2/2}}{N_2(t)} \sum_{m=0}^{\infty} \frac{-{\rm i}\alpha^m}{\sqrt{m!}} \sin(\Omega_m t)|m\rangle, \qquad (8)$$

with

$$|N_1(t)|^2 = {\rm e}^{-|\alpha|^2} \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{m!} \cos(\Omega_m t)^* \cos(\Omega_m t), \qquad (9)$$

$$|N_2(t)|^2 = {\rm e}^{-|\alpha|^2} \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{m!} \sin(\Omega_m t)^* \sin(\Omega_m t) \qquad (10)$$

are normalized coefficients.

If the LD parameters are small enough, one can consider only the terms relating to second order of η in Eq.(3), i.e. reduce Ω_m to $\Omega_m^{\rm L} = \Omega_0 m({\rm i}\eta)^2$ in the LD approximation. With this approximation, a typical SCS can be obtained to be[14]

$$\begin{aligned} |S\rangle &= \left(\frac{1+{\rm e}^{-2|\alpha|^2}}{2}\right)^{1/2} |S_+\rangle|g\rangle \\ &\quad + \left(\frac{1-{\rm e}^{-2|\alpha|^2}}{2}\right)^{1/2} |S_-\rangle|e\rangle \end{aligned} \qquad (11)$$

by setting the laser driving duration $t\Omega_0 = \pi/(2\eta^2)$.[15,16] Here,

$$|S_\pm\rangle = (2 \pm 2{\rm e}^{-2|\alpha|^2})^{-1/2}(|{\rm i}\alpha\rangle \pm |-{\rm i}\alpha\rangle) \qquad (12)$$

are known as even (marked with "+") coherent state and odd (denoted with symbol "–") coherent state, and are correlated with the ground state and the excited state of ion, respectively. Obviously, the probability with which the ion is in the internal state $|g\rangle$ is $P_{|g\rangle}^S = [1 + \exp(-2|\alpha|^2)]/2$. Specifically, for $|\alpha|^2 \gg 1$, we have $P_{|g\rangle}^S = 1/\sqrt{2}$.

Note that the above SCS is obtained in the LD approximation. We now discuss the feasibility of the practical generating of SCS in this approximation. Combination of the desirable external vibrational even and odd coherent states (Eq.(12)) and the resulting

quantum states (Eqs.(7) and (8)) of the ion leads to the following two time-dependent functions:

$$f_1 = |\langle S_+|\varphi_1(t)|\rangle| = \Big|\frac{\sqrt{2}}{2N_1(t)\sqrt{\mathrm{e}^{2|\alpha|^2}+1}}\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{m!} \times[\mathrm{i}^m+(-\mathrm{i})^m]^*\cos(\Omega_m t)\Big|, \quad (13)$$

$$f_2 = |\langle S_-|\varphi_2(t)\rangle| = \Big|\frac{\sqrt{2}}{2N_2(t)\sqrt{\mathrm{e}^{2|\alpha|^2}-1}}\sum_{m=0}^{\infty}\frac{-\mathrm{i}|\alpha|^{2m}}{m!} \times[\mathrm{i}^m-(-\mathrm{i})^m]^*\sin(\Omega_m t)\Big|. \quad (14)$$

Obviously, the fidelity of the generated SCS in the LD approximation could be defined as the smaller one in the above expressions,[30,31] i.e. the fidelity $F(t\Omega_0,\eta,\alpha)=\min(f_1,f_2)$ (as a function of $t\Omega_0$, η and α).

For laser driving duration $t_L\Omega_0=\pi/(2\eta^2)$ (chosen in the LD approximation,[15,16]) we present some fidelities of the generated SCS in Table 1. Indeed, with the smaller values of LD parameter η and coherent state amplitude α, the SCS could be realized well in the LD approximation. For example, the fidelity can reach 0.9959 with $\eta=0.10$ and $\alpha=2$. However, for the larger values of η and α, the fidelity significantly decreases. Specially, for the typical values of experimental parameters $\eta=0.20$ and $\alpha=3$,[14] the fidelity $F(t_L\Omega_0,0.20,3)=0.3232$, which is rather low to generate the desirable SCS. Also, it is seen from Figs.1 and 2 that the phonon probability distributions $P_i(m)=|\langle m|\varphi_i(t)\rangle|^2$ $(i=1,2)$ of the generated vibrational states deviate a lot from the desirable distributions $P_\pm(m)=|\langle m|S_\pm\rangle|^2$ of even and odd coherent states. Thus, it is rather difficult to generate the desirable SCS in experiment with the duration $t_L\Omega_0=\pi/(2\eta^2)$.

Is there any appropriate laser driving duration for generating the desirable SCS around the experimental parameters with a higher fidelity? By using the numerical method, the appropriate duration, say $t_N\Omega_0$, could be chosen for the fidelity F approaching 1, i.e. the duration $t_N\Omega_0$ can be obtained from the condition $F(t_N\Omega_0,\eta,\alpha)\approx 1$. Indeed, our computations as presented in Table 2 show that the SCS could be better generated around the experimental LD parameters and coherent state amplitudes by setting an appropriate laser driving duration. It is seen that the fidelities of the generated SCS are high (All of them are larger

Table 1. Laser driving duration $t_L\Omega_0=\pi/(2\eta^2)$, and fidelity of generated SCS.

α	η	$t_L\Omega_0=\pi/(2\eta^2)$	F
2	0.05	628.3185	0.9997
	0.10	157.0796	0.9959
	0.15	69.81317	0.9798
	0.20	39.26991	0.9401
	0.25	25.13274	0.8688
	0.30	17.45329	0.7673
3	0.05	628.3185	0.9954
	0.10	157.0796	0.9301
	0.15	69.81317	0.7002
	0.20	39.26991	0.3232
4	0.05	628.3185	0.9612
	0.10	157.0796	0.5029
	0.15	69.81317	0.2487

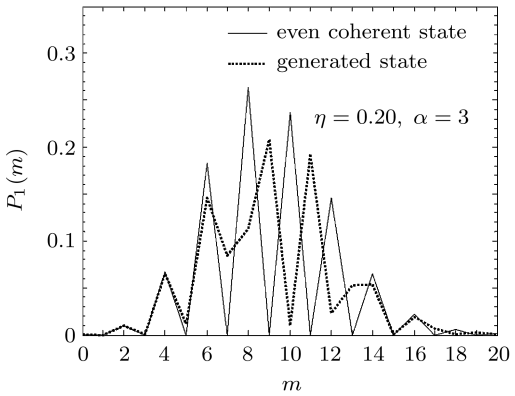


Fig.1. Phonon probability distributions $P_1(m)=|\langle m|\varphi_1(t_L)\rangle|^2$ (dashed curve) of the generated state and $P_+(m)=|\langle m|S_+\rangle|^2$ (solid curve) of the even coherent state for $\eta=0.20$, $\alpha=3$ and $t_L\Omega_0=\pi/(2\eta^2)$.

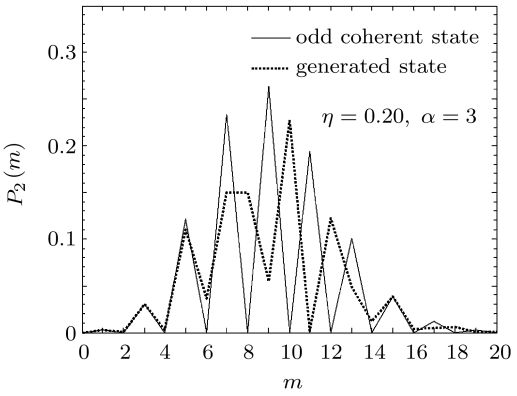


Fig.2. Phonon probability distributions $P_2(m)=|\langle m|\varphi_2(t_L)\rangle|^2$ (dashed curve) of the generated state and $P_-(m)=|\langle m|S_-\rangle|^2$ (solid curve) of the odd coherent state for $\eta=0.20$, $\alpha=3$ and $t_L\Omega_0=\pi/(2\eta^2)$.

Table 2. Some experimental parameters for high fidelity SCS.

α	η	$t_N\Omega_0$	F	$P^N_{\|g\rangle}$	$P^S_{\|g\rangle}$
	0.05	628.990	0.9999	0.5002	0.5002
	0.10	158.965	0.9994	0.5002	0.5002
	0.15	71.6283	0.9967	0.5002	0.5002
2	0.20	41.1549	0.9900	0.5001	0.5002
	0.25	27.0177	0.9758	0.5000	0.5002
	0.30	19.4779	0.9512	0.4998	0.5002
	0.05	632.200	0.9996	0.5000	0.5000
3	0.10	161.007	0.9937	0.5000	0.5000
	0.15	73.7646	0.9690	0.5000	0.5000
	0.20	43.2283	0.9088	0.5000	0.5000
	0.05	632.100	0.9909	0.5000	0.5000
4	0.10	163.840	0.9671	0.5000	0.5000
	0.15	85.8628	0.9684	0.5000	0.5000

than 90%). Typically, for the experimental LD parameters $\eta = 0.20$[14] and 0.25,[32] the fidelities can reach $F(41.1549, 0.20, 2) = 0.9900$ and $F(27.0177, 0.25, 2) = 0.9758$, respectively. Indeed, from Figs.3 and 4 it follows that for the fidelity $F(43.2283, 0.20, 3) = 0.9008$ (corresponding to the typical values of $\eta = 0.20$ and $\alpha = 3$) in Table 2, the phonon probability distribution $P_i(m)$ approaches to the standard $P_{\pm}(m)$ of even and odd coherent states. Also, the probability $P^N_{|g\rangle} = |N_1(t)|^2$ provided in the table, reflecting interferences between two coherent states,[14] is close to the standard $P^S_{|g\rangle} = [1 + \exp(-2|\alpha|^2)]/2$. So far, our computations show that the SCS could be better generated in experiment by setting an appropriate laser driving duration (instead of $t_L\Omega_0 = \pi/(2\eta^2)$ in the previous approaches).

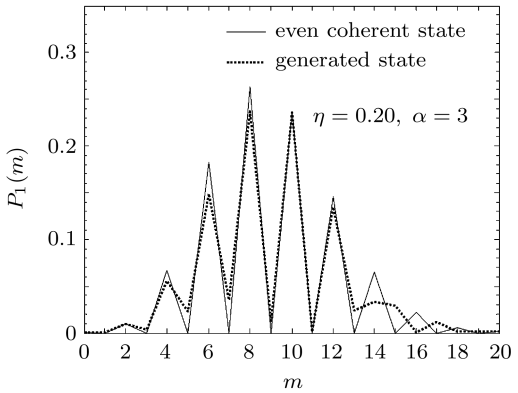


Fig.3. Phonon probability distribution $P_1(m) = |\langle m|\varphi_1(t_N)\rangle|^2$ (dashed curve) of the generated state and the desirable $P_+(m) = |\langle m|S_+\rangle|^2$ (solid curve) of even coherent state for $\eta = 0.20$, $\alpha = 3$ and $t_N\Omega_0 = 43.2283$, showing that they are close to each other.

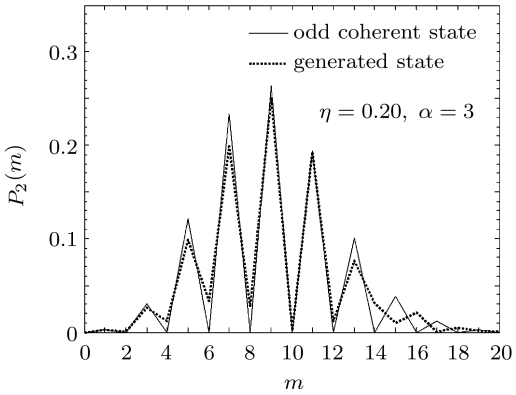


Fig.4. Phonon probability distribution $P_2(m) = |\langle m|\varphi_2(t_N)\rangle|^2$ (dashed curve) of the generated state and the desirable $P_-(m) = |\langle m|S_-\rangle|^2$ (solid curve) of odd coherent state for $\eta = 0.20$, $\alpha = 3$ and $t_N\Omega_0 = 43.2283$, showing that they are close to each other.

An experimental challenge to the generation of SCS is how to design the applied laser pulse with such a short duration (see Table 2, for example.). Fortunately, this difficulty could be overcome by using the current femto-second laser technique.[33,34] Another experimental challenge to the practical existence of decoherence is that the duration of the present generation of SCS should be shorter than the decoherence time.[9,35] It has been shown that the SCS can be maintained for up to $22\,\mu$s.[14] This allow us to generate the desirable SCS within the corresponding decoherence time, e.g. for the experimental parameters $\Omega_0 = 2\pi \times 500\,$kHz,[28] $\eta = 0.25$,[32] and $\alpha = 2$, the duration t for generating the SCS is about $8\,\mu$s. Furthermore, by increasing the value of Ω_0 via enhancing the power of the applied laser beam, the duration t for generating the SCS could be, in principle, significantly reduced. As a consequence, the limit of decoherence could be significantly reduced. In fact, the power of the laser applied experimentally to drive the ion is controllable: it can be adjusted in a range from a few μW to a few hundred μW.[32,36] Therefore, the generation of SCS in this paper is feasible experimentally.

In summary, we have discussed the fidelity of the generated SCS of a single trapped ion in the LD approximation. The results show that the fidelity significantly decreases with the values of LD parameter η and coherent state amplitude α increasing. For $\eta = 0.20$ and $\alpha = 3$, the typical values of experimental parameters, the fidelity is rather low (30%). Also, we have shown, by using the numerical method, that the SCS could be better generated (fidelity $> 90\%$) with these experimental parameters by setting an appropriate laser driving duration. The approach proposed here is feasible experimentally.

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