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Effects of time delay on stochastic resonance of a periodically driven linear system with multiplicative and periodically modulated additive white noises*

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Stochastic resonance (SR) of a periodically driven time-delayed linear system with multiplicative white noise and periodically modulated additive white noise is investigated. In the condition of small delay time, an approximate analytical expression of output signal-to-noise ratio (SNR) is obtained. The analytical results indicate that (1) there exists a resonance peak in the curve for SNR versus time delay; (2) the time delay will suspend the SR dramatically for SNR versus other parameters of the system, such as noise intensity, correlation intensity, and signal frequency, once a certain value is reached, the SR phenomenon disappears.

Keywords: linear system, time delay, stochastic resonance
PACC: 0540, 0545

1. Introduction

The stochastic system with time delay is ubiquitous in nature.[1] The effect of time delay on the stochastic system has received more and more attention.[2−15] The results of previous work manifested that the combination of noise and time delay could considerably change the dynamics of the stochastic system, such as time delay induced coherence resonance,[2,3] excitability,[4] symmetric break,[5,6] Hopf bifurcation,[7,8] and suppression of the population explosion.[9] Numerical simulations were employed when time delay in stochastic systems was large.[5,7] Analytical approximation for time-delayed stochastic systems was developed in the limit of small time delay.[10−14] Understanding the behaviour of time-delayed dynamical systems is the first step to improve the knowledge of memory in general, whose analysis is important especially in medicine, biology, and control theory.

Stochastic resonance (SR) is one of the most appealing aspects of the research on stochastic dynamics.[16−29] Particularly, the SR in linear systems has been investigated extensively,[30−42] many exactly solvable models were established and applied to physics, chemistry, and biology. Recently Cao and Wu studied the constructive role of periodically modulated noise in the stochastic resonance in addition to the input periodical signals of a membrane-protein system.[41] The system they investigated can be used to describe the intercellular signal transmission processes and various enzymatic reaction processes. However, the transmission takes some time, which implies that the input signals are related to the output signals at earlier times.[11−14] It is of interest to examine the effects of time delay on the SR of the system in which the non-delay case has been studied by Cao and Wu.[41] In the present paper, the effects of time delay on the SR in a periodically driven linear system with multiplicative and periodically modulated additive white noises are investigated theoretically.

The remainder of the present paper is organized as follows. In Section 2, an approximate analytical expression of output signal-to-noise ratio is derived. The effect of time delay, a discussion of the results, and conclusions are given in Section 3.

2. Model and output signal-to-noise ratio

A periodically driven linear system with multiplicative noise and periodically modulated additive noise is described by the following Langevin

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equation:[41]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(a+\xi(t))x + (A_1\cos(\Omega_1 t))\eta(t) + A\cos(\Omega t), \tag{1}$$

where $\xi(t)$ and $\eta(t)$ are Gaussian white noises with

$$\langle\xi(t)\rangle = \langle\eta(t)\rangle = 0, \tag{2}$$

$$\langle\xi(t)\xi(t')\rangle = Q_0\delta(t-t'), \tag{3}$$

$$\langle\eta(t)\eta(t')\rangle = D\delta(t-t'), \tag{4}$$

and

$$\langle\xi(t)\eta(t')\rangle = \langle\eta(t)\xi(t')\rangle = \lambda\sqrt{DQ_0}\delta(t-t'). \tag{5}$$

In Eq.(1), $\xi(t)$ and $\eta(t)$ origin from external and intrinsic fluctuations of the system respectively; A is the amplitude of the periodic signal; Ω is the frequency; a is attenuation constant; A_1 and Ω_1 are the periodically modulated amplitude and frequency of the noise $\eta(t)$. In Eqs.(2)–(5) Q_0 and D are the intensities of the noises; λ is the correlated intensity between the noises. However Eq.(1) describes an intercellular signal transmission process, and the transmission lasts some time.[11−13] In this case, the input variables should be described in terms of time-delayed output variables. Equation (1) should be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax_\tau - x_\tau\xi(t) + (A_1\cos(\Omega_1 t))\eta(t) + A\cos(\Omega t), \tag{6}$$

and $x_\tau = x(t-\tau)$, where τ denotes the transmission time.

By using the small delay approximation of stochastic delay differential equations, the equivalent non-delay differential equation of Eq.(6) can be given as[10,14]

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} =& -a(1-a\tau)x - (1-\tau)x\xi(t) + (A_1\cos\Omega_1 t)\eta(t)\\ &+ A\cos(\Omega t).\end{aligned} \tag{7}$$

Equation (7) gives the formal solution

$$\begin{aligned}x(t) =& x_0\mathrm{e}^{-a(1-a\tau)}\mathrm{e}^{-(1-\tau)\int_0^t\xi(s)\mathrm{d}s} + A_1\int_0^t\mathrm{e}^{-a(1-a\tau)(t-s)}\mathrm{e}^{-(1-\tau)\int_s^t\xi(s')\mathrm{d}s'}\cos(\Omega_1 s)\eta(s)\mathrm{d}s\\ &+ A\int_0^t\mathrm{e}^{-a(1-a\tau)(t-s)}\mathrm{e}^{-(1-\tau)\int_s^t\xi(s')\mathrm{d}s'}\cos(\Omega s)\mathrm{d}s.\end{aligned} \tag{8}$$

The correlation function in the stationary state is defined as

$$C(t_0) = \lim_{t\to\infty}\overline{\langle x(t+t_0)x(t)\rangle}, \tag{9}$$

where

$$\overline{\langle x(t+t_0)x(t)\rangle} = \frac{\Omega}{2\pi}\int_t^{t+2\pi/\Omega}\langle x(t+t_0)x(t)\rangle\,\mathrm{d}t. \tag{10}$$

From Eqs.(1)–(9), the correlation function in the stationary state is given by

$$\begin{aligned}C(t_0) =& \frac{(A^2 - 2AA_1\lambda\sqrt{DQ})\left\{\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+\Omega^2\right]\cos(\Omega t_0) - \Omega Q\sin(\Omega t_0)\right\}}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+\Omega^2\right]}\\ &+ A_1^2\left\{\frac{D}{4(\overline{a}-Q)} + \frac{\lambda^2 QD\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+\Omega^2\right]}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+\Omega^2\right]}\right\}\mathrm{e}^{-\frac{Qt_0}{2}}\\ &- \frac{A_1^2\lambda DQ[\cos(\Omega t_0) - \mathrm{e}^{-(\overline{a}-\frac{Q}{2})t_0}]}{8\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega^2\right]}\end{aligned} \tag{11}$$

for $\Omega = \Omega_1$, and

$$
\begin{aligned}
C(t_0) =& \frac{A^2\left\{\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+\Omega^2\right]\cos(\Omega t_0)-\Omega Q\sin(\Omega t_0)\right\}}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+\Omega^2\right]} \\
&+A_1^2\left\{\frac{D}{4(\overline{a}-Q)}+\frac{\lambda^2QD\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+\Omega_1^2\right]}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+{\Omega_1}^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+{\Omega_1}^2\right]}\right\}\mathrm{e}^{-\frac{Qt_0}{2}} \\
&-\frac{A_1^2\lambda DQ}{8\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega_1^2\right]}[\cos(\Omega_1 t_0)-\mathrm{e}^{-(\overline{a}-\frac{Q}{2})t_0}]
\end{aligned}
\tag{12}
$$

for $\Omega \neq \Omega_1$, respectively, with $\overline{a} = a(1-a\tau)$ and $Q = (1-\tau)^2 Q_0$.

The Fourier transform of expressions (11) and (12) leads to the power spectrum

$$
S(\omega) = \int_{-\infty}^{\infty} C(t_0)\mathrm{e}^{-\mathrm{i}\omega t_0}\mathrm{d}t_0 = S_\mathrm{s}(\omega) + S_\mathrm{n}(\omega), \tag{13}
$$

with the output power of the signal $W_\mathrm{s} = \int_0^\infty S_\mathrm{s}(\omega)\mathrm{d}\omega$ and the output power of the noise $W_\mathrm{n} = \int_0^\infty S_\mathrm{n}(\omega)\mathrm{d}\omega$ given by

$$
W_\mathrm{s} = \frac{A^2}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+\Omega^2\right]}\left\{\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+\Omega^2-\Omega Q\right]\pi\right\}, \tag{14}
$$

and

$$
\begin{aligned}
W_\mathrm{n} =& A_1^2\left\{\frac{D}{4(\overline{a}-Q)}+\frac{\lambda^2QD\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+\Omega^2\right]}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+\Omega^2\right]}\right\}\pi \\
&-\frac{2AA_1\lambda\sqrt{DQ}\left\{\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+\Omega^2-\Omega Q\right]\pi\right\}}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+\Omega^2\right]}
\end{aligned}
\tag{15}
$$

for $\Omega = \Omega_1$, and

$$
W_\mathrm{s} = \frac{A^2}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+\Omega^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+\Omega^2\right]}\left\{\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+\Omega^2-\Omega Q\right]\pi\right\},
$$

$$
W_\mathrm{n} = A_1^2\left\{\frac{D}{4(\overline{a}-Q)}+\frac{\lambda^2QD\left[\left(\overline{a}-\frac{Q}{2}\right)\left(\overline{a}-\frac{3Q}{2}\right)+{\Omega_1}^2\right]}{2\left[\left(\overline{a}-\frac{Q}{2}\right)^2+{\Omega_1}^2\right]\left[\left(\overline{a}-\frac{3Q}{2}\right)^2+{\Omega_1}^2\right]}\right\}\pi \tag{17}
$$

for $\Omega \neq \Omega_1$, respectively.

The SNR defined as

$$
\mathrm{SNR} = \frac{W_\mathrm{s}}{W_\mathrm{n}} \tag{18}
$$

can be obtained from Eqs.(14) and (15) for $\Omega = \Omega_1$ and Eqs.(16) and (17) for $\Omega \neq \Omega_1$.

3. Discussion and conclusions

The effects of time delay on the SNR for the case of $\Omega = \Omega_1$ are described separately in Figs.1–4, where all quantities plotted are dimensionless as shown in Ref.[41].

The curves for SNR against time delay τ for different values of correlation intensity λ are plotted in Fig.1, and the existence of a maximum in the curve of SNR-τ is the identifying characteristic of the SR phenomenon. The height of the peak increases with the value of λ increasing. It means that the increase of correlation intensity between noises can make the SR phenomenon prominent. This behaviour corresponds to that of Ref.[41].

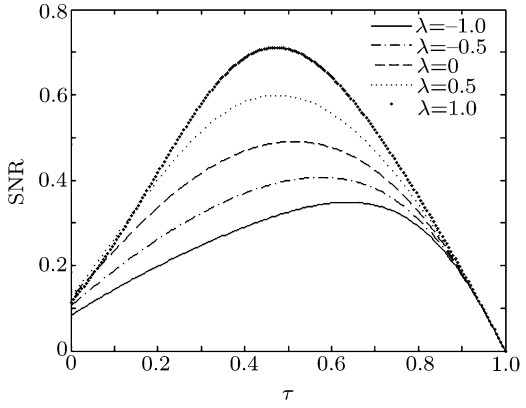


Fig.1. Curves for SNR vs delay time τ for different values of correlated intensity τ for the case of $\Omega = \Omega_1$. The values of the other parameters are $a = 1$, $A = 1$, $A_1 = 1$, $D = 1$, $Q_0 = 0.69$, and $\Omega = \Omega_1 = 1$.

The curves for SNR versus multiplicative noise intensity Q_0 for different values of time delay τ are plotted in Fig.2. The same SR phenomenon appears for the case of $\tau < 0.5$. When other parameters are fixed, the maximum of SNR decreases as the delay time increases, and the position of this peak shifts from large values of Q_0 to small values of Q_0. Once the delay time increases to 0.5, the peak in the curve of SNR-Q_0 disappears. The time delay can cause the SR phenomenon to disappear in this case.

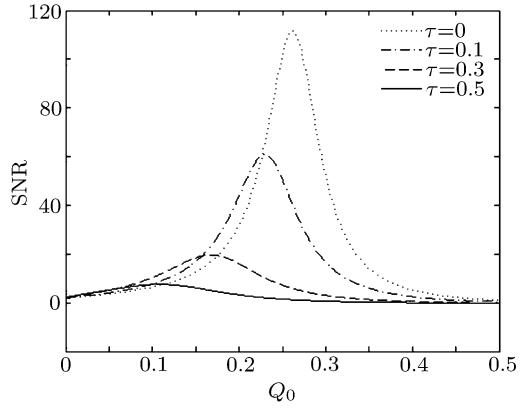


Fig.2. Curves for SNR vs noise intensity Q_0 for different values of time delay for the case of $\Omega = \Omega_1$. The values of the other parameters are $a = 1$, $A = 1$, $A_1 = 1$, $D = 1$, $\lambda = 0.85$, and $\Omega = \Omega_1 = 0.5$.

The curves for SNR with respect to correlation intensity λ between noises for different values of time delay τ are plotted in Fig.3. The same SR phenomenon appears for the case of $\tau < 0.15$. When other parameters are fixed, the maximum of SNR decreases as the delay time increases, but the position of this peak does not change. The time delay can lead the SR phenomenon to disappear in this case.

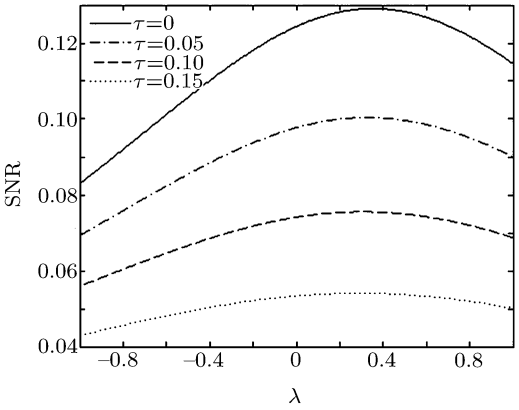


Fig.3. Curves for SNR vs noise correlated intensity λ for different values of time delay for the case of $\Omega = \Omega_1$. The values of the other parameters are $a = 1$, $A = 1$, $A_1 = 1$, $D = 1$, $Q_0 = 0.69$, and $\Omega = \Omega_1 = 1$.

The curves for SNR versus signal frequency Ω for different values of time delay τ are plotted in Fig.4.

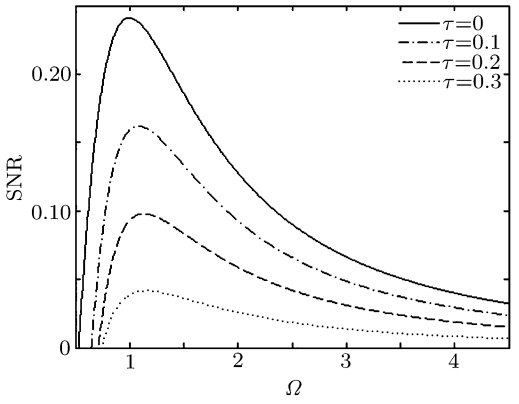


Fig.4. Curves for SNR vs signal frequency Ω for different values of time delay for the case of $\Omega = \Omega_1$. The values of the other parameters are $a = 1$, $A = 1$, $A_1 = 1$, $D = 1$, $Q_0 = 0.62$, and $\lambda = 0.5$.

There exists a peak structure in the curve of SNR-Ω, and the height of this peak becomes lower as delay time increases. The position of the peak does not change as delay time increases. It means that the SR existing in SNR-Ω can also be suppressed by increasing the time delay.

The effect of time delay on SNR in the case of $\Omega \neq \Omega_1$ is plotted in Figs.5 and 6, separately. All quantities plotted are dimensionless as shown in Ref.[41].

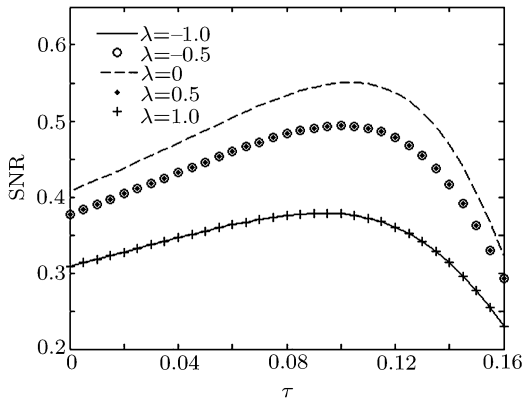


Fig.5. Curves for vs delay time τ for different values of correlated intensity λ for the case of $\Omega \neq \Omega_1$. The values of the other parameters are $a = 4.8$, $A = 1$, $A_1 = 1$, $D = 1$, $Q_0 = 0.69$, $\Omega = 1.5$, and $\Omega_1 = 1$.

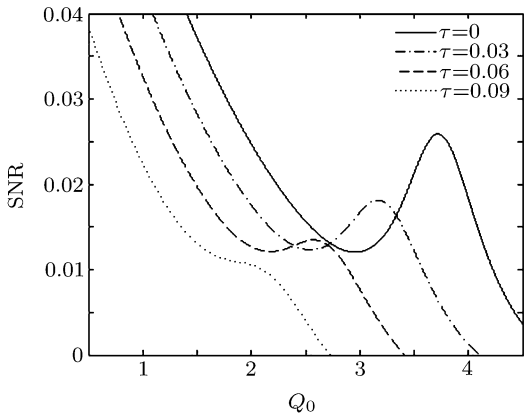


Fig.6. Curves for SNR vs noise intensity Q_0 for different values of time delay for the case of $\Omega \neq \Omega_1$. The values of the other parameters are $a = 4.8$, $A = 1$, $A_1 = 1$, $D = 1$, $\lambda = 0.85$, $\Omega = 9$, and $\Omega_1 = 1$.

The curves for SNR versus time delay τ for different values of correlation intensity λ are plotted in Fig.5, the existence of a maximum in the curve of SNR $- \tau$ is the identifying characteristic of the SR phenomenon. The height of the peak increases as the value of λ increases from -1 to zero, but decreases as λ increases from zero to 1. The sign of λ has no effect on height of peak. This behaviour is very different from that in the case for $\Omega = \Omega_1$ which is shown in Fig.1.

The curves for SNR against multiplicative noise intensity Q_0 for different values of time delay τ are plotted in Fig.6. A minimum and maximum exist in the curve of SNR-Q_0 for the case of $\tau < 0.09$. The positions of the two extrema move from large values of Q_0 to small values of Q_0, and the height of maximum becomes lower as the value of τ increases. Finally the two extrema disappear simultaneously as τ increases to 0.09, and the curve for SNR-Q_0 becomes monotonically descending. The behaviour of SNR in this case is more complex than that in the case for $\Omega = \Omega_1$ which is shown in Fig.2.

In the present paper, we study the effects of the time delay on the stochastic resonance of a linear system driven by multiplicative and periodically modulated additive white noises. The analytic expression for the SNR of the system is derived in the small time delay approximation. By using the expression of SNR, the effects of time delay on stochastic resonance of the system are investigated for the cases of $\Omega = \Omega_1$ and $\Omega \neq \Omega_1$. For the case of $\Omega = \Omega_1$, λ enforces the stochastic resonance of the system about τ. For the case of $\Omega \neq \Omega_1$, λ enforces the stochastic resonance of the system about τ for $\lambda < 0$, and restrains it for $\lambda > 0$. The stochastic resonances of the system about Q_0, λ, and Ω all disappear with the value of τ increasing for the two cases. In conclusion, the SR phenomenon is regarded as a typical example of the construction role of environment fluctuations in stochastic systems, particularly in biological systems. The finite time delay weakens the construction role of noise in our study.

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