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Theoretical analysis of quantum game in cavity QED*

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Recent years, several ways of implementing quantum games in different physical systems have been presented. In this paper, we perform a theoretical analysis of an experimentally feasible way to implement a two player quantum game in cavity quantum electrodynamic(QED). In the scheme, the atoms interact simultaneously with a highly detuned cavity mode with the assistance of a classical field. So the scheme is insensitive to the influence from the cavity decay and the thermal field, and it does not require the cavity to remain in the vacuum state throughout the procedure.

Keywords: entanglement, quantum game, cavity QED
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1. Introduction

Recent development in quantum computation and quantum information theory[1,2] has extended the scope of game theory to quantum world. Eisert *et al*[3] first introduced quantum entanglement to the two-player prisoner's dilemma and showed that the game ceased to pose a dilemma if quantum strategies were allowed for. The properties of prisoner's dilemma were presented in Ref.[4], and the investigations on multiplayer quantum game[5,6] and continuous variable quantum game[7] were conducted. Recently, quantum games with decoherence have been discussed.[8−10] Quantum game theory is useful for investigating quantum communication, quantum algorithm, quantum cryptography, etc, so finding an efficient way of implementing quantum games in different physical systems is important for the quantum information theory.

The prisoner's dilemma game was first realized on a nuclear magnetic resonance quantum computer.[11] Then two proposals for optically realizing quantum games[12,13] were presented. However, in their optical proposals, the Kerr medium and the technology of single photons were both difficult in practical experiments. While a scheme for quantum games in ion traps has been suggested recently.[14] In the scheme, the decoherence is also a big problem in practical experiments. As is well known, cavity quantum electrodynamics (QED) is one of important candidates for quantum information processing(QIP) and attracts much attention. Various schemes have been proposed for implementing several quantum algorithms in cavity QED, such as quantum Fourier transform,[15] Deutsch–Jozsa algorithm,[16] quantum dense coding,[17] Grover search algorithm,[18] quantum clock synchronization algorithm[19] and so on. However, the quantum game in cavity QED is little investigated. In this paper, we present the scheme for implementing the two-player quantum game in cavity QED.

2. Quantum prisoner dilemma

We first briefly recall the scheme of the quantum game presented in Ref.[3]. There are two players, and each player can choose two possible strategies: coop-

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erate ($\widehat{C}$) and defect ($\hat{D}$). Classically the dominant strategy for both players is to defect(the Nash equilibrium), for no player can improve his payoff by unilaterally changing his own strategy, even though the Pareto optimal is for both players to cooperate. This is the dilemma. In quantum version, each player possesses a qubit $|C\rangle$ ($|D\rangle$). The qubits are prepared first in the initial state $|C\rangle\,|C\rangle$, then the entangling gate $\hat{J}$ acts on it. The initial state of the quantum game is $|\psi_i\rangle = \widehat{J}\,|CC\rangle$, where $\hat{J}$ is $\exp(\mathrm{i}\gamma D \otimes D/2)$, with $\gamma \in [0, \pi/2]$ measuring the entanglement of the state $|\psi_i\rangle$:

$$|\psi_i\rangle = \cos\left(\frac{\gamma}{2}\right)|C\rangle|C\rangle + \mathrm{i}\sin\left(\frac{\gamma}{2}\right)|D\rangle|D\rangle. \quad (1)$$

Then the unitary transformations $\hat{U}_A$ and $\hat{U}_B$ are chosen from a two-parameter set of unitary matrices $2 \otimes 2$ as:

$$\hat{U}\,(\theta,\phi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi}\cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \mathrm{e}^{-\mathrm{i}\phi}\cos(\theta/2) \end{pmatrix} \quad (2)$$

for all $0 \le \theta \le \pi$ and $0 \le \phi \le \pi/2$; indeed, the strategies C and D can be associated with $U(0,0)$ and $U(\pi,0)$ respectively. Finally, the disentangling gate $\widehat{J}^{+}$ is operated and the system is measured in the computational basis, giving rise to one of the four outcomes $|CC\rangle$, $|CD\rangle$, $|DC\rangle$, $|DD\rangle$. Alice's and Bob's payoffs are $\$_A = 3P_{CC} + 5P_{DC} + P_{DD}$ and $\$_B = 3P_{CC} + 5P_{CD} + P_{DD}$ respectively, where $P_{\sigma\sigma'} = \langle\sigma\sigma'|\,\rho\,|\sigma\sigma'\rangle$ is the probability of finding the state $|\sigma\sigma'\rangle$ (with $\sigma, \sigma' \in \{C, D\}$).

3. Scheme for implementing quantum game in cavity QED

As one of possible candidates for QIP, cavity QED attracts much attention. Decoherence of the cavity field is still a big obstacle for QIP in cavity QED. Recently, Zheng and Guo[20] proposed two atoms interacting with a nonresonant cavity, in which the two atoms could be entangled without information transfer between the atoms and the cavity. However, it requires the cavity to be in the vacuum state initially. Osnaghi *et al*[21] have experimentally demonstrated this scheme. Then in Ref.[22], with a strong resonant classical field added, the photon-number-dependent Stark shift could be cancelled. It does not require that the cavity be in the vacuum state initially, and so the scheme is insensitive to both the cavity decay and the thermal field.

In order to implement the quantum game in cavity QED, we encode the qubits $|C\rangle$ ($|D\rangle$) in the forms: $|C\rangle = |g\rangle$, and $|D\rangle = |e\rangle$. First we consider how to realize the entangling of gate $\hat{J}$, which is the most important operation in the quantum game, for it induces the entanglement. In the subspace spanned by $|C_1\rangle|C_2\rangle$, $|C_1\rangle|D_2\rangle$, $|D_1\rangle|C_2\rangle$, and $|D_1\rangle|D_2\rangle$, we can obtain gate $\hat{J}$ as

$$\hat{J} = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) & 0 & 0 & \mathrm{i}\sin\left(\frac{\gamma}{2}\right) \\ 0 & \cos\left(\frac{\gamma}{2}\right) & -\mathrm{i}\sin\left(\frac{\gamma}{2}\right) & 0 \\ 0 & -\mathrm{i}\sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) & 0 \\ \mathrm{i}\sin\left(\frac{\gamma}{2}\right) & 0 & 0 & \cos\left(\frac{\gamma}{2}\right) \end{bmatrix}. \quad (3)$$

We consider two identical two-level atoms simultaneously interacting with a single-mode cavity field, driven by a classical field. The Hamiltonian (assuming $\hbar = 1$) in the rotating-wave approximation reads[22]

$$H = \frac{1}{2}\sum_{j=1}^{2}\omega_0\sigma_{z,j} + \omega_a a^{+}a\sum_{j=1}^{2}[g(a^{+}\sigma_j^{-} + a\sigma_j^{+}) + \Omega(\sigma_j^{+}\mathrm{e}^{-\mathrm{i}\omega t} + \sigma_j^{-}\mathrm{e}^{\mathrm{i}\omega t})], \quad (4)$$

where $\sigma_{z,j} = |e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|$, $\sigma_j^{+} = |e_j\rangle\langle g_j|$, and $\sigma_j^{-} = |g_j\rangle\langle e_j|$, with $|e_j\rangle$ ($|g_j\rangle$) being the excited (ground) state of the jth atom; ω_0, ω_a and ω are the frequency for atomic transition, cavity mode and classical field respectively; a^{+} and a are the creation operator and annihilation operator for the cavity mode; g is the atom–cavity coupling strength; Ω is the Rabi frequency of the classical field. Assuming $\omega_0 = \omega$ and

$\delta = \omega_0 - \omega_a$, we have the following Hamiltonian in the interaction picture:[22]

$$H_{\rm I} = \sum_{j=1}^{2}[\Omega(\sigma_j^+ + \sigma_j^-) + g(\mathrm{e}^{-\mathrm{i}\delta t}a^+\sigma_j^- + \mathrm{e}^{\mathrm{i}\delta t}a\sigma_j^+)]. \quad (5)$$

When $\Omega \gg \delta$, g and $\delta \gg g$, we can obtain the evolution operator of the system in the interaction picture,[22] i.e.

$$U_I(t) = \mathrm{e}^{-\mathrm{i}H_0 t}\mathrm{e}^{-\mathrm{i}H_e t} \quad (6)$$

with

$$H_0 = \sum_{j=1}^{2}(\Omega(\sigma_j^+ + \sigma_j^-), \quad (7)$$

and

$$\begin{aligned} H_{\rm e} =& \lambda[\frac{1}{2}\sum_{j=1}^{2}(|e_j\rangle\langle e_j| + |g_j\rangle\langle g_j|) \\ &+ (\sigma_1^+\sigma_2^+ + \sigma_1^+\sigma_2^- + \text{H.c.})] \end{aligned} \quad (8)$$

with $\lambda = g^2/(2\delta)$. It is obvious from expression that (5), the phonon-number dependent Stark shift has been cancelled by an additional strong resonant classical field. If we define $J_x = \frac{1}{2}\sum_{j=1}^{2}(\sigma_j^+ + \sigma_j^-)$, expressions (4) and (5) reduce to $H_0 = 2\Omega J_x$, and $H_{\rm e} = 2\lambda J_x^2$ respectively and expression (6) becomes

$$U_I(t) = \mathrm{e}^{-\mathrm{i}2\Omega J_x}\mathrm{e}^{-\mathrm{i}2\lambda t J_x^2} = \mathrm{e}^{-\mathrm{i}b(J_x^2 + hJ_x)} \quad (9)$$

with $b = 2\lambda t$ and $h = \dfrac{\Omega}{\lambda}$. In the subspace spanned by $|e_1\rangle|e_2\rangle$, $|e_1\rangle|g_2\rangle$, $|g_1\rangle|e_2\rangle$, and $|g_1\rangle|g_2\rangle$, we define the two-qubit Hadamard gate as

$$\begin{aligned} H^{\otimes 2} =& \prod_{i=1}^{2} H_i = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ =& \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \end{aligned} \quad (10)$$

where H_i is the Hadamard gate acting on the ith atom, transforming states as $|g_i\rangle \rightarrow \dfrac{1}{\sqrt{2}}(|g_i\rangle + |e_i\rangle)$, and $|e_i\rangle \rightarrow \dfrac{1}{\sqrt{2}}(|g_i\rangle - |e_i\rangle)$. The $U_{\rm I}(t)$ can be expressed in the same basis as

$$\begin{bmatrix} \frac{1}{2} + \frac{1}{2}\cos(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{-\mathrm{i}}{2}\sin(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{-\mathrm{i}}{2}\sin(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{1}{2}\cos(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} \\ \frac{-\mathrm{i}}{2}\sin(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{1}{2} + \frac{1}{2}\cos(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{-1}{2} + \frac{1}{2}\cos(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{-\mathrm{i}}{2}\sin(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} \\ \frac{-\mathrm{i}}{2}\sin(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{-1}{2} + \frac{1}{2}\cos(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{1}{2} + \frac{1}{2}\cos(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{-\mathrm{i}}{2}\sin(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} \\ \frac{-1}{2} + \frac{1}{2}\cos(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{-\mathrm{i}}{2}\sin(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{-\mathrm{i}}{2}\sin(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} & \frac{1}{2} + \frac{1}{2}\cos(2\Omega t)\mathrm{e}^{-\mathrm{i}2\lambda t} \end{bmatrix}. \quad (11)$$

If we choose $\Omega t = m\pi$ (m is an integer), we have

$$U_I(t) = \mathrm{e}^{-\mathrm{i}\lambda t}\begin{bmatrix} \cos(\lambda t) & 0 & 0 & -\mathrm{i}\sin(\lambda t) \\ 0 & \cos(\lambda t) & -\mathrm{i}\sin(\lambda t) & 0 \\ 0 & -\mathrm{i}\sin(\lambda t) & \cos(\lambda t) & 0 \\ -\mathrm{i}\sin(\lambda t) & 0 & 0 & \cos(\lambda t) \end{bmatrix}. \quad (12)$$

Choosing $t_2 = \dfrac{\pi}{4\lambda}$ and $\dfrac{\Omega}{\lambda} = 4m + 3$ yields

$$H^{\otimes 2}U_{\rm I}(t_2)H^{\otimes 2} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (13)$$

Two atoms are simultaneously prepared in the ground state $|g_1\rangle|g_2\rangle$. First, the two atoms undergo the operation as shown in Eq.(12), then after the $H^{\otimes 2}U_{\rm I}(t_2)H^{\otimes 2}$ operation, the atomic state evolves into the state of $\cos(\theta)|g_1\rangle|g_2\rangle + \mathrm{i}\sin(\theta)|e_1\rangle|e_2\rangle$, which is the initial state of quantum game. So the entangling

of gate $\hat{J}$ can be easily realized in cavity QED.

The unitary transformations in the form of $U(\theta,\phi)$, which is the quantum game player's strategy, can be simply implemented by passing the atom through a classical laser field. The Hamiltonian in the rotating-wave approximation is given, in the interaction picture, as

$$H = \frac{\hbar\Omega}{2}(\mathrm{e}^{-\mathrm{i}\phi}|e\rangle\langle g| + \mathrm{e}^{\mathrm{i}\phi}|g\rangle\langle e|), \tag{14}$$

where Ω is the Rabi frequency. The corresponding time evaluating unitary operator is given by

$$\begin{aligned} U_{\mathrm{I}}(t) =& \exp(-\mathrm{i}Ht) \\ =& \begin{pmatrix} \cos\dfrac{\Omega_R t}{2} & \mathrm{i}\sin\dfrac{\Omega_R t}{2}\mathrm{e}^{-\mathrm{i}\phi} \\ \mathrm{i}\sin\dfrac{\Omega_R t}{2}\mathrm{e}^{-\mathrm{i}\phi} & \cos\dfrac{\Omega_R t}{2} \end{pmatrix}. \end{aligned} \tag{15}$$

Setting $\phi = \dfrac{3\pi}{2}$ yields

$$U_{\mathrm{I}}(t) = \begin{pmatrix} \cos\dfrac{\Omega_R t}{2} & \sin\dfrac{\Omega_R t}{2} \\ -\sin\dfrac{\Omega_R t}{2} & \cos\dfrac{\Omega_R t}{2} \end{pmatrix},$$

which is the $R_y(-\theta)$ operation.

In order to realize the operation of $R_Z(\theta)$, we can let the atom cross three classical laser fields. Let the atom interact with the first classical field by choosing the amplitude and the phase of the classical field respectively as $\Omega t = \pi/2$ and $\phi = 3\pi/2$. And then, let the atom cross the second classical field by choosing $\Omega t = \theta$ and $\phi = 0$. Finally, let the atom cross the third classical field by choosing $\Omega t = \pi/2$ and $\phi = \pi/2$. After completing the above three steps, we achieve the transition: $U_I(t) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}$, which is the rotation around the z axis by θ. So the unitary operator of the player's strategy can be realized by using the following identity:

$$\hat{U}(\theta,\phi) = R_Z(\phi)R_Y(-\theta)R_Z(\phi), \tag{16}$$

which can be easily realized in cavity QED.

4. Conclusion

In this paper, we have presented a simple scheme to implement the quantum prisoner's dilemma in cavity QED. By adding a strong resonant classical field during the qubit operation, we can cancel the phonon-number dependent Stark shift. Thus the scheme is immune to the influence from both cavity decay and thermal field. We hope that the realization of quantum game can give a further insight into the study on quantum network.

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