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New approach for analysing master equations of generalized phase diffusion models in the entangled state representation*

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By virtue of the well-behaved properties of the bipartite entangled states representation, this paper analyse and solves some master equations for generalized phase diffusion models, which seems concise and effective. This method can also be applied to solve other master equations.

Keywords: generalized phase diffusion, entangled state representation, master equations
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1. Introduction

A great task in quantum optics is solving master equations which describe interaction between radiation field and atoms (or reservoir),[1−4] so that we can know how the field damps or diffuses. For this purpose, one usually converts density operator equations into c-number equations by using the P-representaiton[5,6] or the Wigner function approach.[7] In this work we point out that by virtue of the bipartite entangled state representation[8] we can conveniently tackle this task in a concise way, this is because the bipartite entangled states possess well-behaved properties. In this way we can have a fresh picture of the time-evolution of density operators from the point of view of entangled state representation, other than the P-representation or the Wigner function. The work is arranged as follows. In Section 2 we briefly review the properties of the entangled state representation $|\tau\rangle$ in which one mode is a fictitious mode introduced for convenience. In Sections 3 and 4 we concentrate on solving density operator equations for the generalized phase diffusion process.

2. Brief review of the entangled state representation

Denoting the light field mode by $a^\dagger$, we introduce the entangled state in two-mode Fock space[8,9]

$$|\tau\rangle = \exp\left[-\frac{1}{2}|\tau|^2 + \tau a^\dagger - \tau^* \tilde{a}^\dagger + a^\dagger \tilde{a}^\dagger\right] |0\tilde{0}\rangle\,, \quad (1)$$

where $\tilde{a}^\dagger$ is a fictitious mode accompanying $a^\dagger$ introduced for convenience, $[a^\dagger, \tilde{a}^\dagger] = 0$, $[\tilde{a}, \tilde{a}^\dagger] = 1$, $|0\tilde{0}\rangle = |0\rangle|\tilde{0}\rangle$, $\tilde{a}|\tilde{0}\rangle = 0$. At this point we mention that the method of introducing fictitious modes was firstly introduced by Takahashi and Umezawa for thermo field dynamics in Ref.[10]. The states $|\tau\rangle$ make up an orthonormal-complete relation

$$\int \frac{\mathrm{d}^2\tau}{\pi} |\tau\rangle\langle\tau| = 1,$$
$$\langle\tau'|\,\tau\rangle = \pi\delta(\tau' - \tau)\,\delta(\tau'^* - \tau^*)\,, \quad (2)$$

and obey the following eigenvector equations

$$\left(a - \tilde{a}^\dagger\right)|\tau\rangle = \tau|\tau\rangle\,, \quad \left(\tilde{a} - a^\dagger\right)|\tau\rangle = -\tau^*|\tau\rangle\,, \quad (3)$$
$$\langle\tau|\left(a^\dagger - \tilde{a}\right) = \tau^*\langle\tau|\,, \quad \langle\tau|\left(a - \tilde{a}^\dagger\right) = \tau\langle\tau|\,. \quad (4)$$

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It then follows

$$a\left|\tau=0\right\rangle=\tilde{a}^{\dagger}\left|\tau=0\right\rangle,$$
$$a^{\dagger}\left|\tau=0\right\rangle=\tilde{a}\left|\tau=0\right\rangle, \tag{5}$$

and

$$a^{\dagger}a\left|\tau=0\right\rangle=\tilde{a}^{\dagger}\tilde{a}\left|\tau=0\right\rangle. \tag{6}$$

Also from Eq.(1) we derive

$$\left\langle\tau\right|\tilde{a}^{\dagger}=\left(\frac{\partial}{\partial\tau^{*}}-\frac{\tau}{2}\right)\left\langle\tau\right|,$$
$$\left\langle\tau\right|a^{\dagger}=-\left(\frac{\partial}{\partial\tau}-\frac{\tau^{*}}{2}\right)\left\langle\tau\right|, \tag{7}$$
$$\left\langle\tau\right|\tilde{a}=-\left(\frac{\partial}{\partial\tau}+\frac{\tau^{*}}{2}\right)\left\langle\tau\right|,$$
$$\left\langle\tau\right|a=\left(\frac{\partial}{\partial\tau^{*}}+\frac{\tau}{2}\right)\left\langle\tau\right|. \tag{8}$$

We also find the remarkable property for $\tau=|\tau|\mathrm{e}^{\mathrm{i}\varphi}$,

$$\begin{aligned}&\left(\tilde{a}^{\dagger}\tilde{a}-a^{\dagger}a\right)\left|\tau\right\rangle\\&=-\left(a^{\dagger}\tau+\tilde{a}^{\dagger}\tau^{*}\right)\\&\quad\times\exp\left[-\frac{1}{2}|\tau|^{2}+\tau a^{\dagger}-\tau^{*}\tilde{a}^{\dagger}+a^{\dagger}\tilde{a}^{\dagger}\right]\left|0\tilde{0}\right\rangle\\&=\mathrm{i}\frac{\partial}{\partial\varphi}\left|\tau\right\rangle,\end{aligned} \tag{9}$$

this brings much convenience for solving some master equations for generalized phase diffusion models. In the next section we shall show that by enlarging the light field operator ρ of real mode to the state $\left|\rho\right\rangle=\rho\left|\tau=0\right\rangle$ with the fictitious mode we can directly transform many density operator equations to c-number equations in the entangled state representation, which provides a new view for analysing phase diffusion process of light field.

3. New method of solving equations for the phase diffusion process

By virtue of these properties we can examine density operator equations in the entangled state representation. For example, for a harmonic oscillator governed by $\omega a^{\dagger}a$, ρ's Heisenberg equation is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=-\mathrm{i}\omega[a^{\dagger}a,\rho]. \tag{10}$$

Operating the both sides of Eq.(10) on the state $\left|\tau=0\right\rangle$, and denoting

$$\left|\rho\right\rangle=\rho\left|\tau=0\right\rangle, \tag{11}$$

then using Eqs.(5) and (6) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\rho\right\rangle=\mathrm{i}\omega\left(\tilde{a}^{\dagger}\tilde{a}-a^{\dagger}a\right)\left|\rho\right\rangle. \tag{12}$$

Projecting Eq.(12) onto the entangled state representation and then using Eq.(9) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\tau\right|\rho\left\rangle&=\mathrm{i}\omega\left\langle\tau\right|\left(\tilde{a}^{\dagger}\tilde{a}-a^{\dagger}a\right)\left|\rho\right\rangle\\&=\omega\frac{\partial}{\partial\varphi}\left\langle\tau\right|\rho\left\rangle,\end{aligned} \tag{13}$$

this form of equation is simple, the solution of Eq.(13) is

$$\left\langle\tau\right|\rho\left\rangle=f\left(\frac{\varphi}{\omega}+t\right)R\left(r\right),\quad r\equiv|\tau|, \tag{14}$$

which indicates that there exists a phase factor $\mathrm{e}^{\mathrm{i}\omega t}$ in the time evolution of ρ.

For the phase diffusion process of a harmonic oscillator, the density operator satisfies the equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=(-\mathrm{i})^{2}\gamma[a^{\dagger}a,[a^{\dagger}a,\rho]]. \tag{15}$$

Operating the both sides of Eq.(15) on $\left|\tau=0\right\rangle$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|\rho\right\rangle&=\gamma[2a^{\dagger}a\rho a^{\dagger}a-\rho\left(a^{\dagger}a\right)^{2}-\left(a^{\dagger}a\right)^{2}\rho]\left|\tau=0\right\rangle\\&=\gamma[2a^{\dagger}a\tilde{a}^{\dagger}\tilde{a}-\left(\tilde{a}^{\dagger}\tilde{a}\right)^{2}-\left(a^{\dagger}a\right)^{2}]\left|\rho\right\rangle\\&=-\gamma\left(\tilde{a}^{\dagger}\tilde{a}-a^{\dagger}a\right)^{2}\left|\rho\right\rangle.\end{aligned} \tag{16}$$

Due to Eq.(9), Eq.(16) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\tau\right|\rho\left\rangle=\gamma\frac{\partial^{2}}{\partial\varphi^{2}}\left\langle\tau\right|\rho\left\rangle. \tag{17}$$

Making the variable separation, $\left\langle\tau\right|\rho\left\rangle=T\left(t\right)\Phi\left(\varphi\right)R\left(r\right)$, and substituting it into Eq.(17) we obtain

$$\frac{T'}{T}=\gamma\frac{\Phi''\left(\varphi\right)}{\Phi\left(\varphi\right)}=-\nu, \tag{18}$$

where ν is the variable-separation constant. Thus we conclude

$$\begin{aligned}&\left\langle\tau\right|\rho\left\rangle\\&=\mathrm{e}^{-\nu t}\left[A\cos\left(\sqrt{\frac{\nu}{\gamma}}\varphi\right)+B\sin\left(\sqrt{\frac{\nu}{\gamma}}\varphi\right)\right]R\left(r\right),\end{aligned} \tag{19}$$

where A, B are two integration constant depending on the initial state, Eq.(19) manifestly exhibits the phase diffusion process. If we take $\nu=\gamma$, then $\left\langle\tau\right|\rho\left\rangle=\mathrm{e}^{-\gamma t}\left[A\cos\varphi+B\sin\varphi\right]R\left(r\right)$.

4. Some generalized phase diffusion models

Now we generalize Eq.(15) in several aspects.

1) When it is extended as

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= (-\mathrm{i})^3\gamma[a^\dagger a,[a^\dagger a,[a^\dagger a,\rho]]]\\ &= \mathrm{i}\gamma[a^\dagger a,(2a^\dagger a\rho a^\dagger a-\rho a^\dagger aa^\dagger a-a^\dagger aa^\dagger a\rho)]\\ &= \mathrm{i}\gamma[2(a^\dagger a)^2\rho a^\dagger a-a^\dagger a\rho(a^\dagger a)^2-(a^\dagger a)^3\rho\\ &\quad -2a^\dagger a\rho(a^\dagger a)^2+\rho(a^\dagger a)^3+(a^\dagger a)^2\rho a^\dagger a], \end{aligned}\tag{20}$$

operating the both sides of Eq.(20) on $|\tau=0\rangle$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle &= \mathrm{i}\gamma[3\widetilde{a}^\dagger\widetilde{a}(a^\dagger a)^2-3(\widetilde{a}^\dagger\widetilde{a})^2a^\dagger a\\ &\quad -(a^\dagger a)^3+(\widetilde{a}^\dagger\widetilde{a})^3|\rho\rangle\\ &= \mathrm{i}\gamma\left(\widetilde{a}^\dagger\widetilde{a}-a^\dagger a\right)^3|\rho\rangle. \end{aligned}\tag{21}$$

Then projecting onto $\langle\tau|$ we obtain a simple equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau|\,\rho\rangle=-\gamma\frac{\partial^3}{\partial\varphi^3}\langle\tau|\,\rho\rangle. \tag{22}$$

By induction we infer that for the following generalized diffusion equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=(-\mathrm{i})^n\gamma[a^\dagger a,[a^\dagger a,[\ldots[a^\dagger a,\rho]]\ldots], \tag{23}$$

where there are n commutators, we can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau|\,\rho\rangle=(-\mathrm{i})^n\gamma\frac{\partial^n}{\partial\varphi^n}\langle\tau|\,\rho\rangle, \tag{24}$$

so

$$\langle\tau|\,\rho\rangle=C\mathrm{e}^{-\nu t}\exp\left[-\sqrt[n]{\frac{-\nu}{\gamma}}R(r)\right], \tag{25}$$

which also exhibits phase diffusion.

2) When ρ's equation is

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= (-\mathrm{i})^2\omega[a^\dagger a,[a^\dagger a,\rho]]\\ &\quad +g\left(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a\right), \end{aligned}\tag{26}$$

where the second round bracket represents dissipation, from Eq.(26) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle &= \left[-\omega\left(\widetilde{a}^\dagger\widetilde{a}-a^\dagger a\right)^2+g\left(2a\tilde{a}-a^\dagger a-\widetilde{a}^\dagger\widetilde{a}\right)\right]|\rho\rangle\\ &= \left[-\omega\left(\widetilde{a}^\dagger\widetilde{a}-a^\dagger a\right)^2+g\left(\left(\tilde{a}-a^\dagger\right)a-\left(\widetilde{a}^\dagger-a\right)\widetilde{a}\right)\right]|\rho\rangle, \end{aligned}\tag{27}$$

then using Eqs.(3), (4), (7) and (8) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau|\,\rho\rangle &= \left[\omega\frac{\partial^2}{\partial\varphi^2}\langle\tau|\,\rho\rangle+g\langle\tau|\,(\tau\widetilde{a}-\tau^*a)\right]|\rho\rangle\\ &= \left[\omega\frac{\partial^2}{\partial\varphi^2}+g\left(\tau\left[-\left(\frac{\partial}{\partial\tau}+\frac{\tau^*}{2}\right)\right]-\tau^*\left(\frac{\partial}{\partial\tau^*}+\frac{\tau}{2}\right)\right)\right]\langle\tau|\,\rho\rangle\\ &= \left[\omega\frac{\partial^2}{\partial\varphi^2}+g\left(-\tau\frac{\partial}{\partial\tau}-\tau^*\frac{\partial}{\partial\tau^*}-\tau^*\tau\right)\right]\langle\tau|\,\rho\rangle. \end{aligned}\tag{28}$$

Noting

$$\begin{aligned}&\frac{\partial}{\partial\tau}=\frac{1}{2}\mathrm{e}^{-\mathrm{i}\varphi}\left(\frac{\partial}{\partial r}-\frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi}\right),\\ &\frac{\partial}{\partial\tau^*}=\frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi}\left(\frac{\partial}{\partial r}+\frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi}\right),\\ &r=|\tau|, \end{aligned}\tag{29}$$

then from Eq.(28) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau|\,\rho\rangle=\left[\omega\frac{\partial^2}{\partial\varphi^2}-g\left(r\frac{\partial}{\partial r}+r^2\right)\right]\langle\tau|\,\rho\rangle, \tag{30}$$

where $\langle\tau|\,\rho\rangle$ is separable, writing $\langle\tau|\,\rho\rangle = T(t)R(r)\Phi(\varphi)$, then from Eq.(30) we obtain

$$\frac{T'}{T}=\omega\frac{\Phi''(\varphi)}{\Phi(\varphi)}-g\left(r\frac{R'}{R}+r^2\right)=-\nu, \tag{31}$$

where $-\nu$ is the variable-separable constant. It is easily seen $T(t)=\mathrm{e}^{-\nu t}$, so

$$\omega\frac{\Phi''(\varphi)}{\Phi(\varphi)}=g\left(r\frac{R'}{R}+r^2\right)-\nu. \tag{32}$$

Let $\Phi(\varphi)=\mathrm{e}^{\mathrm{i}k\varphi}$, from Eq.(32) we obtain

$$\frac{\nu-\omega k^2}{rg}-r=\frac{R'}{R}=d\ln R,$$

$$R = \exp\left\{\int \mathrm{d}r \left(\frac{\nu - \omega k^2}{rg} - r\right)\right\}, \qquad (33)$$

so

$$\langle \tau |\ \rho\rangle = \mathrm{e}^{\mathrm{i}k\varphi - \frac{r^2}{2} - \nu t} r^{\left(\nu - \omega k^2\right)/g}. \qquad (34)$$

Thus one can see that the entangled state representation brings much convenience for solving master equations of generalized phase diffusion models.

3) When ρ's equation is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathrm{i}\lambda\left[a^\dagger + a, \rho\right] + (-\mathrm{i})^2\omega[a^\dagger a, [a^\dagger a, \rho]], \qquad (35)$$

where the first term represents an action of external force, then acting Eq.(35) on $|\tau = 0\rangle$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle \ = \ &\{\mathrm{i}\lambda\left(a^\dagger + a - \tilde{a}^\dagger - \tilde{a}\right) \\ &-\omega\left(\widetilde{a}^\dagger\widetilde{a} - a^\dagger a\right)^2\}|\rho\rangle\,. \end{aligned} \qquad (36)$$

Using Eqs.(3) and (30) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau|\ \rho\rangle = \left\{2\mathrm{i}\lambda r\cos\varphi + \omega\frac{\partial^2}{\partial\varphi^2}\right\}\langle\tau|\ \rho\rangle\,. \qquad (37)$$

One can also use variable separation method to solve this equation, $\langle\tau|\ \rho\rangle = T(t)\,\Phi(\varphi)\,R(r)$,

$$\frac{T'(t)}{T(t)} = \left\{2\mathrm{i}\lambda r\cos\varphi + \omega\frac{\Phi''(\varphi)}{\Phi(\varphi)}\right\} = -\nu, \qquad (38)$$

where

$$\omega\,\Phi''(\varphi) + (\nu + 2\mathrm{i}\lambda r\cos\varphi)\,\Phi(\varphi) = 0 \qquad (39)$$

is a Mathieu equation whose solution has been discussed profoundly in Ref.[11]. Therefore we reach an interesting conclusion that phase diffusion under an action of external force may lead to a process governed by the Mathieu equation.

In summary, we have shown that the remarkable property of the entangled state

$$\langle\tau|\left(\tilde{a}^\dagger\tilde{a} - a^\dagger a\right) = -\mathrm{i}\frac{\partial}{\partial\varphi}\langle\tau| \qquad (40)$$

helps us to analyse master equations for generalized phase diffusion models in terms of the wavefunction $\langle\tau|\ \rho\rangle$, which brings much convenience. This is an alternate approach conveniently tackling this task in a concise way, and provides us with a fresh view of the time-evolution of density operators, other than the P-representation or the Wigner function approaches.

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