Vol 18 No 3, March 2009
1674-1056/2009/18(03)/0902-08
Chinese Physics B
© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

Wigner function of the thermo number states*

Hu Li-Yun(胡利云)[a)†] and Fan Hong-Yi(范洪义)[a)b)]

[a)] *Department of Physics, Shanghai Jiaotong University, Shanghai* 200030, *China*

[b)] *Department of Material Science and Engineering, University of Science and Technology of China, Hefei* 230026, *China*

(Received 27 June 2008; revised manuscript received 27 July 2008)

Based on thermo field dynamics (TFD) and using the thermo Wigner operator in the thermo entangled state representation we derive the Wigner function of number states at finite temperature (named thermo number states). The figure of Wigner function shows that its shape gets smoothed as the temperature rises, implying that the quantum noise becomes larger.

Keywords: thermo Wigner operator, entangled state representation, thermo number states, thermo field dynamics

PACC: 0365, 4250

1. Introduction

Wigner distribution function[1] of quantum states is an important topic in quantum mechanics, quantum statistics and quantum optics. In Ref.[2], Vogel and Risken pointed out that the Wigner distribution can be obtained by tomographic inversion of a set of measured probability distributions $P_\phi(x_\phi)$ of the quadrature amplitude. Later, Smithey *et al*[3] also pointed out that once the distribution $P_\phi(x_\phi)$ are obtained, one can use the inverse Radon transformation, familiar in tomographic imaging to obtain the Wigner distribution and density matrix.[3] For a very good reading material about Wigner function in quantum optics theory we cite Schleich's book 'Quantum Optics in Phase Space'.[4] It has been known that for a density operator ρ the Wigner function is defined as[1,4,5]

$$W(p,q) = \mathrm{Tr}\left[\rho\Delta(p,q)\right], \tag{1}$$

where $\Delta(p,q)$ is the Wigner operator. For one freedom case, the Wigner function is defined as[4]

$$W(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle q+\frac{v}{2}\right|\rho\left|q-\frac{v}{2}\right\rangle \mathrm{e}^{-\mathrm{i}pv}\mathrm{d}v, \tag{2}$$

where $|q\rangle$ is the coordinate eigenvector of Q, $Q|q\rangle = q|q\rangle$. Correspondingly, the Wigner operator is expressed as

$$\Delta(p,q) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}v\left|q-\frac{v}{2}\right\rangle\left\langle q+\frac{v}{2}\right|\mathrm{e}^{-\mathrm{i}pv}. \tag{3}$$

$W(p,q)$ in phase space has its physical meaning that its marginal distribution gives probability of measurement in coordinate space and momentum space. This can be clearly seen from the explicit normally ordered form of $\Delta(p,q)$

$$\begin{aligned}\Delta(p,q) =& \frac{1}{\pi}{:}\,\mathrm{e}^{-(q-Q)^2-(p-P)^2}\,{:}\\ =& \frac{1}{\pi}{:}\exp[-2(a^\dagger-\alpha^*)(a-\alpha)]{:}\,,\end{aligned} \tag{4}$$

where $\alpha = \frac{1}{\sqrt{2}}(q+\mathrm{i}p)$, $:\ :$ denotes normal ordering, $a^\dagger$ and a are Bose creation and annihilation operators, respectively, which satisfy $[a,a^\dagger]=1$ and are related to Q and P by $Q=(a+a^\dagger)/\sqrt{2}$ and $P=(a-a^\dagger)/(\mathrm{i}\sqrt{2})$. Equation (4) can be directly derived by performing the integration in Eq.(3) by virtue of the technique of integration within an ordered product (IWOP) of operators[6,7] and the vacuum projector $|0\rangle\langle 0| =: \exp\left(-a^\dagger a\right):$. The Wigner function of number state is well known,

$$\begin{aligned}W_n(q,p) =& \langle n|\,\Delta(p,q)\,|n\rangle\\ =& \frac{(-1)^n}{\pi}\exp\left[-2|\alpha|^2\right]\mathrm{L}_n\left(4|\alpha|^2\right),\end{aligned} \tag{5}$$

where $\mathrm{L}_n(x)$ is the Laguerre polynomials, whose definition is[8]

$$\mathrm{L}_m(x) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{x^k}{k!}. \tag{6}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos 10775097 and 10874174).

†Corresponding author. E-mail: hlyun2008@126.com

In nature, most systems are not isolated, but are immersed in a 'thermal reservoir'; excitation and de-excitation processes of a system are influenced by its energy exchange with reservoirs. For an excitation process, a question arises naturally: what is the Wigner function of number state at finite temperature? So far as we know, this question has not been clearly reported in the literature before. In order to clearly understand the effect of temperature on the Wigner function of number state, we introduce thermo Wigner operator, expressed in terms of the thermo entangled state representation, into thermo field dynamics (TFD) theory of Umezawa–Takahashi.[9]

This paper is arranged as follows. In Section 2 after briefly reviewing the TFD formalism[10] we introduce the thermo squeezing operator in the coherent thermal state representation (thermo entangled state representation). In Section 3 we introduce the thermo Wigner operator in the same representation, which brings great convenience for calculating the thermo Wigner functions. In Section 4 we calculate the Wigner function of thermo number state; by thermo number state we mean the number states at finite temperature, for different particle numbers, and analyse the effect of temperature. By drawing figures of Wigner function we see that their shapes get smoothed with rising temperature, implying that the quantum noise becomes larger. The quantum fluctuation of quadrature amplitudes in thermo number state is derived in Section 5.

2. Thermo squeezing operator in thermo entangled state representation

The TFD theory was invented by Takahashi and Umezawa to describe the thermal communication between systems and reservoirs more conveniently. In order to calculate the thermal reservoir effect in a natural way, Takahashi and Umezawa proposed the TFD theory, which can convert the calculations of ensemble averages at finite temperature in equivalent expectation values with a pure state, i.e.

$$\langle A\rangle = Z^{-1}(\beta)\,\mathrm{Tr}\left(A\mathrm{e}^{-\beta H}\right), \tag{7}$$

into the pure state average[9]

$$\langle A\rangle = \langle 0(\beta)|\,A\,|0(\beta)\rangle\,, \tag{8}$$

where $Z(\beta)$ is the partition function, H is the system's Hamiltonian. The worthwhile convenience is at the expense of introducing a fictitious field (or called a tilde-conjugate field, denoted as operator $\tilde{a}^{\dagger}$) in the extending Hilbert space, thus the original optical field state $|n\rangle$ in the Hilbert space $\mathcal{H}$ is accompanied by a tilde state $|\tilde{n}\rangle$ in $\tilde{\mathcal{H}}$. A similar rule holds for operators: every annihilation operator a acting on $\mathcal{H}$ has an image $\tilde{a}$ acting on $\tilde{\mathcal{H}}$. For a harmonic oscillator system,

$$|0(\beta)\rangle = \exp\left[a^{\dagger}\tilde{a}^{\dagger}\tanh\theta\right]|0,\tilde{0}\rangle = S(\theta)\,|0,\tilde{0}\rangle\,, \tag{9}$$

where

$$S(\theta) \equiv \exp\left[-\theta\left(a\tilde{a} - a^{\dagger}\tilde{a}^{\dagger}\right)\right] \tag{10}$$

is the thermo squeezing operator which transforms the zero-temperature vacuum $|0,\tilde{0}\rangle$ into the thermo vacuum state $|0(\beta)\rangle$, and $\beta = 1/kT$, k is the Boltzmann constant, T is the system temperature; θ is related to the Bose distribution by

$$\tanh\theta = \exp\left(-\frac{\hbar\omega}{2kT}\right). \tag{11}$$

At finite temperature, according to TFD, the number state $|n\rangle$ is replaced by $|n,\tilde{n}\rangle$. In Ref.[10] we proposed the coherent thermo state representation

$$|\tau\rangle = \exp\left[-\frac{1}{2}|\tau|^2 + \tau a^{\dagger} - \tau^{*}\tilde{a}^{\dagger} + a^{\dagger}\tilde{a}^{\dagger}\right]|0,\tilde{0}\rangle\,. \tag{12}$$

The completeness and orthogonality of $|\tau\rangle$ are, respectively,

$$\int\frac{\mathrm{d}^2\tau}{\pi}\,|\tau\rangle\langle\tau| = 1,\ \ \langle\tau'|\,\tau\rangle = \pi\delta(\tau-\tau')\delta(\tau^{*}-\tau'^{*}), \tag{13}$$

where $|\tau\rangle$ obeys the eigenvector equations

$$\left(a-\tilde{a}^{\dagger}\right)|\tau\rangle = \tau\,|\tau\rangle\,,\ \ \left(\tilde{a}-a^{\dagger}\right)|\tau\rangle = \tau^{*}\,|\tau\rangle\,. \tag{14}$$

Comparing the form of Eq.(14) with the bipartite entangled state of EPR[11−13] we can also name $|\tau\rangle$ the thermo entangled state. With use of $|\tau\rangle$ the thermo squeezing operator is neatly expressed as[14,15]

$$S(\theta) = \int\frac{\mathrm{d}^2\tau}{\pi\mu}\,|\tau/\mu\rangle\langle\tau|\,,\ \mu^2 = \frac{1+\tanh\theta}{1-\tanh\theta}, \tag{15}$$

which leads naturally to

$$S(\theta)\,|\tau\rangle = \frac{1}{\mu}\,|\tau/\mu\rangle\,, \tag{16}$$

and $|0(\beta)\rangle$ is the thermo vacuum state,

$$\begin{aligned}|0(\beta)\rangle &= \int\frac{\mathrm{d}^2\tau}{\pi\mu}\,|\tau/\mu\rangle\langle\tau\,|0,\tilde{0}\rangle\\ &= \int\frac{\mathrm{d}^2\tau}{\pi\mu}\,|\tau/\mu\rangle\,\mathrm{e}^{-\frac{1}{2}|\tau|^2}.\end{aligned} \tag{17}$$

Equation (15) brings convenience for calculating the Wigner function of the thermo number state $S(\theta)|n,\tilde{n}\rangle$ (number states at finite temperature).

3. Thermo Wigner operator in thermo entangled state representation

Recall that the explicit normally ordered form of Wigner operator $\Delta(p,q)$ is

$$\Delta(p,q)=\frac{1}{\pi}\colon \exp[-2(a^{\dagger}-\alpha^{*})(a-\alpha)]\colon . \tag{18}$$

By virtue of the coherent thermo state representation we can introduce the thermo Wigner operator

$$\Delta_T(\sigma,\gamma)=\int\frac{\mathrm{d}^2\tau}{\pi^3}|\sigma-\tau\rangle\langle\sigma+\tau|\exp(\tau\gamma^{*}-\gamma\tau^{*}), \tag{19}$$

where $\gamma=\alpha+\varepsilon^{*}$, $\sigma=\alpha-\varepsilon^{*}$. By using the IWOP technique, from Eq.(19), we can obtain

$$\begin{aligned}\Delta_T(\sigma,\gamma)=&\pi^{-2}\colon \exp\left[-2(a^{\dagger}-\alpha^{*})(a-\alpha)\right.\\&\left.-2(\tilde{a}^{\dagger}-\varepsilon^{*})(\tilde{a}-\varepsilon)\right]\colon ,\end{aligned} \tag{20}$$

from which we see that $\Delta_T(\sigma,\gamma)$ includes the fictitious part, so we have to take the trace for 'tilde' field, denoted as Tr_t. By virtue of the coherent state representation in the 'tilde' mode, we have

$$\begin{aligned}&2\mathrm{Tr}_t\left[\Delta_T(\sigma,\gamma)\right]\\=&2\mathrm{Tr}_t\left[\Delta_T(\sigma,\gamma)\int\frac{\mathrm{d}^2\tilde{z}}{\pi}|\tilde{z}\rangle\langle\tilde{z}|\right]\\=&\pi^{-2}2\int\frac{\mathrm{d}^2\tilde{z}}{\pi}\langle\tilde{z}|\colon\exp\left[-2(a^{\dagger}-\alpha^{*})(a-\alpha)\right.\\&\left.-2(\tilde{a}^{\dagger}-\varepsilon^{*})(\tilde{a}-\varepsilon)\right]\colon|\tilde{z}\rangle\\=&\frac{1}{\pi}\colon\exp[-2(a^{\dagger}-\alpha^{*})(a-\alpha)]\colon\\=&\Delta(p,q)=2\int\mathrm{d}^2\varepsilon\Delta_T(\sigma,\gamma).\end{aligned} \tag{21}$$

From Eq.(4) we know that the relationship of $\Delta_T(\sigma,\gamma)$ with $\Delta(p,q)$ is

$$\Delta(p,q)=2\mathrm{Tr}_t\left[\Delta_T(\sigma,\gamma)\right]=2\int\mathrm{d}^2\varepsilon\Delta_T(\sigma,\gamma), \tag{22}$$

so

$$\begin{aligned}W(p,q)=&2\int\mathrm{d}^2\varepsilon W_T(\sigma,\gamma),\\W_T(\sigma,\gamma)=&\mathrm{Tr}\left[\rho\Delta_T(\sigma,\gamma)\right].\end{aligned} \tag{23}$$

We name $W_T(\sigma,\gamma)$ the thermo Wigner function whose trace over the 'tilde' component leads to the Wigner function of quantum states of the system.

4. Wigner function of the thermo number state

Wigner function of quantum states in phase space is an important topic in quantum mechanics, quantum statistics and quantum optics. Recently, Wigner functions of some quantum states have been calculated by using the coherent (entangled) state representation of Wigner operator.[16,17] In this section, using the coherent thermo state representation Eq.(22), we calculate the Wigner function of thermo excited state (thermo number state). For this purpose, we begin with expanding the coherent thermo state $|\tau\rangle$ as

$$|\tau\rangle=\mathrm{e}^{-|\tau|^2/2}\sum_{s,t=0}^{\infty}\frac{(-1)^t\mathrm{H}_{s,t}(\tau,\tau^{*})}{\sqrt{s!t!}}\left|s,\tilde{t}\right\rangle, \tag{24}$$

where $\left|s,\tilde{t}\right\rangle=\dfrac{a^{\dagger s}\tilde{a}^{\dagger t}}{\sqrt{s!t!}}\left|0\tilde{0}\right\rangle$ is the two-mode number state and $\mathrm{H}_{s,t}(\tau,\tau^{*})$ is the two-variable Hermite polynomial, defined as

$$\mathrm{H}_{m,n}(\epsilon,\varepsilon)=\sum_{k=0}^{\min(m,n)}\frac{(-1)^k m!n!\epsilon^{m-k}\varepsilon^{n-k}}{k!(m-k)!(n-k)!}, \tag{25}$$

whose generating function is

$$\exp\left[-tt'+\epsilon t+\varepsilon t'\right]=\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}\mathrm{H}_{m,n}(\epsilon,\varepsilon), \tag{26}$$

or

$$\mathrm{H}_{m,n}(\epsilon,\varepsilon)=\frac{\partial^{m+n}}{\partial t^m\partial t'^n}\exp\left[-tt'+\epsilon t+\varepsilon t'\right]|_{t=t'=0}. \tag{27}$$

From Eq.(24), we find

$$\langle\tau\,|n,\tilde{n}\rangle=\mathrm{e}^{-|\tau|^2/2}\frac{(-1)^n\mathrm{H}_{n,n}(\tau,\tau^{*})}{n!}. \tag{28}$$

On the other hand, from Eqs.(13), (15) and (19), it is easy to see that

$$\begin{aligned}S^{-1}(\theta)\Delta_T(\sigma,\gamma)S(\theta)=&\Delta_T\left(\mu\sigma,\frac{\gamma}{\mu}\right),\\\tanh\theta=&\frac{\mu^2-1}{\mu^2+1}.\end{aligned} \tag{29}$$

Then using Eqs.(26) and (29), and the integral formula,

$$\int\frac{\mathrm{d}^2\beta}{\pi}\mathrm{e}^{\varsigma|\beta|^2+\xi\beta+\eta\beta^{*}}=-\frac{1}{\varsigma}\mathrm{e}^{-\frac{\xi\eta}{\varsigma}},\quad \mathrm{Re}\varsigma<0, \tag{30}$$

we can calculate the thermo Wigner function

$$\begin{aligned}
&W_T(\sigma,\gamma)\\
=&\langle n,\tilde{n}|\, S^{-1}\Delta_T(\sigma,\gamma)\, S\,|n,\tilde{n}\rangle\\
=&\langle n,\tilde{n}|\,\Delta_T\left(\mu\sigma,\frac{\gamma}{\mu}\right)|n,\tilde{n}\rangle\\
=&\frac{\mathrm{e}^{-\mu^2|\sigma|^2-\frac{|\gamma|^2}{\mu^2}}}{\pi^2 n!n!}\frac{\partial^{2n}}{\partial t^n\partial r^n}\frac{\partial^{2n}}{\partial t'^n\partial r'^n}\exp\left[-rt-r't\right]\\
&\times\exp\left[r\left(\frac{1}{\mu}\gamma^*+\mu\sigma^*\right)+t'\left(\frac{\gamma}{\mu}+\sigma\mu\right)\right.\\
&+t\left(\mu\sigma^*-\frac{1}{\mu}\gamma^*\right)\\
&\left.+r'\left(\sigma\mu-\frac{\gamma}{\mu}\right)\right]_{t'=r'=t=r=0}.
\end{aligned}\tag{31}$$

Employing the generation function of two-variable Hermite polynomial (26) we obtain

$$\begin{aligned}
W_T(\sigma,\gamma)=&\frac{\mathrm{e}^{-\mu^2|\sigma|^2-\frac{|\gamma|^2}{\mu^2}}}{\pi^2 n!n!}\mathrm{H}_{n,n}\left(\frac{1}{\mu}\gamma^*+\mu\sigma^*,\frac{\gamma}{\mu}+\sigma\mu\right)\\
&\times\mathrm{H}_{n,n}\left(\mu\sigma^*-\frac{1}{\mu}\gamma^*,\sigma\mu-\frac{\gamma}{\mu}\right).
\end{aligned}\tag{32}$$

Then noting the relationship between $\mathrm{H}_{m,m}$ and the Laguerre polynomials L_m,

$$\mathrm{H}_{m,m}(r,r^*)=m!\,(-1)^m\,\mathrm{L}_m\left(|r|^2\right),\tag{33}$$

we can further rewrite Eq.(32) as

$$\begin{aligned}
W_T(\sigma,\gamma)\equiv&\langle n,\tilde{n}|\, S^{-1}\Delta_T(\sigma,\gamma)\, S\,|n,\tilde{n}\rangle\\
=&\frac{\mathrm{e}^{-\mu^2|\sigma|^2-\frac{|\gamma|^2}{\mu^2}}}{\pi^2}\mathrm{L}_n\left(\left|\frac{\gamma}{\mu}+\sigma\mu\right|^2\right)\\
&\times\mathrm{L}_n\left(\left|\sigma\mu-\frac{\gamma}{\mu}\right|^2\right).
\end{aligned}\tag{34}$$

From another point of view, using $\gamma=\alpha+\varepsilon^*,\ \ \sigma=\alpha-\varepsilon^*$, we can also rewrite Eq.(32) as ($W_T(\sigma,\gamma)\to W_T(\alpha,\varepsilon)$)

$$\begin{aligned}
&W_T(\alpha,\varepsilon)\\
=&\frac{1}{\pi^2 n!n!}\frac{\partial^{2n}}{\partial r^n\partial t^n}\frac{\partial^{2n}}{\partial r'^n\partial t'^n}\exp\left[-rt-r't'\right.\\
&-2\left(|\alpha|^2+|\varepsilon|^2\right)\cosh(2\theta)+2\left(\alpha\varepsilon+\alpha^*\varepsilon^*\right)\sinh(2\theta)\\
&+2\left(t'\alpha^*+r'\alpha-r\varepsilon-t\varepsilon^*\right)\sinh\theta\\
&\left.+2\left(r\alpha^*+t\alpha-t'\varepsilon-r'\varepsilon^*\right)\cosh\theta\right]_{t'=r'=t=r=0},
\end{aligned}\tag{35}$$

where we have set $\mu=\mathrm{e}^{\theta}$,

$$\sinh\theta=\frac{\mu^2-1}{2\mu},\quad\cosh\theta=\frac{\mu^2+1}{2\mu}.\tag{36}$$

From Eq.(23) we obtain the Wigner function of $S\,|n,\tilde{n}\rangle$, ($\alpha=(q+\mathrm{i}p)/\sqrt{2}$)

$$\begin{aligned}
W(\alpha)=&2\int\mathtt{d}^2\varepsilon W_T(\alpha,\varepsilon)\\
=&\frac{\mathrm{sech}(2\theta)}{\pi n!n!}\exp\left[-2|\alpha|^2\,\mathrm{sech}(2\theta)\right]\frac{\partial^{2n}}{\partial r^n\partial t^n}\\
&\times\frac{\partial^{2n}}{\partial r'^n\partial t'^n}\exp\left[(rr'+tt')\tanh(2\theta)\right]\\
&\times\exp\left\{\mathrm{sech}(2\theta)\left[r't'-rt\right.\right.\\
&+2\left(r\alpha^*+t\alpha\right)\cosh\theta\\
&\left.\left.-2\left(t'\alpha^*+r'\alpha\right)\sinh\theta\right]\right\}_{t'=r'=t=r=0}.
\end{aligned}\tag{37}$$

Expanding the exponential item $\exp\left[(rr'+tt')\tanh(2\theta)\right]$ by series and then using the formula

$$\frac{\partial^{l+k}}{\partial\xi^l\partial\eta^k}\mathrm{H}_{m,n}(\xi,\eta)=\frac{m!n!\mathrm{H}_{m-l,n-k}(\xi,\eta)}{(m-l)!\,(n-k)!},\tag{38}$$

we can finally obtain

$$\begin{aligned}
W(\alpha)=&\frac{(-1)^n\,n!n!}{\pi\cosh^{2n+1}(2\theta)}\exp\left[-2|\alpha|^2\,\mathrm{sech}(2\theta)\right]\\
&\times\sum_{l,k=0}^{n}\frac{1}{l!k!}\frac{\left(-\mathrm{i}\sinh(2\theta)\right)^{l+k}}{\left[(n-l)!\,(n-k)!\right]^2}\\
&\times\mathrm{H}_{n-l,n-k}\left(\mathrm{i}E,\mathrm{i}E^*\right)\mathrm{H}_{n-k,n-l}\left(F,F^*\right),
\end{aligned}\tag{39}$$

where we have set

$$E=\frac{2\alpha\sinh\theta}{\sqrt{\cosh(2\theta)}},\quad F=\frac{2\alpha\cosh\theta}{\sqrt{\cosh(2\theta)}}.\tag{40}$$

When $n=0$, noticing that $\tanh\theta=\mathrm{e}^{-\frac{1}{2}\hbar\omega\beta}$, $\cosh^2\theta=\dfrac{1}{1-\mathrm{e}^{-\beta\hbar\omega}}$, $\sinh^2\theta=\dfrac{\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}$, Eq.(39) reduces to

$$W(\alpha)=\frac{1-\mathrm{e}^{-\beta\hbar\omega}}{\pi(1+\mathrm{e}^{-\beta\hbar\omega})}\mathrm{e}^{-2|\alpha|^2\frac{1-\mathrm{e}^{-\beta\hbar\omega}}{1+\mathrm{e}^{-\beta\hbar\omega}}},\tag{41}$$

which is just the Wigner function of thermo vacuum state $|0(\beta)\rangle$.

In particular, when $T\to0$, $\mathrm{e}^{-\beta\hbar\omega}\to\mathrm{e}^{-\infty}\to0$, $\sinh\theta\to0,\cosh\theta\to1$, $E\to0,F\to2\alpha$, we see

$$\begin{aligned}
W(\alpha)_{T\to0}=&\frac{1}{\pi n!}\mathrm{e}^{-2|\alpha|^2}\mathrm{H}_{n,n}(2\alpha,2\alpha^*)\\
=&\frac{(-1)^n}{\pi}\mathrm{e}^{-2|\alpha|^2}\mathrm{L}_n\left(4|\alpha|^2\right),
\end{aligned}\tag{42}$$

which is just the Wigner function of number state $|n\rangle$ at zero temperature Eq.(5).

Next, we discuss the changes in the Wigner function (WF) with variation of the parameters n and θ, (see Figs.1–3). We plot the Wigner function of single-mode number states in the phase space in Fig.1 for different values of n, which corresponds to zero temperature case $T=0$ ($\theta=0$). It is obvious that there is some negative region of the WF in the phase space, an evidence of the nonclassicality of the state.

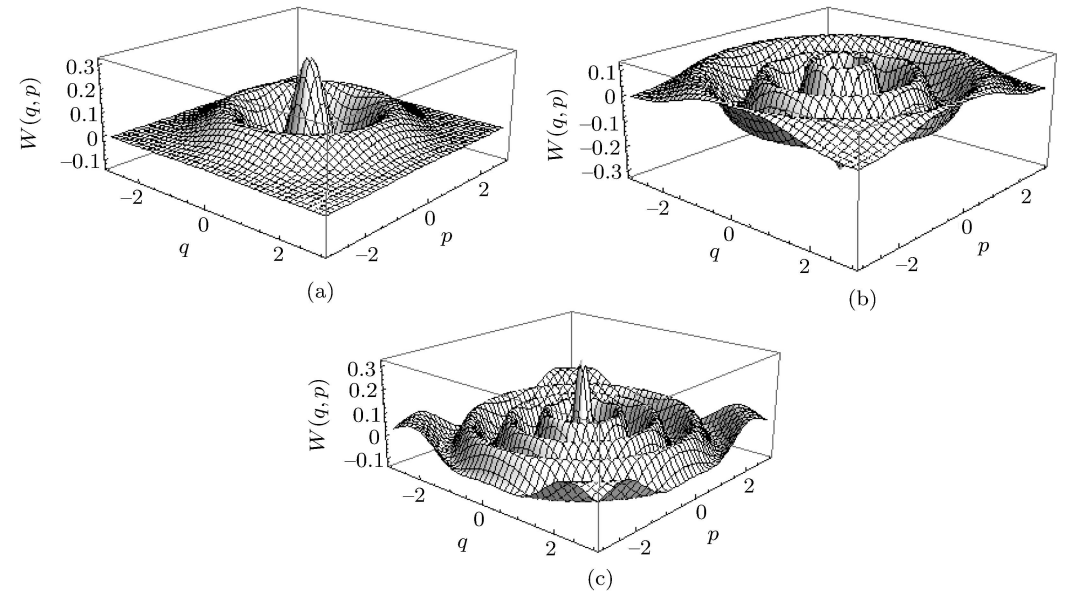


Fig.1. Wigner function of single-mode number states for different values of n. (a) n=2, (b) n=5, (c) n=8.

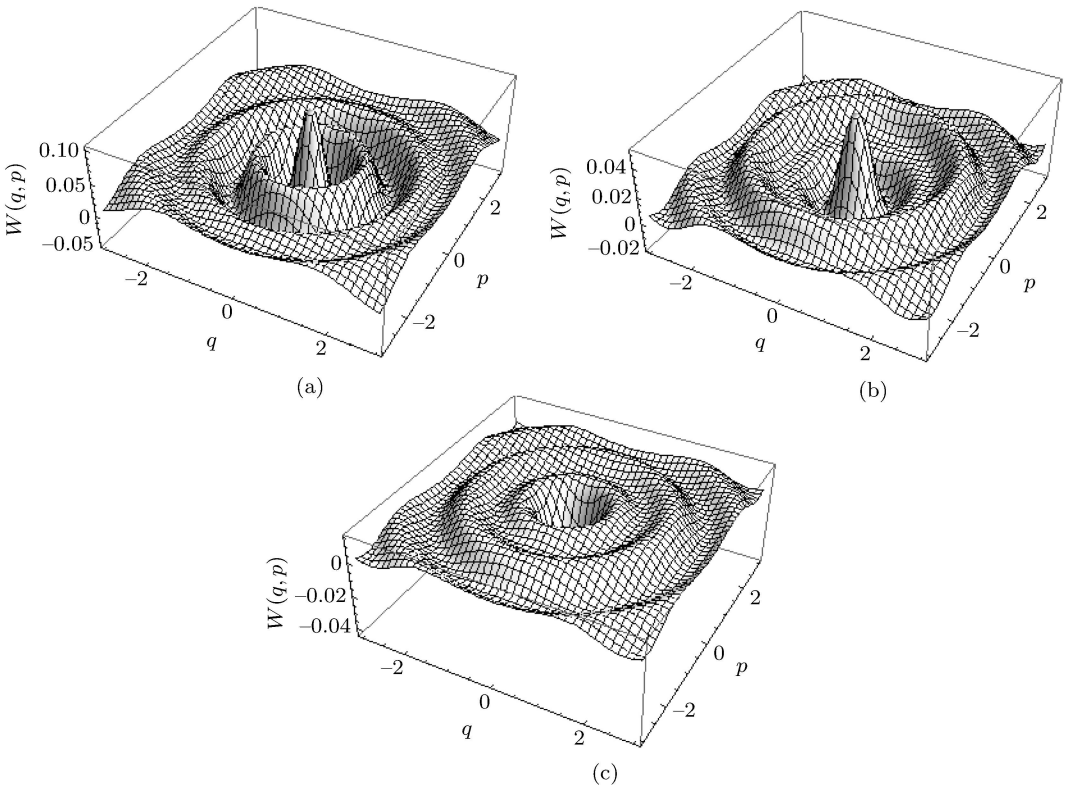


Fig.2. Wigner function of the thermo number states for $n = 5$ and different values of θ. (a) θ=0.2, (b) θ=0.5, (c) θ=0.8.

The case is true for Figs.2 and 3. From Fig.1 one can see that the 'oscillating frequency' of WF becomes larger with the increase of n. We plot the WF of thermo number states in the phase space in Fig.2 for $n = 5$ and different values of θ, from which one can see that the 'oscillating action' of WF becomes inconspicuous with the increasing temperature. Especially, when $\theta = 0.8$, a 'flat nappy' shape appears. For $\theta = 0.5$ and different values of n, we can draw a similar conclusion, see Fig.3 similar to Fig.1. These imply that the quantum noise becomes larger with the increase of T and n.

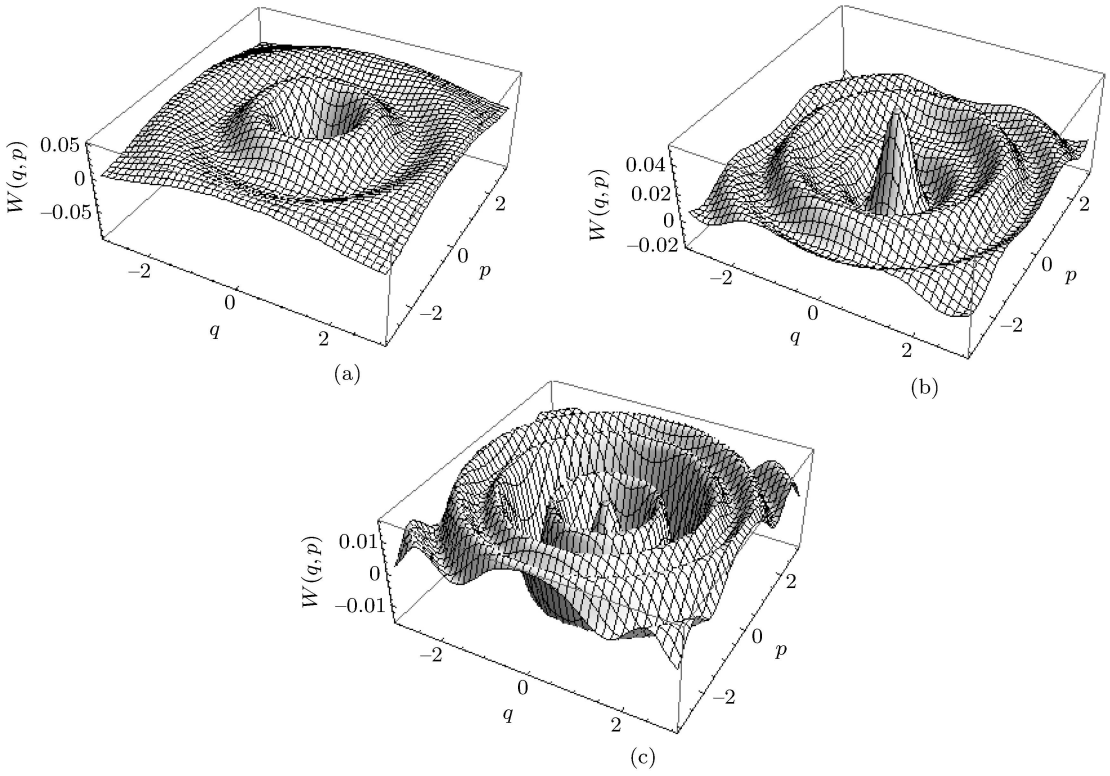


Fig.3. Wigner function of the thermo number states for $\theta = 0.5$ and different values of n. (a) n=2, (b) n=5, (c) n=8.

5. Quantum fluctuation of quadrature amplitudes in $S\left|n,\tilde{n}\right\rangle$

For the thermo excited system, we can introduce the quadrature phase amplitudes

$$q = \left(a + a^{\dagger}\right)/\sqrt{2},\ p = \mathrm{i}\left(a^{\dagger} - a\right)/\sqrt{2},$$
$$\tilde{q} = \left(\tilde{a} + \tilde{a}^{\dagger}\right)/\sqrt{2},\ \tilde{p} = \mathrm{i}\left(\tilde{a}^{\dagger} - \tilde{a}\right)/\sqrt{2}. \tag{43}$$

Noticing the transform relation

$$\begin{aligned} S^{\dagger}aS =& a\cosh\theta + \tilde{a}^{\dagger}\sinh\theta, \\ S^{\dagger}\tilde{a}S =& \tilde{a}\cosh\theta + a^{\dagger}\sinh\theta, \end{aligned} \tag{44}$$

we see that

$$\begin{aligned} S^{\dagger}\left(q - \tilde{q}\right)S &= \frac{1}{\mu}\left(q - \tilde{q}\right), \\ S^{\dagger}\left(p + \tilde{p}\right)S &= \frac{1}{\mu}\left(p + \tilde{p}\right). \end{aligned} \tag{45}$$

Thus the fluctuations of $(q - \tilde{q})$ and $(p + \tilde{p})$ in thermo excited number state $S\left|n,\tilde{n}\right\rangle$ are

$$\begin{aligned} \left[\Delta\left(q - \tilde{q}\right)\right]^2 &= \left\langle n,\tilde{n}\right| S^{\dagger}\left(q - \tilde{q}\right)^2 S\left|n,\tilde{n}\right\rangle \\ &= \frac{1}{\mu^2}\left\langle n,\tilde{n}\right|\left(q - \tilde{q}\right)^2\left|n,\tilde{n}\right\rangle \\ &= (2n+1)\,\frac{1 - \mathrm{e}^{-\hbar\omega/(2kT)}}{1 + \mathrm{e}^{-\hbar\omega/(2kT)}}, \end{aligned} \tag{46}$$

and

$$\left[\Delta\left(p + \tilde{p}\right)\right]^2 = (2n+1)\,\frac{1 - \mathrm{e}^{-\hbar\omega/(2kT)}}{1 + \mathrm{e}^{-\hbar\omega/(2kT)}}. \tag{47}$$

On the other hand, we notice that

$$\begin{aligned} &\left\langle n,\tilde{n}\right| S^{\dagger}q\tilde{q}S\left|n,\tilde{n}\right\rangle \\ =& \frac{1}{2}\left\langle n,\tilde{n}\right| S^{\dagger}\left(a + a^{\dagger}\right)\left(\tilde{a} + \tilde{a}^{\dagger}\right)S\left|n,\tilde{n}\right\rangle \\ =& (2n+1)\,\frac{\mathrm{e}^{-\hbar\omega/(2kT)}}{1 - \mathrm{e}^{-\hbar\omega/(kT)}}, \end{aligned} \tag{48}$$

and

$$\left\langle n,\tilde{n}\right| S^{\dagger}p\tilde{p}S\left|n,\tilde{n}\right\rangle = -\,(2n+1)\,\frac{\mathrm{e}^{-\hbar\omega/(2kT)}}{1 - \mathrm{e}^{-\hbar\omega/(kT)}}. \tag{49}$$

From Eqs.(46) and (47), it then follows that

$$\begin{aligned}\langle n,\tilde{n}|\, S^{\dagger}q^{2}S\,|n,\tilde{n}\rangle &= \langle n,\tilde{n}|\, S^{\dagger}\tilde{q}^{2}S\,|n,\tilde{n}\rangle \\ &= \frac{2n+1}{2}\frac{1+\mathrm{e}^{-\omega\hbar/(Tk)}}{1-\mathrm{e}^{-\omega\hbar/(Tk)}}, \end{aligned} \quad (50)$$

and

$$\langle n,\tilde{n}|\, S^{\dagger}p^{2}S\,|n,\tilde{n}\rangle = \frac{2n+1}{2}\frac{1+\mathrm{e}^{-\omega\hbar/(Tk)}}{1-\mathrm{e}^{-\omega\hbar/(Tk)}}. \quad (51)$$

Thus

$$\Delta p\Delta q = \frac{2n+1}{2}\frac{1+\mathrm{e}^{-\omega\hbar/(Tk)}}{1-\mathrm{e}^{-\omega\hbar/(Tk)}}, \quad (52)$$

which arises with the increase of n and T. In particularly, when $|n,\tilde{n}\rangle = \left|0,\tilde{0}\right\rangle$, $S\,|n,\tilde{n}\rangle$ reduces to $|0\,(\beta)\rangle$, and Eq.(52) gives the fluctuation of thermo vacuum state.

On the other hand, using Eq.(44) we have

$$\langle n,\tilde{n}|\, S^{\dagger}a^{\dagger}aS\,|n,\tilde{n}\rangle = n\cosh^{2}\theta+(1+n)\sinh^{2}\theta, \quad (53)$$

and

$$\begin{aligned}&\langle n,\tilde{n}|\, S^{\dagger}\left(a^{\dagger}a\right)^{2}S\,|n,\tilde{n}\rangle \\ =&n^{2}\cosh^{4}\theta+(1+n)^{2}\sinh^{4}\theta \\ &+(2n+1)^{2}\sinh^{2}\theta\cosh^{2}\theta, \end{aligned} \quad (54)$$

then

$$\begin{aligned}\Delta\left(a^{\dagger}a\right) =&\sqrt{2n^{2}+2n+1}\sinh\theta\cosh\theta \\ =&\frac{\mathrm{e}^{-\frac{\omega\hbar}{2kT}}}{1-\mathrm{e}^{-\frac{\omega\hbar}{kT}}}\sqrt{2n^{2}+2n+1}. \end{aligned} \quad (55)$$

It is easy to see that the fluctuation of particle number becomes larger when n and T are increased.

In fact, using the relation $p^{2}+q^{2}=2a^{\dagger}a+1$, from Eqs.(50) and (51) we can also derive Eq.(53). In particular, when $n=0$, Eqs.(54) and (55) reduce to the case of thermo vacuum state. On the other hand, it is interesting to notice that the average value in Eq.(53) is similar in form to that in single-mode squeezed number state $S_{1}\,|n\rangle = \exp\left[\frac{\lambda}{2}\left(a^{2}-a^{\dagger 2}\right)\right]|n\rangle$,

$$\langle n|\, S_{1}^{\dagger}a^{\dagger}aS_{1}\,|n\rangle = n\cosh^{2}\lambda+(n+1)\sinh^{2}\lambda, \quad (56)$$

but Eq.(54) is different from

$$\begin{aligned}&\langle n|\, S_{1}^{\dagger}\left(a^{\dagger}a\right)^{2}S_{1}\,|n\rangle \\ =&n^{2}\cosh^{4}\lambda+(n+1)^{2}\sinh^{4}\lambda \\ &+\left((2n+1)^{2}+1\right)\sinh^{2}\lambda\cosh^{2}\lambda, \end{aligned} \quad (57)$$

it follows

$$\Delta_{1}\left(a^{\dagger}a\right) = \sqrt{2n^{2}+2n+2}\sinh\lambda\cosh\lambda. \quad (58)$$

which is different from the number fluctuation in (55). In addition, the fluctuation of quadrature phase amplitudes in $S_{1}\,|n\rangle$ can be calculated as

$$\begin{aligned}\left\langle q^{2}\right\rangle =&\langle n|\, S_{1}^{\dagger}q^{2}S_{1}\,|n\rangle = \frac{1}{\nu^{2}}\langle n|\, q^{2}\,|n\rangle \\ =&\frac{1}{\nu^{2}}\frac{2n+1}{2},\ \nu=\mathrm{e}^{\lambda}, \\ \left\langle p^{2}\right\rangle =&\nu^{2}\frac{2n+1}{2}, \\ \Delta p\Delta q =&\frac{2n+1}{2}, \end{aligned} \quad (59)$$

which is also different from Eq.(52). It is obvious that, as expected, the fluctuation of quadrature phase amplitudes in the thermo number state is larger than that in the single-mode squeezed number state due to temperature effect.

In summary, in the context of TFD, by using the neat expression of the thermo Wigner operator in thermo entangled state representation, we have derived the Wigner function of thermo number states. This could be the easiest method in treating the thermo effect. We also found that the oscillating frequency of the shape of Wigner function rises with the increase of number n and that the oscillating action of WF becomes inconspicuous with the increase of temperature T. In addition, we have calculated the quantum fluctuation of quadrature amplitudes in $S\,|n,\tilde{n}\rangle$. A similar conclusion can be drawn for the fluctuation of particle number. These indicate that the quantum noise becomes larger accompanying the increase of $T\,(\theta)$ and n. For an operational definition of Wigner function and the discussion about quantum decoherence we refer to Refs.[18] and [19].

References

[1] Wigner E 1932 *Phys. Rev.* **40** 749

[2] Vogel K and Risken H 1989 *Phys. Rev.* A **40** 2847

[3] Smithey D T, Beck M and Raymer M G 1993 *Phys. Rev. Lett.* **70** 1244

[4] Wolfgang P Schleich 2001 *Quantum Optics in Phase Space* (Birlin: Wiley-VCH)

[5] Fan H Y 2005 *From Quantum Mechanics to Quantum Optics—Development of the Mathematical Physics* (Shanghai: Jiaotong University Press) p107 (in Chinese)
[6] Fan H Y, Lu H L and Fan Y 2006 *Ann. Phys.* **321** 480
Fan H Y 2003 *J. Opt.* B: *Quant. Semiclass. Opt.* **5** R147
[7] Wünsche A 1999 *J. Opt.* B: *Quant. Semiclass. Opt.* **1** R11
[8] Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* 3rd Ed. (New York: Springer Verlag)
[9] Takahashi Y and Umezawa H 1975 *Collective Phenomena* **2** 55
[10] Fan H Y and Fan Y 1998 *Phys. Lett.* A **246** 242
[11] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[12] Fan H Y and Klauder J R 1994 *Phys. Rev.* A **49** 704
[13] Fan H Y and Fan Y 1996 *Phys. Rev.* A **54** 958
[14] Fan H Y and Fan Y 1998 *Phys. Lett.* A **246** 242
[15] Fan H Y and Fan Y 2001 *Phys. Lett.* A **282** 269
[16] Meng X G, Wang J S, Liang B L and Li H Q 2008 *Chin. Phys.* B **17** 1791
[17] Wang J S and Meng X G 2008 *Chin. Phys.* B **17** 1254
[18] Lougovski P, Solano, Zhang Z M, Walther H, Mack H and Schleich W P 2003 *Phys. Rev. Lett.* **91** 010401
[19] Fan H Y and and Hu L Y 2008 *Mod. Phys. Lett.* B **22** 2435
Hu L Y and and Fan H Y 2008 *J. Opt. Soc. Am.* B **25** 1955