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Reversible Carnot cycle outside a black hole*

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A Carnot cycle outside a Schwarzschild black hole is investigated in detail. We propose a reversible Carnot cycle with a black hole being the cold reservoir. In our model, a Carnot engine operates between a hot reservoir with temperature T_1 and a black hole with Hawking temperature $T_{\rm H}$. By naturally extending the ordinary Carnot cycle to the black hole system, we show that the thermal efficiency for a reversible process can reach the maximal efficiency $1-\frac{T_{\rm H}}{T_1}$. Consequently, black holes can be used to determine the thermodynamic temperature by means of the Carnot cycle. The role of the atmosphere around the black hole is discussed. We show that the thermal atmosphere provides a necessary mechanism to make the process reversible.

Keywords: black hole, thermodynamics, Carnot cycle
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1. Introduction

The discovery of Hawking radiation confirms that a black hole can emit particles with the same spectrum as that emitted by a black body with temperature $T_{\rm H}=\kappa/2\pi$, where κ is the surface gravity of the black hole. Since the 1970s, the connections between black hole mechanics and ordinary thermodynamics have been widely studied. An important issue is the generalized second law (GSL) of black hole thermodynamics, which states that the total entropy, i.e. the entropy of the black hole and the entropy of ordinary matter outside the black hole, never decreases. The validity of the GSL was first challenged by Geroch's gedanken experiment.[1] In this experiment, an agent lowers a box of matter slowly down to the horizon of a black hole and then releases it into the black hole. Owing to the gravitational redshift, the energy of the box, measured from infinity, vanishes at the horizon. According to the first law of black hole mechanics, the area of the black hole remains unchanged after the box is released. Therefore, the box loses all its entropy while the entropy of the black hole does not change, leading to a violation of the GSL. To rescue the GSL, Bekenstein argued that the entire box could not be lowered arbitrarily close to the horizon of the black hole due to the finite size of the box. By proposing a universal entropy bound $S\leq 2\pi ER$[2] for the matter, Bekenstein then showed that the GSL did hold in the Geroch's gedanken experiment. An alternative solution was given by Unruh and Wald,[3] with the thermal atmosphere around the black hole taken into account.

An equivalent statement of the second law of thermodynamics is that perpetual motion of the second kind is forbidden. For a Carnot engine operating between heat sources with temperatures T_1 and T_2 $(T_1>T_2)$ separately, the maximum thermal efficiency is

$$\eta=1-\frac{T_2}{T_1}. \tag{1}$$

As shown by Bekenstein,[1] the Geroch process is no more efficient than its 'equivalent reversible heat engine'. Thus, no violation of the second law is entailed in the Geroch process.

It seems that the issue of the Geroch process has been solved satisfactorily. However, in our point of view, problems remain, even at the fundamental level, especially when the process is treated in the context of Carnot cycle. A reversible Carnot cycle is important to black hole thermodynamics. If such a process does not exist, the close analogue between the ordinary thermodynamics and the black hole mechanics will break down, leading to a crisis for black hole thermodynamics. Moreover, the well-known Hawking temperature is read off from the thermal spectrum.

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It is not clear whether a black hole can be associated with a thermodynamic temperature determined by a Carnot cycle. To the best of our knowledge, such a question has not been clearly answered. Therefore, we shall construct a Carnot cycle in a black hole system based on the Geroch process.

Recall that a typical Carnot cycle comprises four distinct processes[4] as follows:

(a) the working substance absorbs heat Q_1 isothermally at temperature T_1;

(b) the working substance expands adiabatically and its temperature changes from T_1 to T_2 ($T_2 < T_1$);

(c) the working substance rejects heat Q_2 through an isothermal compression;

(d) the working substance is compressed adiabatically and returns to its initial state.

Correspondingly, the original Geroch process can be divided into the following four processes:

(a) the box obtains energy E at infinity.

(b) an agent lowers the box slowly down to the surface of the black hole.

(c) the box is released and all its substances fall into the black hole.

(d) The empty box is brought back to infinity.

Although the four processes look similar to those in a typical Carnot cycle, one will see, after a careful examination, that their differences are quite significant. In the Geroch process, the working substance exchanges energy with the heat sources in a direct manner. No work is done in the two steps. What is more, there is no explicit isothermal process in the Geroch process. In addition, the working substance in a typical Carnot cycle is sealed in a box, i.e. no particle exchange is allowed in the whole process. It is certainly not the case in the Geroch process since the box rejects all its substances into the black hole. Finally, as we shall show in Section 2, the Geroch process cannot be a reversible process because it involves non-quasistatic processes.

In view of the arguments above, we propose a modified Geroch process to describe the Carnot cycle outside a Schwarzschild black hole. In this model, working substance is some matter with vanishing chemical potential, e.g. the photon gas. Instead of letting the matter drop suddenly into the black hole, we release the matter isothermally and quasi-statically. This is an extension of the ordinary Carnot cycle, but this extension is by no means trivial and straightforward. To accomplish this, we need to specify the temperature outside the black hole. Since the thermal atmosphere is not taken into account in Bekenstein's arguments, it is not obvious how to specify the local temperature outside the black hole. We solve this problem by requiring that the total entropy keep unchanged, which is a necessary condition for a reversible Carnot cycle. To fulfil this requirement, we find that the temperature of the box at the dropping point must be equal to $T_{\rm H}/\chi$, where χ is the redshift factor. This is an interesting result because it coincides with the local temperature of the thermal atmosphere in equilibrium with the black hole. But our derivation holds even without the thermal atmosphere. Finally, we show that the cycle achieves the maximum thermal efficiency. This result verifies the thermodynamic nature of black holes in relation to the Carnot cycle. Our derivation in Section 3 is performed in the absence of thermal atmosphere. In Section 4, we show that the same thermal efficiency holds when taking into account the thermal atmosphere in equilibrium with the black hole.

2. The Geroch process

In this section, we discuss whether the Geroch process is a reversible Carnot cycle. Suppose a box at infinity is filled with thermal radiation. The radiation has energy E and temperature T. Then an agent lowers the box slowly down to the horizon and drops all the substances into the black hole. If the red-shift factor at the dropping point is χ, the energy measured from infinity is χE.[5] Therefore, after the box has been dropped, the energy obtained by the black hole is χE and the energy obtained by the agent at infinity is $E - \chi E$. If this process is considered to be a Carnot cycle, it is natural to identify E with Q_1, the heat absorbed from a hot reservoir at infinity, and χE with Q_2, the heat rejected to the black hole. The thermal efficiency is given by

$$\eta = \frac{W}{Q_1} = \frac{E - \chi E}{E} = 1 - \chi, \tag{2}$$

where W is the work done in the lowering process. If the box can be dropped exactly at the horizon, where $\chi = 0$, the process will have a unit efficiency, which is in violation of the Carnot's theorem. However, as Bekenstein[1] pointed out, χ could not be zero due to the finite size of the box. What we show now is that the Geroch process cannot be a reversible process. It is easy to see that in the Geroch process, the first step in a typical Carnot cycle is skipped, i.e. absorbing

heat isothermally from a hot reservoir. We may make it up by imagining that the energy E of the box is obtained from a hot reservoir filled with thermal radiation of temperature T_1. According to the first law of thermodynamics, we have $Q_1 = E$, where Q_1 is the heat flowing from the reservoir to the box and E is the energy that the box finally absorbs. On the other hand, from the integrated form of the Gibbs–Duhem relation[6]

$$TS = E + PV, \tag{3}$$

we have $E < TS = \int T\mathrm{d}S$. The equality holds because we assume that the process is isothermal. Then we have $Q_1 < \int T\mathrm{d}S$, showing that the process is not reversible. By parallel argument, we see that the process where the box rejects energy to the black hole is not reversible either. Thus, the increase of entropy in the Geroch process is inevitable. In the next section, we will construct an improved Carnot cycle in which every process is made reversible.

3. A reversible Carnot cycle

Now we propose a reversible Carnot cycle based on the Geroch process. Our model involves some important and interesting thermodynamic natures. We also divide the process into the following four steps:

Step (a) Fill a hot reservoir at infinity with thermal radiation of temperature T_1. We require the chemical potential of the thermal radiation to vanish. We also require $T_1 > T_{\mathrm{H}}$, since the black hole is used as a cold reservoir. The initial state of the working substance is an empty box. The mass of the empty box is negligible. We fill the box with the thermal radiation from the hot reservoir. As shown in Section 2, if the box takes energy without doing work, the process cannot be reversible. In our model, we replace one side of the box with a piston and put it into the reservoir. Then we pull the piston slowly down to the other side of the box such that the box finally is full of thermal radiation of temperature T_1 (see Fig.1). In this way, we have a box of substance with energy E_1. Since the energy is obtained quasi-statically and isothermally, we can employ the first law of thermodynamics in the form

$$Q_1 = T_1 S_1 = E_1 + W_1, \tag{4}$$

where Q_1 is the heat flowing to the box, S_1 is the entropy of the matter in the box, and W_1 is the work done by the thermal radiation. Note that in writing Eq.(4), we have made the assumption that the chemical potential $\mu = 0$. In a typical Carnot cycle, the working substance is sealed in a box with constant number of particles. Hence, only energy exchange occurs. In our treatment (also in the model discussed by Geroch and Bekenstein), both energy and particle are allowed to change. This is an extension of the typical Carnot cycle. However, due to the requirement $\mu = 0$, their difference does not manifest itself in Eq.(4). By comparing with the Geroch process (a), we see that the heat Q_1 is no longer equal to the energy E_1 because we construct a quasi-static process in which work is done by the radiation.

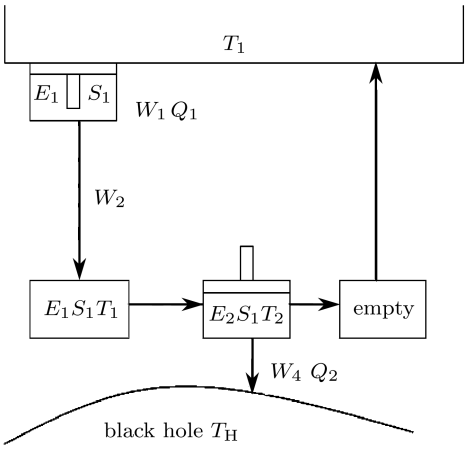


Fig.1. A reversible Carnot cycle outside a black hole.

Step (b) Adiabatic process. This process actually comprises two sub-steps.

Sub-step (b1) The box is lowered slowly down to some point near the black hole with the red-shift factor χ. This step is nothing but repeating the corresponding Geroch process. In this step, the substance inside feels no change. All locally measured quantities, such as energy, entropy, temperature volume, etc, remain unchanged. However, the energy measured by the agent holding the string at infinity becomes χE, which means that the work done in the lowering process is

$$W_2 = E_1(1 - \chi). \tag{5}$$

It is worth pointing out that this is not thermodynamic work done by the substance in the box since there is no change in the volume of the substance. It is indeed equivalent to the work done by gravity in a pure mechanical process. A much closer analogue of the adiabatic process in an ordinary Carnot cycle is given in sub-step (b2) below.

Sub-step (b2) This sub-step plays an essential role in our model, which is not included at all in the Geroch process. In a typical Carnot cycle, the working substance has to experience an adiabatic expansion to cool down to temperature T_2, the temperature of the cold reservoir. As we just illustrated in sub-step (b1), the adiabatic lowering process does not change the temperature of the substance. So it is necessary to add another adiabatic process to cool down the substance from T_1 to T_2. But what is T_2? The answer is rather subtle. Since the box is not in direct contact with the black hole, the temperature at any point outside the black hole is not well-defined. Now we use a plausible criterion to specify T_2 as follows. Suppose that the box cools down to T_2 via an adiabatic expansion. The entropy of box remains $S_1 = Q_1/T_1$ (see Eq.(4)). Then the box gives up all its energy to the black hole as shown in step (c) below. The energy that the black hole finally takes is $Q_2 = \chi T_2 S_1$ (see Eq.(10)). Thus, the change in the total entropy is

$$\Delta S = -S_1 + \frac{\chi T_2 S_1}{T_{\rm H}}. \tag{6}$$

Since a necessary condition for a reversible process is that the total entropy does not change, we assume $\Delta S = 0$ and find immediately from Eq.(6)

$$T_2 = \frac{T_{\rm H}}{\chi}. \tag{7}$$

One can recognize this relation as the well-known Tolman law when thermal atmosphere outside a black hole is in equilibrium.[6] But Eq.(7) was derived completely in the absence of thermal atmosphere! This result shows that the box should be adjusted to the same local temperature as if it is surrounded by the thermal atmosphere. It is worth noticing that $T_1 > T_2$ holds in a typical Carnot cycle so the box should reduce its temperature through an adiabatic expansion. In our model, although we have required $T_1 > T_{\rm H}$, it does not guarantee $T_1 > T_2 = \dfrac{T_{\rm H}}{\chi}$. Instead, if the box is too close to the horizon, we must have $T_1 < T_2$. In Bekenstein's argument,[1] this cannot happen because the local temperature of the box must have a lower limit $T \sim 1/R$, where R is the size of the box. Otherwise, the hydrodynamic description will break down. This limit is crucial in Benkenstein's argument to prove the GSL. However, this limit is irrelevant to our model. If $T_1 < T_2$, we just use an adiabatic compression to increase the temperature to T_2. Without loss of generality, we assume $T_1 > T_2$ in the rest of the present paper.

Different from sub-step (b1), sub-step (b2) changes the parameter set of the box from (E_1, T_1) to (E_2, T_2) while the entropy S_1 keeps unchanged. Because the adiabatic expansion takes place in the same location, the work measured from infinity is given by

$$W_3 = \chi(E_1 - E_2). \tag{8}$$

Step (c) Release the substances from the box into the black hole. We use the piston again, but in the opposite way, to push out the thermal radiation in a quasistatic and isothermal manner. We have

$$Q_2' = T_2 S_1 = E_2 - W_4', \tag{9}$$

which is similar to Eq.(4). In Eq.(9), prime is used to label locally measured quantity. By our convention, E_2 also represents the local energy. We use Q_2' to denote the absolute value of heat, while the value of W_4' is negative because the volume of the box reduces to zero. The corresponding quantities measured at infinity are

$$Q_2 = \chi Q_2' = \chi T_2 S_1,$$

and

$$W_4 = \chi W_4'. \tag{10}$$

As seen by a local observer, the energy released from the box is Q_2'. By assuming that all the energy goes to the black hole, we find that the energy absorbed by the black hole is Q_2, instead of Q_2'.

Step (d) Lift the empty box to infinity and return to its initial state. Nothing happens in this process since the mass of the empty box is negligible.

Now we are ready to compute the thermal efficiency for the complete Carnot cycle. It is not difficult to check that the total work $W \equiv W_1 + W_2 + W_3 + W_4$ satisfies the familiar relation in a typical Carnot cycle

$$W = Q_1 - Q_2. \tag{11}$$

By definition, the efficiency is

$$\eta = \frac{W}{Q_1} = 1 - \frac{Q_2}{Q_1}. \tag{12}$$

By substituting Eqs.(4) and (10), and using Eq.(7), we obtain

$$\eta = 1 - \frac{T_{\rm H}}{T_1}. \tag{13}$$

This is the desired efficiency for a reversible Carnot engine operating between two heat sources with temperatures T_1 and $T_{\rm H}$, separately. Thus the thermodynamic nature of the black hole is confirmed in the

framework of Carnot cycle. Equation (13) also indicates that the black hole can be used to determine the thermodynamic temperature by means of Carnot cycle.

4. Considering the thermal atmosphere

In the previous section, we follow the framework provided by Bekenstein, neglecting the quantum effect of the black hole. There is a different prescription given by Unruh and Wald,[3] where the thermal atmosphere around the black hole plays an important role. They considered a black hole surrounded by a thermal bath in equilibrium. The thermal bath has a quantum origin as discussed in Ref.[3]. Then a box will feel an effective buoyancy force from the thermal bath. Now we repeat the four steps in the previous section, with the thermal bath taken into account. Step 1. The box interacts with the hot reservoir and the thermal bath is irrelevant. Denote the volume of the box by V_1. Step 2(1). In the lowering process, the box feels the buoyancy force from the thermal atmosphere. So the energy flowing to the agent via work is smaller than W_2 in Eq.(5). But we can see that this additional work is cancelled out when the empty box is returned to infinity in step (d). Step 2(2). The box experiences an adiabatic expansion and the volume changes from V_1 to V_2. The thermal bath also contributes to work in this process due to the change of volume. Step 3. The substance is pumped out of the box and we are left with an empty box with volume V_2. Note that the thermal bath does not change any calculation in this step. Step 4. The empty box is dragged back to infinity. In the absence of thermal atmosphere, as we have discussed in the previous section, nothing really happens. But when the thermal bath is considered, work has to be done even if the box is empty. This work is exactly what we need to cancel the work done by the thermal atmosphere in Steps 2(1) and 2(2). Since the box has to return to its initial state, we first change the volume of the empty box form V_2 to V_1 by compressing one side of the box slowly. Without thermal atmosphere, this step is totally unnecessary. This is why we did not mention it in Section 3. But now it is the reversal of Step 2(2) for the thermal atmosphere. After that we return the box to infinity. Since the box has the same volume V_1 as that in the lowering process, the work done by the thermal atmosphere in Step 2(1) is cancelled. It is then not difficult to see that the basic relation (11) holds and we obtain the same efficiency as that shown in Eq.(13).

Note that Eq.(7) now is precisely the condition satisfied by the thermal bath in equilibrium with the black hole.[6] The thermal bath acts as a medium connecting the black hole and the box. Indeed, the thermal bath can be viewed as a thermodynamic extension of the black hole. In the presence of thermal bath, it becomes natural to cool down the substance in the box to T_2, which is equal to the local temperature of the thermal bath around it. Another important role of the thermal bath is concerned with the reversibility. Our model in the previous section describes a reversible Carnot cycle in the sense that the total entropy does not change and the efficiency is the same as that of any equivalent reversible Carnot engine. However, we have used the black hole only as a perfect absorber. This is the classical aspect of the black hole. If we wish to reverse the whole process, the black hole needs to lose mass by emitting particles. Without the thermal bath, one cannot take energy directly from the black hole. But this can be easily done with the help of the thermal bath. One may reduce the mass of the black hole by taking energy from the thermal bath since the combination of the black hole and its thermal atmosphere is actually one thermodynamic system.[6] Therefore, the thermal bath plays an essential role in reversing the Carnot cycle.

5. Concluding remarks

In this paper, we have proposed a reversible Carnot cycle outside a Schwarzschild black hole. The efficiency of this cycle confirms that the black hole behaves as a thermodynamic object with the Hawking temperature $T_{\rm H}$. Our model is kind of combination of the Geroch process and a typical Carnot cycle. Nevertheless, this combination is made in a subtle manner. We have modified the Geroch process by considering quasistatic process similar to that in an ordinary Carnot cycle. To construct an isothermal process near the black hole, we have determined a local temperature T_2 by requiring that the total entropy be unchanged. We then find that T_2 obeys the Tolman Law. Finally, we have proved that the efficiency holds in the presence of thermal bath surrounding the black hole. The thermal bath also plays an essential role in the reversible process.

In an ordinary Carnot cycle, the working substance is a closed system. Only energy may flow in and out. In our model, the closed system is extended to an open system, allowing the transfer of matter. But this extension is on condition that the chemical potential must vanish. It is not difficult to apply our derivation to a closed system with nonvanishing chemical potential as has been used in an ordinary Carnot cycle. The two cases, i.e. a closed system with nonvanishing μ and an open system with vanishing μ, give rise to the same thermal efficiency (13) because the chemical potential term does not appear in the first law of thermodynamics in the two cases. For an open system with non-zero chemical potential, the first law will include the contribution from the chemical potential and the basic work-heat relation (11) in an ordinary Carnot cycle will no longer hold. Such an issue is beyond the scope of the present paper.

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