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A third-order asymptotic solution of nonlinear standing water waves in Lagrangian coordinates

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Asymptotic solutions up to third-order which describe irrotational finite amplitude standing waves are derived in Lagrangian coordinates. The analytical Lagrangian solution that is uniformly valid for large times satisfies the irrotational condition and the pressure $p = 0$ at the free surface, which is in contrast with the Eulerian solution existing under a residual pressure at the free surface due to Taylor's series expansion. In the third-order Lagrangian approximation, the explicit parametric equation and the Lagrangian wave frequency of water particles could be obtained. In particular, the Lagrangian mean level of a particle motion that is a function of vertical label is found as a part of the solution which is different from that in an Eulerian description. The dynamic properties of nonlinear standing waves in water of a finite depth, including particle trajectory, surface profile and wave pressure are investigated. It is also shown that the Lagrangian solution is superior to an Eulerian solution of the same order for describing the wave shape and the kinematics above the mean water level.

Keywords: Lagrangian, nonlinear standing waves, particle trajectory, Lagrangian wave frequency

PACC: 0340G, 4735, 9210F

1. Introduction

The motion of fluid flow may be described either by observing the fluid velocity at fixed positions or by the trajectories of particles that are carried along with the fluid flow. The two descriptions are respectively called Eulerian method and Lagrangian method. For an incompressible fluid the Eulerian approach is clearly preferable, because the corresponding continuity equation is linear. As is well known, the Eulerian description at the free surface is always expressed as a Taylor series at a fixed water level, in which it is implicitly assumed that the surface profile is a differentiable single-valued function. This Taylor series expansion with respect to the fixed free surface suggests the use of a perturbation method and the first successful application was based on the Stokes wave theory.[1] It provides a basis of regular asymptotic techniques for higher order permanent standing wave solutions at a uniform water depth or other problems of practical application.[2−11]

In the Lagrangian approach, surface elevation is described through the position of the surface particles, i.e. the particles whose vertical parameter $b = 0$. Unlike the Eulerian surfaces, which are given as an implicit function, the Lagrangian form is described through the parametric representation of particle motion. The Lagrangian description is more suitable for limiting free surface motion whereas these features cannot be described by classical Eulerian solutions.[12−15] The first water wave theory in Lagrangian coordinates was obtained by Gerstner[16] where the flow possessed a finite vorticity. Sainflou[17] and Miche[18] proposed perturbation Lagrangian solutions for a standing wave in water of a finite depth. Sekerzh-Zenkovich *et al*,[19,20] and Shingareva and Lizerraga-Celaya[21] developed analytical frequency-amplitude relationships in Lagrangian variables for capillary-gravity waves or standing waves in water of a finite depth. The above-mentioned theories are rotational, which is inconsistent with Kelvin's circulation theorem.[22]

It would be preferable to establish a theory in which waves are irrotational and perfectly constructed

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in the Lagrangian framework and they can also be easily extended to three-dimensional problems. As a first step toward this goal, a set of governing equations in Lagrangian coordinates is derived for two-dimensional standing gravity waves in water of a uniform depth. Expanding the unknown function in a small perturbation parameter expansion related to the wave steepness, we deduce the complete and systematic asymptotic equations in Lagrangian variables by using the Lindstedt–Poincaré perturbation method. Next, it will be shown that the third-order Lagrangian trajectory solution for the particles can be solved sequentially for each order of approximation. The Lagrangian mean level is derived for all particles over the whole range of levels, and it is more general than the expression which is applicable only to the particles at the free surface.[23] The mean level of the particle motion in Lagrangian form differing from that in Eulerian form has been deduced and explained by Longuet-Higgins[24] for progressive water waves. We also find that the second-order Lagrangian wave period is the same as that of the Eulerian wave[3] which differs from the case of progressive water waves.[24] The third-order trajectory solution for water particles exhibits that the motion of particle moves along inclined curves oscillating with the fluid between a node and anti-node. Beneath the nodes the particle trajectories are nearly horizontal, and under the antinodes the orbits are vertical. Comparing the wave profile obtained from vertical label $b = 0$ at the free surface in which the pressure $p = 0$ is exactly satisfied with that obtained under a residual pressure at the free surface, shows that the Lagrangian approximation provides better results for steep waves than the Eulerian solution of the same order.

2. Formulation of the problem

We consider a two-dimensional standing gravity wave in a uniform horizontal impermeable bed. The x-axis is taken at the still water level and y-axis is positive vertically upward. d is the still water depth, k is the wave number $k = 2\pi/L$ and $\sigma_{\rm w} = 2\pi/T_{\rm w}$ is the wave frequency where L is the wave length and $T_{\rm w}$ is the wave period. The fluid motion in the Lagrangian representation is described by tracing individual fluid particles. For the two-dimensional flow, fluid particles are distinguished by the horizontal parameter a and the vertical parameter b that are known as Lagrangian labels. These labels have one-to-one correlation with the initial positions (x_0, y_0) of particles, which has been shown in, say, §16 of the book by Lamb.[22] The motion is described by a set of trajectories $x(a, b, t)$ and $y(a, b, t)$, where x and y are Cartesian coordinates. The dependent variables x and y represent the position of particle at time t, and each of them is a function of independent variables a, b and t. The system of Lagrangian governing equations and the boundary conditions for two-dimensional irrotational standing water waves are summarized below:[25]

$$x_a y_b - x_b y_a = \frac{\partial(x, y)}{\partial(a, b)} = 1, \tag{1}$$

$$\begin{aligned} & x_{at} y_b - x_{bt} y_a + x_a y_{bt} - x_b y_{at} \\ = \; & \frac{\partial(x_t, y)}{\partial(a, b)} + \frac{\partial(x, y_t)}{\partial(a, b)} = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} & x_{at} x_b - x_{bt} x_a + y_{at} y_b - y_{bt} y_a \\ = \; & \frac{\partial(x_t, x)}{\partial(a, b)} + \frac{\partial(y_t, y)}{\partial(a, b)} = 0, \end{aligned} \tag{3}$$

$$\frac{\partial \phi}{\partial a} = x_t x_a + y_t y_a, \quad \frac{\partial \phi}{\partial b} = x_t x_b + y_t y_b, \tag{4}$$

$$\frac{p}{\rho} = -\frac{\partial \phi}{\partial t} - gy + \frac{1}{2}(x_t^2 + y_t^2), \tag{5}$$

$$p = 0, \quad b = 0, \tag{6}$$

$$v = y_t = 0, \quad y = b = -d. \tag{7}$$

$$x = a = \frac{nL}{2}, \quad n \in I \quad \text{(integer)}, \tag{8}$$

where the subscripts a, b, and t denote partial differentiation with respect to the specified variable, $p(a, b, t)$ is the water pressure, and $\phi(a, b, t)$ is the velocity potential function in the Lagrangian system. Except for Eqs.(4) and (5), the fundamental physical relationships determining the above equations have been derived by many authors.[22,26] Equation (1) is the continuity equation which is established at $b = 0$ of the vertical label marking the particles at the free surface, thereby imposing the invariant condition on the volume of a Lagrangian particle; Eq.(2) is the differentiation of Eq.(1) with respect to time. Equations (3) and (4) describe the irrotational flow condition and the corresponding Lagrangian velocity potential, respectively. Equation (5) is the Bernoulli equation for the irrotational flow in the Lagrangian description. Equation (6) describes the dynamic boundary condition of zero pressure at the free surface. Equation (7) is the bottom boundary condition of zero vertical velocity. Equation (8) describes the fluid particle that moves up and down but not horizontally at the side walls.

3. Asymptotic solutions

To solve the nonlinear equations (1)–(8), we use the Lindstedt–Poincaré technique[27] that yields uniform expansions to obtain the solutions in the Lagrangian system. In the Lagrangian approach, the particle positions x and y, the potential function ϕ and the pressure p each are considered as a function of independent variables a, b and time t. Following Pierson,[27] Chen[25] and Piedra-Cueva,[28] these solutions are sought in perturbation series in a small parameter ε related to wave steepness. This ordering parameter ε is also inserted to identify the order of the associated term.

$$x(a,b,t) = a + \sum_{n=1}^{\infty} \varepsilon^n [f_n(a,b,\sigma t) + f'_n(a,b,\sigma_0 t)], \tag{9}$$

$$y(a,b,t) = b + \sum_{n=1}^{\infty} \varepsilon^n [g_n(a,b,\sigma t) + g'_n(a,b,\sigma_0 t)], \tag{10}$$

$$\phi(a,b,t) = \sum_{n=1}^{\infty} \varepsilon^n [\phi_n(a,b,\sigma t) + \phi'_n(a,b,\sigma_0 t)], \tag{11}$$

$$p(a,b,t) = -\rho g b + \sum_{n=1}^{\infty} \varepsilon^n p_n(a,b,\sigma t), \tag{12}$$

$$\sigma = \sigma_0 + \sum_{n=1}^{\infty} \varepsilon^n \sigma_n = 2\pi/T_L, \tag{13}$$

where $x(a,b,t)$ and $y(a,b,t)$ are the particle positions and the Lagrangian variables (a, b) are two arbitrary characteristic parameters. In these expressions, f_n, g_n, ϕ_n and p_n are expected to be associated with nth-order harmonic solution. f'_n, g'_n and ϕ'_n are non-periodic functions that increase linearly with time. $\sigma = 2\pi/T_L$ is the angular frequency of particle motion or the Lagrangian angular frequency for a particle reappearing at the same elevation. T_L is the period of particle motion. Substituting expressions (9)–(13) into Eqs.(1)–(8), and collecting terms of equal ε order, we obtain a sequence of nonhomogeneous governing equations that can be solved successively as shown in the following sections.

3.1. First-order approximation

Collecting terms of order ε, the governing equations (1)–(8) can be obtained as

$$f_{1a} + f'_{1a} + g_{1b} + g'_{1b} + [\sigma_{0a}(f_{1\sigma t} + f'_{1\sigma_0 t}) + \sigma_{0b}(g_{1\sigma t} + g'_{1\sigma_0 t})]t = 0, \tag{14a}$$

$$\sigma_0(f_{1a\sigma t} + f'_{1a\sigma_0 t} + g_{1b\sigma t} + g'_{1b\sigma_0 t}) + \sigma_{0a}(f_{1\sigma t} + f'_{1\sigma_0 t}) + \sigma_{0b}(g_{1\sigma t} + g'_{1\sigma_0 t}) + \sigma_0\{\sigma_{0a}[f_{1(\sigma t)^2} + f'_{1(\sigma_0 t)^2}] + \sigma_{0b}[g_{1(\sigma t)^2} + g'_{1(\sigma_0 t)^2}]\}t = 0, \tag{14b}$$

$$\sigma_0(f_{1b\sigma t} + f'_{1b\sigma_0 t} - g_{1a\sigma t} - g'_{1a\sigma_0 t}) + \sigma_{0b}(f_{1\sigma t} + f'_{1\sigma_0 t}) - \sigma_{0a}(g_{1\sigma t} + g'_{1\sigma_0 t}) + \sigma_0\{\sigma_{0b}[f_{1(\sigma t)^2} + f'_{1(\sigma_0 t)^2}] - \sigma_{0a}[g_{1(\sigma t)^2} + g'_{1(\sigma_0 t)^2}]\}t = 0, \tag{14c}$$

$$\phi_{1a} + \phi'_{1a} + \sigma_{0a}(\phi_{1\sigma t} + \phi'_{1\sigma_0 t})t = \sigma_0(f_{1\sigma t} + f'_{1\sigma_0 t}), \tag{14d}$$

$$\phi_{1b} + \phi'_{1b} + \sigma_{0b}(\phi_{1\sigma t} + \phi'_{1\sigma_0 t})t = \sigma_0(g_{1\sigma t} + g'_{1\sigma_0 t}), \tag{14e}$$

$$\frac{p_1}{\rho} = -\sigma_0(\phi_{1\sigma t} + \phi'_{1\sigma_0 t}) - g(g_1 + g'_1), \tag{14f}$$

$$p_1 = 0, \quad \text{at} \quad b = 0, \tag{14g}$$

$$g_1 = g'_1 = 0, \quad \text{and} \quad g_{1\sigma t} = g'_{1\sigma_0 t} = 0, \quad \text{on} \quad b = -d, \tag{14h}$$

$$f_1 = f'_1 = 0, \quad \text{at} \quad a = nL/2, n = 0, \pm 1, \pm 2, ... \tag{14i}$$

The flow is assumed to be periodic with a crest at $a = 0$ and $t = 0$. The solution is a bounded periodic function only if the coefficient of time t is set to zero to avoid the occurrence of the time-growing secular terms, which makes the solution non-uniformly valid for large times. The first-order solution can be easily written as

$$f_1 = -\alpha \frac{\cosh(k(b+d))}{\cosh(kd)} \sin(ka) \cdot \cos(\sigma t), \quad f'_1 = 0, \tag{15}$$

$$g_1 = \alpha \frac{\sinh(k(b+d))}{\cosh(kd)} \cos(ka) \cdot \cos(\sigma t), \quad g'_1 = 0, \tag{16}$$

$$\sigma_0^2 = gk \tanh(kd) = gkT, \quad T = \tanh(kd), \tag{17}$$

$$\phi_1 = -\alpha\frac{\sigma_0}{k}\frac{\cosh(k(b+d))}{\cosh(kd)}\cos(ka)\cdot\sin(\sigma t)+D_1, \quad \phi_1' = D_1', \tag{18}$$

$$\frac{p_1}{\rho} = -\alpha g\frac{\sinh(kb)}{\cosh^2(kd)}\cos(ka)\cdot\cos(\sigma t), \tag{19}$$

where the parameter α represents a dimensionless amplitude function of particle displacement. D_1 and D_1' are the arbitrary constants and we can take $D_1 = D_1' = 0$ without loss of generality. $\phi_1(a,b,t)$ is the first-order Lagrangian velocity potential and $p_1(a,b,t)$ is the wave dynamical pressure in the Lagrangian form with pressure $p_1 = 0$ at the free surface $b = 0$. Expressions (15)–(19) exactly satisfy all the hydrodynamic equations formulated in Lagrangian terms including the irrotational condition. The dispersion relation shows that the first-order Lagrangian wave frequency (σ_0) is the same as the first-order Stokes wave frequency in the Eulerian approach. The first-order free surface in Lagrangian coordinates is given by setting $b = 0$ in expression (16), and it is similar to the expression for the wave profile found from the first-order Eulerian equations.[3]

3.2. Second-order approximation

After substituting the first order solution, the second-order governing equations in term of ε, including the continuity equation, the irrotational condition and the energy equation are given by

$$f_{2a}+f_{2a}'+g_{2b}+g_{2b}'+f_{1a}g_{1b}-f_{1b}g_{1a}+(\sigma_{1a}f_{1\sigma t}+\sigma_{1b}g_{1\sigma t})t=0, \tag{20a}$$

$$\sigma_0(f_{2a\sigma t}+f_{2a\sigma_0 t}'+g_{2b\sigma t}+g_{2b\sigma_0 t}')+\sigma_1(f_{1a}+g_{1b})_{\sigma t}+\sigma_0(f_{1a}g_{1b}-f_{1b}g_{1a})_{\sigma t}+\sigma_{1a}f_{1\sigma t}+\sigma_{1b}g_{1\sigma t}$$
$$+\sigma_0[\sigma_{1a}f_{1(\sigma t)^2}+\sigma_{1b}g_{1(\sigma t)^2}]t=0, \tag{20b}$$

$$\sigma_0(f_{2b\sigma t}+f_{2b\sigma_0 t}'-g_{2a\sigma t}-g_{2a\sigma_0 t}')+\sigma_1(f_{1b}-g_{1a})_{\sigma t}+\sigma_{1b}f_{1\sigma t}-\sigma_{1a}g_{1\sigma t}$$
$$+\sigma_0(f_{1a}f_{1b\sigma t}-f_{1a\sigma t}f_{1b}+g_{1a}g_{1b\sigma t}-g_{1b}g_{1a\sigma t})+\sigma_0[\sigma_{1b}f_{1(\sigma t)^2}-\sigma_{1a}g_{1(\sigma t)^2}]t=0, \tag{20c}$$

$$\phi_{2a}+\phi_{2a}'=\sigma_0(f_{2\sigma t}+f_{2\sigma_0 t}')+\sigma_1 f_{1\sigma t}+\sigma_0(f_{1a}f_{1\sigma t}+g_{1a}g_{1\sigma t})-\sigma_{1a}t\phi_{1\sigma t}, \tag{20d}$$

$$\phi_{2b}+\phi_{2b}'=\sigma_0(g_{2\sigma t}+g_{2\sigma_0 t}')+\sigma_1 g_{1\sigma t}+\sigma_0(f_{1b}f_{1\sigma t}+g_{1b}g_{1\sigma t})-\sigma_{1b}t\phi_{1\sigma t}, \tag{20e}$$

$$\frac{p_2}{\rho}=-[\sigma_0(\phi_{2\sigma t}+\phi_{2\sigma_0 t}')+g(g_2+g_2')]-\sigma_1\phi_{1\sigma t}+\frac{1}{2}\sigma_0^2(f_{1\sigma t}^2+g_{1\sigma t}^2), \tag{20f}$$

and the boundary conditions are

$$p_2=0, \quad \text{at} \quad b=0, \tag{20g}$$

$$g_2=g_2'=0, \quad \text{and} \quad g_{2\sigma t}=g_{2\sigma_0 t}'=0, \quad \text{on} \quad b=-d, \tag{20h}$$

$$f_2=f_2'=0, \quad \text{at } a=nL/2, n=0,\pm1,\pm2,.... \tag{20i}$$

To solve the differentiated equations for f_2, f_2', g_2 and g_2', we first insert the first-order quantities given by expressions (15)–(19) into Eqs.(20a)–(20c) and then simplify the results. Finally we obtain

$$f_{2a}+f_{2a}'+g_{2b}+g_{2b}' = \frac{1}{4}\alpha^2k^2\left[\frac{\cosh(2k(b+d))}{\cosh^2(kd)}+\frac{\cos(2ka)}{\cosh^2(kd)}\right](1+\cos(2\sigma t))$$
$$-\alpha\left[\sigma_{1a}\frac{\cosh(k(b+d))}{\cosh(kd)}\sin(ka)-\sigma_{1b}\frac{\sinh(k(b+d))}{\cosh(kd)}\cos(ka)\right]t\sin(\sigma t), \tag{21}$$

$$\sigma_0(f_{2a\sigma t}+f_{2a\sigma_0 t}'+g_{2b\sigma t}+g_{2b\sigma_0 t}')$$
$$=-\frac{1}{2}\alpha^2k^2\sigma_0\left[\frac{\cosh(2k(b+d))}{\cosh^2(kd)}+\frac{\cos(2ka)}{\cosh^2(kd)}\right]\sin(2\sigma t)$$
$$-\alpha\left[\sigma_{1a}\frac{\cosh(k(b+d))}{\cosh(kd)}\sin(ka)-\sigma_{1b}\frac{\sinh(k(b+d))}{\cosh(kd)}\cos(ka)\right](\sin(\sigma t)+\sigma_0 t\cos(\sigma t)), \tag{22}$$

$$\sigma_0(f_{2b\sigma t}+f'_{2b\sigma_0 t}-g_{2a\sigma t}-g'_{2a\sigma_0 t})$$
$$=-\alpha\left[\sigma_{1a}\frac{\sinh(k(b+d))}{\cosh(kd)}\cos(ka)+\sigma_{1b}\frac{\cosh(k(b+d))}{\cosh(kd)}\sin(ka)\right](\sin(\sigma t)+\sigma_0 t\cos(\sigma t)). \tag{23}$$

For standing waves of permanent type, the terms that increase linearly with time are set to zero to evade the development of the resonance. We have $\sigma_{1a}=\sigma_{1b}=0$ or $\sigma_1=w_1=$ constant. As a result, the second-order horizontal and vertical trajectories which satisfy the boundary conditions (20h) and (20i) can be written as

$$f_2=-\frac{\cosh(2k(b+d))}{\cosh^2(kd)}\sin(2ka)\cdot(\beta_{2220}+\beta_{2222}\cos(2\sigma t))+\frac{1}{8}\alpha^2 k\frac{1}{\cosh^2(kd)}\sin(2ka)\cdot(1+\cos(2\sigma t))$$
$$-\beta_{2111}\frac{\cosh(k(b+d))}{\cosh(kd)}\sin(ka)\cdot\cos(\sigma t),\quad f'_2=0, \tag{24}$$
$$g_2=\frac{\sinh(2k(b+d))}{\cosh^2(kd)}\cos(2ka)\cdot(\beta_{2220}+\beta_{2222}\cos(2\sigma t))+\frac{1}{8}\alpha^2 k\frac{\sinh(2k(b+d))}{\cosh^2(kd)}(1+\cos(2\sigma t))$$
$$+\beta_{2111}\frac{\sinh(k(b+d))}{\cosh(kd)}\cos(ka)\cdot\cos(\sigma t),\quad g'_2=0. \tag{25}$$

Inserting the first-order solutions given by Eqs.(15)–(19), (24) and (25) into the second-order irrotational equations (20d) and (20e) yields

$$\phi_{2a}=\sigma_0\left[2\beta_{2222}\frac{\cosh(2k(b+d))}{\cosh^2(kd)}-\frac{1}{2}\frac{\alpha^2 k}{\cosh^2(kd)}\right]\sin(2ka)\cdot\sin(2\sigma t)$$
$$+(\alpha w_1+\sigma_0\beta_{2111})\frac{\cosh(k(b+d))}{\cosh(kd)}\sin(ka)\cdot\sin(\sigma t)-\phi'_{2a}, \tag{26a}$$
$$\phi_{2b}=\sigma_0\left[-2\beta_{2222}\cos(2ka)-\frac{1}{2}\alpha^2 k\right]\frac{\sinh(2k(b+d))}{\cosh^2(kd)}\sin(2\sigma t)$$
$$-(\alpha w_1+\sigma_0\beta_{2111})\frac{\sinh(k(b+d))}{\cosh(kd)}\cos(ka)\cdot\sin(\sigma t)-\phi'_{2b}. \tag{26b}$$

Note that the secular terms in Eq.(26) must be omitted, so we obtain $\alpha w_1+\sigma_0\beta_{2111}=0$. The second-order Lagrangian velocity potential is obtained by integrating over the Lagrangian variables a and b as

$$\phi_2=\sigma_0\left[-\frac{1}{k}\beta_{2222}\cos(2ka)-\frac{1}{4}\alpha^2\right]\frac{\cosh(2k(b+d))}{\cosh^2(kd)}\sin(2\sigma t)+\frac{1}{4}\frac{\alpha^2\sigma_0}{\cosh^2(kd)}\cos(2ka)\cdot\sin(2\sigma t)+D_2(\sigma t), \tag{27}$$
$$\alpha w_1+\sigma_0\beta_{2111}=0,\quad \phi'_2=D'_2(\sigma_0 t), \tag{28}$$

where $D_2(\sigma t)$ ($D'_2(\sigma_0 t)$) is an undetermined constant that is a function of σt ($\sigma_0 t$).

Substituting the solutions up to the second order into the energy equation (20f) and applying the zero pressure condition at the free surface (20g), the unknown coefficients w_1, β_{2111},β_{2222}, β_{2220}, $D_2(\sigma t)$ and $D'_2(\sigma_0 t)$ are obtained as

$$w_1=\beta_{2111}=0,\quad \beta_{2222}=\frac{3}{16}\alpha^2 k(T^{-2}-1),$$
$$\beta_{2220}=-\frac{1}{16}\alpha^2 k(1-T^2),$$
$$D_2(\sigma t)=\frac{1}{16}\alpha^2\sigma_0(1+3T^2)\sin(2\sigma t),$$
$$\phi'_2=D'_2(\sigma_0 t)=-\frac{1}{8}\alpha^2\sigma_0^2(1-T^2)t. \tag{29}$$

Now that all constants have been determined, the second-order Lagrangian solutions are completely determined. They are given by

$$f_2=\alpha^2 k\left\{\frac{\cosh(2k(b+d))}{\cosh(2kd)}\left[-\frac{3}{16}(T^{-2}-T^2)\cos(2\sigma t)+\frac{1}{16}(1-T^4)\right]\right.$$
$$\left.+\frac{1}{8}(1-T^2)(1+\cos(2\sigma t))\right\}\sin(2ka),\quad f'_2=0, \tag{30}$$

$$g_2 = \alpha^2 k\frac{\sinh(2k(b+d))}{\cosh(2kd)}\left\{\left[\frac{3}{16}(T^{-2}-T^2)\cos(2\sigma t)-\frac{1}{16}(1-T^4)\right]\cdot\cos(2ka)\right.$$
$$\left.+\frac{1}{8}(1+T^2)(1+\cos(2\sigma t))\right\},\quad g_2'=0, \tag{31}$$

$$\phi_2 = \alpha^2\sigma_0\left\{\frac{\cosh(2k(b+d))}{\cosh(2kd)}\left[-\frac{3}{16}(T^{-2}-T^2)\cos(2ka)-\frac{1}{4}(1+T^2)\right]\right.$$
$$\left.+\frac{1}{4}(1-T^2)\cos(2ka)+\frac{1}{16}(1+3T^2)\right\}\sin(2\sigma t), \tag{32}$$

$$\phi_2' = -\frac{1}{8}\alpha^2\sigma_0^2(1-T^2)t, \tag{33}$$

$$\frac{p_2}{\rho} = gk\alpha^2\left\{\frac{3}{16}\left[2(T^{-1}-T^3)\frac{\cosh(2k(b+d))}{\cosh(2kd)}-(T^{-2}-T^2)\frac{\sinh(2k(b+d))}{\cosh(2kd)}\right]-\frac{3}{8}(T-T^3)\right\}\cos(2ka)\cdot\cos(2\sigma t)$$
$$+gk\alpha^2\left\{\frac{3}{8}(T+T^3)\frac{\cosh(2k(b+d))}{\cosh(2kd)}-\frac{1}{8}(1+T^2)\frac{\sinh(2k(b+d))}{\cosh(2kd)}-\frac{1}{8}(T+3T^3)\right\}\cos(2\sigma t)$$
$$+gk\alpha^2\left\{\frac{1}{16}(1-T^4)\frac{\sinh(2k(b+d))}{\cosh(2kd)}-\frac{1}{8}(T-T^3)\right\}\cos(2ka)$$
$$+gk\alpha^2\left\{\frac{1}{8}\left[(T+T^3)\frac{\cosh(2k(b+d))}{\cosh(2kd)}-(1+T^2)\frac{\sinh(2k(b+d))}{\cosh(2kd)}\right]+\frac{1}{8}(T-T^3)\right\}, \tag{34}$$

$$\sigma_1 = 0. \tag{35}$$

The horizontal particle trajectory in expression (30) is a spatial periodic function, implying that a particle moves up and down in space and time. It is unlike the progressive water waves that form a non-closed orbital motion. The three components of a second-order vertical trajectory in expression (31) include a time-dependent term that fluctuates in-phase, in proportion to $\cos(2\sigma t)$, a space-dependent term that fluctuates out of phase, and a vertical shift correction term decaying with water depth, which is a function of b, independent of time and space.

Taking the average of vertical trajectory g_2 from expression (31) over a given period of a particle motion, the present theory derives a second order Lagrangian mean level of standing wave $\bar{y}(a,b,t)$ for all particles at different values of vertical marked level b.

$$\bar{y}(a,b,t) = \frac{1}{T_L}\int(b+\varepsilon g_1+\varepsilon^2 g_2)\mathrm{d}t$$
$$= b+\alpha^2 k\frac{\sinh(2k(b+d))}{\cosh(2kd)}\times\left[-\frac{1}{16}(1-T^4)\cos(2ka)+\frac{1}{8}(1+T^2)\right]. \tag{36}$$

By setting $b=0$ in expression (36), the Lagrangian mean level at the free surface $\bar{\eta}_L$ is obtained as

$$\bar{y}(a,b=0)=\bar{\eta}_L = \frac{1}{8}\alpha^2 k[-(T-T^3)\cos(2ka)+2T)]. \tag{37}$$

Expression (37) is the same as that derived by Ng,[23] which holds true only at the free surface for pure standing waves. However, Ng[23] deduced the free surface setup (37) in a Lagrangian expression by using a time-averaged method resembling the linear solution which cannot be extended to higher order nonlinear solutions. In contrast, the present method can be easily generalized to the deduction of higher order Lagrangian solution. Moreover, it also deserves to be mentioned that the Lagrangian set-up in Ng's equation (57) is inconsistent with that of Longuet-Higgins[24] for progressive water waves. The Lagrangian mean elevation has been explained before by Longuet-Higgins[24] and the interested readers could see the relevant references for further details.

The Eulerian mean level $\bar{\eta}_{\mathrm{w}}$ for two-dimensional standing waves has been obtained by Tadjbakhsh and Keller[3] and also by Longuet-Higgins and Stewart[29] as

$$\bar{\eta}_{\mathrm{w}} = a^2 k\cdot\coth(2kd)\cdot\cos(2kx). \tag{38}$$

The Eulerian mean level is slightly raised at the antinodes and correspondingly lower at the nodes.

However, a reverse behaviour is found in the Lagrangian mean level. We further remark that the average of Lagrangian mean level in a horizontal Lagrangian variable a is not identically zero, while the horizontal spatial average of Eulerian one with respect to x over a wavelength vanishes due to the periodicity.

3.3. Third-order approximation

Once the problem of $O(\varepsilon^2)$ is solved, the third-order continuity, irrotational and energy equations can be obtained by applying the first- and second-order approximations to the governing equations of $O(\varepsilon^3)$ as

$$f_{3a}+f'_{3a}+g_{3b}+g'_{3b}+f_{1a}g_{2b}+f_{2a}g_{1b}-f_{1b}g_{2a}-f_{2b}g_{1a}+(\sigma_{2a}f_{1\sigma t}+\sigma_{2b}g_{1\sigma t})t=0, \tag{39a}$$

$$\sigma_0(f_{3a\sigma t}+f'_{3a\sigma_0 t}+g_{3b\sigma t}+g'_{3b\sigma_0 t})+\sigma_2(f_{1a\sigma t}+g_{1b\sigma t})+\sigma_{2a}f_{1\sigma t}+\sigma_{2b}g_{1\sigma t}+\sigma_0[\sigma_{2a}f_{1(\sigma t)^2}+\sigma_{2b}g_{1(\sigma t)^2}]t$$
$$+\sigma_0[f_{1a\sigma t}g_{2b}+f_{2a\sigma t}g_{1b}+f_{1a}g_{2b\sigma t}+f_{2a}g_{1b\sigma t}-f_{1b\sigma t}g_{2a}-f_{1b}g_{2a\sigma t}-f_{2b\sigma t}g_{1a}-f_{2b}g_{1a\sigma t}]=0, \tag{39b}$$

$$\sigma_0(f_{3b\sigma t}+f'_{3b\sigma_0 t}-g_{3a\sigma t}-g'_{3a\sigma_0 t})+\sigma_2(f_{1b\sigma t}-g_{1a\sigma t})+\sigma_{2b}f_{1\sigma t}-\sigma_{2a}g_{1\sigma t}+\sigma_0[\sigma_{2b}f_{1(\sigma t)^2}-\sigma_{2a}g_{1(\sigma t)^2}]t$$
$$+\sigma_0[f_{1a}f_{2b\sigma t}+f_{2a}f_{1b\sigma t}-f_{2a\sigma t}f_{1b}-f_{1a\sigma t}f_{2b}+g_{1a}g_{2b\sigma t}+g_{2a}g_{1b\sigma t}-g_{1b}g_{2a\sigma t}-g_{2b}g_{1a\sigma t}]=0, \tag{39c}$$

$$\phi_{3a}+\phi'_{3a}=\sigma_0(f_{3\sigma t}+f'_{3\sigma_0 t})+\sigma_2 f_{1\sigma t}+\sigma_0[f_{1\sigma t}f_{2a}+f_{2\sigma t}f_{1a}+g_{1\sigma t}g_{2a}+g_{2\sigma t}g_{1a}]-\sigma_{2a}t\phi_{1\sigma t}, \tag{39d}$$

$$\phi_{3b}+\phi'_{3b}=\sigma_0(g_{3\sigma t}+g'_{3\sigma_0 t})+\sigma_2 g_{1\sigma t}+\sigma_0[f_{1\sigma t}f_{2b}+f_{2\sigma t}f_{1b}+g_{1\sigma t}g_{2b}+g_{2\sigma t}g_{1b}]-\sigma_{2b}t\phi_{1\sigma t}, \tag{39e}$$

$$\frac{p_3}{\rho}=-[\sigma_0(\phi_{3\sigma t}+\phi'_{3\sigma_0 t})+g(g_3+g'_3)]-\sigma_2\phi_{1\sigma t}+\sigma_0^2[f_{1\sigma t}f_{2\sigma t}+g_{1\sigma t}g_{2\sigma t}], \tag{39f}$$

and the boundary conditions are

$$p_3=0 \quad \text{at} \quad b=0, \tag{39g}$$

$$g_3=g'_3=0 \quad \text{and} \quad g_{3\sigma t}=g'_{3\sigma_0 t}=0 \quad \text{on} \quad b=-d, \tag{39h}$$

$$f_3=f'_3=0 \quad \text{at} \quad a=nL/2, n=0,\pm1,\pm2,.... \tag{39i}$$

The procedure to obtain the solutions at this order is similar to that at the order of $O(\varepsilon^2)$. The secular terms that grow increasingly with time are set to zero; after a lengthy but straightforward manipulation, the second-order Lagrangian wave frequency and third-order solutions of horizontal and vertical trajectories, Lagrangian potential function and wave pressure are obtained as

$$\sigma_2 = \frac{1}{64}\alpha^2k^2(9T^{-2}-12-3T^2-2T^4)\sigma_0, \tag{40}$$

$$\begin{aligned} f_3 = {} & \alpha^3k^2\frac{\cosh(3k(b+d))}{\cosh(3kd)}\left\{\sin(3ka)\left[-\frac{1}{256}(9T^{-4}+5T^{-2}-53+39T^2)\cos(3\sigma t)\right.\right. \\ & \left.-\frac{1}{256}(9T^{-2}+27-15T^2-39T^4+18T^6)\cos(\sigma t)\right]+\sin(ka)\left[-\frac{1}{64}(3T^{-2}+6-9T^2)\right. \\ & \left.\left.\times\cos(3\sigma t)-\frac{1}{64}(15T^{-2}+32-41T^2-6T^4)\cos(\sigma t)\right]\right\} \\ & +\alpha^3k^2\frac{\cosh(k(b+d))}{\cosh(kd)}\left\{\sin(3ka)\times\left[\frac{1}{192}(15T^{-2}-34+19T^2)\cos(3\sigma t)\right.\right. \\ & \left.+\frac{1}{64}(9T^{-2}-24+17T^2-2T^4)\cos(\sigma t)\right]+\sin(ka)\left[\frac{1}{768}(9T^{-2}+78-47T^2)\cos(3\sigma t)\right. \\ & \left.\left.+\frac{1}{64}(9T^{-2}-16+T^2-2T^4)\cos(\sigma t)\right]\right\}, \quad f'_3=0, \end{aligned} \tag{41}$$

$$\begin{aligned} g_3 = {} & \alpha^3k^2\frac{\sinh(3k(b+d))}{\cosh(3kd)}\left\{\cos(3ka)\left[\frac{1}{256}(9T^{-4}+5T^{-2}-53+39T^2)\cos(3\sigma t)\right.\right. \\ & \left.+\frac{1}{256}(9T^{-2}+27-15T^2-39T^4+18T^6)\cos(\sigma t)\right]+\cos(ka)\left[\frac{1}{192}(15T^{-2}+34-33T^2)\cos(3\sigma t)\right. \\ & \left.\left.+\frac{1}{64}(9T^{-2}+20-19T^2+6T^4)\cos(\sigma t)\right]\right\}+\alpha^3k^2\frac{\sinh(k(b+d))}{\cosh(kd)}\left\{\cos(3ka)\left[-\frac{1}{64}(3T^{-2}-6+3T^2)\right.\right. \end{aligned}$$

$$\times\cos(3\sigma t)-\frac{1}{64}(15T^{-2}-28+11T^2+2T^4)\cos(\sigma t)\Big]+\cos(ka)\left[-\frac{1}{768}(9T^{-2}+78-47T^2)\cos(3\sigma t)\right.$$
$$\left.-\frac{1}{64}(9T^{-2}-16+T^2-2T^4)\cos(\sigma t)\Big]\right\},\quad g_3'=0, \tag{42}$$

$$\phi_3=\alpha^3k\sigma_0\frac{\cosh(3k(b+d))}{\cosh(3kd)}\left\{\cos(3ka)\left[-\frac{1}{256}(9T^{-4}+5T^{-2}-53+39T^2)\sin(3\sigma t)\right.\right.$$
$$-\frac{1}{256}(3T^{-2}+9-5T^2-13T^4+6T^6)\sin(\sigma t)\Big]+\cos(ka)\left[-\frac{1}{64}(9T^{-2}+22-15T^2)\sin(3\sigma t)\right.$$
$$\left.\left.-\frac{1}{64}(3T^{-2}+8-T^2+6T^4)\sin(\sigma t)\Big]\right\}+\alpha^3k\sigma_0\frac{\cosh(k(b+d))}{\cosh(kd)}\left\{\cos(3ka)\left[\frac{1}{64}(9T^{-2}-22+13T^2)\right.\right.$$
$$\left.\left.\times\sin(3\sigma t)+\frac{1}{64}(3T^{-2}-12+11T^2-2T^4)\sin(\sigma t)\Big]+\frac{1}{256}(9T^{-2}+94-63T^2)\cos(ka)\cdot\sin(3\sigma t)\right\}, \tag{43}$$

$$\frac{p_3}{\rho}=g\alpha^3k^2\left\{\left[(3T+9T^3)\frac{\cosh(3k(b+d))}{\cosh(3kd)}-(1+3T^2)\frac{\sinh(3k(b+d))}{\cosh(3kd)}\right]\cdot\frac{1}{256}(9T^{-4}-22T^{-2}+13)\right.$$
$$\left.-\frac{1}{64}\left[(21T^{-1}-50T+29T^3)\frac{\cosh(k(b+d))}{\cosh(kd)}-(3T^{-2}-6+3T^2)\frac{\sinh(k(b+d))}{\cosh(kd)}\right]\right\}\cos(3ka)\cdot\cos(3\sigma t)$$
$$+g\alpha^3k^2\left\{\left[\frac{1}{3}(T+3T^3)\frac{\cosh(3k(b+d))}{\cosh(3kd)}-(1+3T^2)\frac{\sinh(3k(b+d))}{\cosh(3kd)}\right]\cdot\frac{1}{256}(9T^{-2}-15T^2+6T^4)\right.$$
$$\left.-\frac{1}{64}\left[(9T^{-1}-28T+21T^3-2T^5)\frac{\cosh(k(b+d))}{\cosh(kd)}-(15T^{-2}-28+11T^2+2T^4)\frac{\sinh(k(b+d))}{\cosh(kd)}\right]\right\}$$
$$\times\cos(3ka)\cdot\cos(\sigma t)+g\alpha^3k^2\left\{\frac{1}{64}(21T^{-1}+50T-39T^3)\frac{\cosh(3k(b+d))}{\cosh(3kd)}-\frac{1}{192}(15T^{-2}-11)(1+3T^2)\right.$$
$$\times\frac{\sinh(3k(b+d))}{\cosh(3kd)}+\left[9T\frac{\cosh(k(b+d))}{\cosh(kd)}-\frac{\sinh(k(b+d))}{\cosh(kd)}\right]\cdot\frac{1}{768}(-9T^{-2}-78+47T^2)$$
$$\left.-\frac{1}{16}(T-T^3)\frac{\cosh(k(b+d))}{\cosh(kd)}\right\}\cos(ka)\cdot\cos(3\sigma t)+g\alpha^3k^2\left\{\frac{1}{64}\left[(9T^{-1}+24T-7T^3+6T^5)\frac{\cosh(3k(b+d))}{\cosh(3kd)}\right.\right.$$
$$-(9T^{-2}-7+2T^2)(1+3T^2)\frac{\sinh(3k(b+d))}{\cosh(3kd)}+(9T^{-1}-20T+5T^3-2T^5)\frac{\cosh(k(b+d))}{\cosh(kd)}$$
$$\left.\left.+(9T^{-2}-16+T^2-2T^4)\frac{\sinh(k(b+d))}{\cosh(kd)}\right]\right\}\cos(ka)\cdot\cos(\sigma t). \tag{44}$$

Expression (40) shows the second-order Lagrangian wave frequency of a particle motion, which is equal to the Eulerian wave frequency obtained by Tadjbakhsh and Keller.[3] It indicates that the Lagrangian wave frequency in standing water waves is irrelevant to the Lagrangian horizontal and vertical labels (a,b) of a particle. The two components of third-order particle trajectory include a time-dependent term that fluctuates in-phase, proportionally to $\cos(3\sigma t)$ or $\cos(\sigma t)$, and a space-dependent term that fluctuates out of phase.

4. Results and discussion

4.1. Surface profile and wave pressure

The surface profile for the highest standing wave in water of an infinite depth is obtained in the present study in the third-order or fifth-order approximation of Penney and Price.[2] And the experimental data obtained by Taylor[30] are depicted in Fig.1. It can be

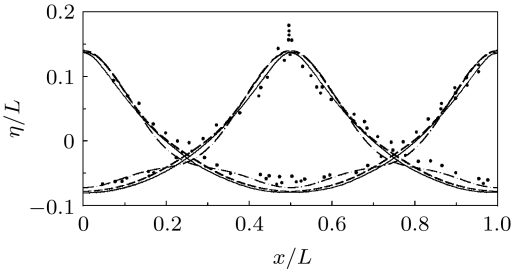


Fig.1. Profiles of standing wave in deep water, where solid points denote the measurements of Taylor;[30] solid line represents the third-order Lagrangian solution; dash-dotted line refers to the results obtained in the third-order approximation of Penny and Price;[2] dashed line shows the results obtained in the fifth-order approximation of Penny and Price.[2]

seen that the results of the third-order Lagrangian solution are in agreement with the experimental data and also with the results obtained in the fifth-order Eulerian approximation except near the wave crest, while a secondary oscillation is found in the third-order wave profile of Penney and Price.[2] Clearly, the third-order Lagrangian solution is superior to the Eulerian solution for describing standing wave profiles, that is, the theoretical Lagrangian solution provides a much better prediction of the free surface for a high amplitude wave profile than the Stokes' expansion of the same order.

The time history of wave pressure at a mean water level can be seen in Fig.2. Figure 2 displays the curves for time history of wave pressure at a mean water level, obtained separately from the numerical model,[6] the third-order Eulerian solution[3] and the third order Lagrangian solution given by the present theory at $H/d = 0.17$ and four different relative water depths, i.e. $d/L = 0.1, 0.3, 0.5, 1.0$, for nonlinear standing waves. Comparing with the numerical calculations[6] shows that the third-order Lagrangian solution is more accurate than that obtained in the third-order Eulerian approximation of Tadjbakhsh and Keller[3] in a deeper water depth.

It is known that the surface profile is not preliminarily specified in the Eulerian analysis, and it has to be derived from a solution that is restricted to a single valued function of the spatial position and the dynamic boundary condition at the free surface, which can be satisfied only in an approximate manner.[6] However, the free surface in the Lagrangian analysis is represented by an explicit parametric function of particles, and this allows a larger flexibility in describing the actual shape and the wave kinematics above the mean water level of the water waves. Moreover, the Lagrangian solution exactly satisfies the dynamic boundary condition and has zero pressure at the free surface.

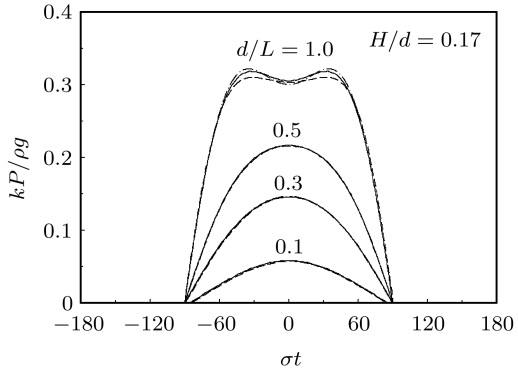
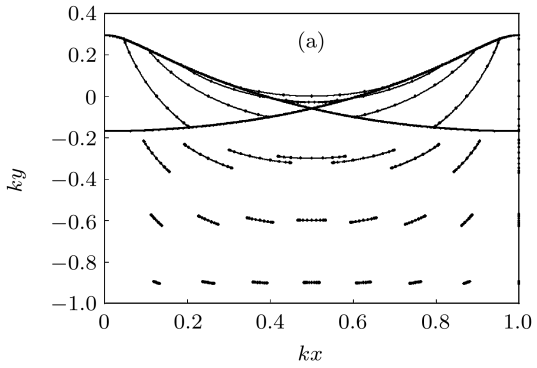
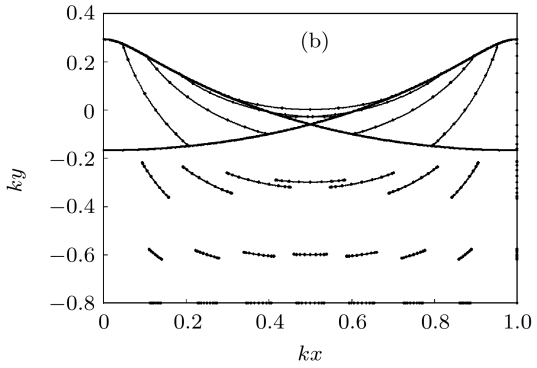


Fig.2. Time history of wave pressure at a mean water level with $H/d = 0.17$ and four different relative water depths: $d/L = 0.1, 0.3, 0.5, 1.0$ for nonlinear standing waves, where dashed line represents the results obtained from the numerical model,[6] dash-dotted line denotes the results given by the third-order Eulerian solution,[3] and solid line refers to the results gained from the third-order Lagrangian solution given by the present theory.

4.2. The particle orbits

Figures 3(a)–3(d) are the plots of particle trajectories for various values of elevation b, including $b = 0$, which corresponds to particles at the free surface, and different values of wave steepness ($H/L = 0.5$, 0.4, 0.3, 0.2) from $t/T_{\rm w} = 0$ to $t/T_{\rm w} = \pi/2$. It can be seen that the particle motion is greatest at the free surface and decreases exponentially with depth below the surface. Furthermore, the surface particle always keeps on the free surface. In each complete orbit, the particle moves along inclined curves oscillating with the fluid between a node and anti-node. The inclined curve is steepest where the vertical excursion is larger than its horizontal excursion at the free surface and becomes horizontal at the bottom. The particle trajectory moves almost horizontally around the nodes, and particle moves purely vertically under the anti-nodes.





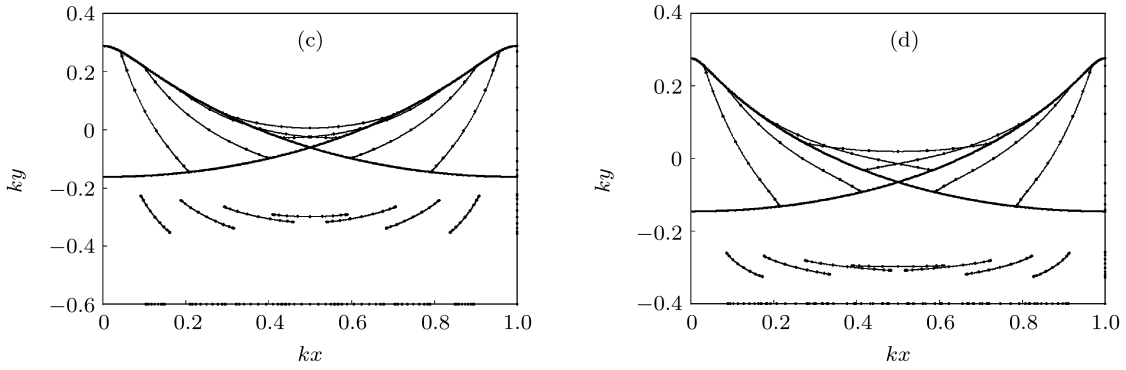


Fig.3. The third-order particle trajectories at different elevations for various wave conditions, where (a) is for the relative water depth $d/L = 0.5$ and the wave steepness $H/L = 0.1$, (b) is for $d/L = 0.4$ and $H/L = 0.1$, (c) is for $d/L = 0.3$ and $H/L = 0.1$, and (d) is for $d/L = 0.2$ and $H/L = 0.1$.

5. Concluding remarks

In this paper we have shown that for irrotational standing gravity waves the Lagrangian solutions of the whole flow field up to the third order are obtained in the complete Lagrangian coordinates. This solution exactly satisfies the original nonlinear governing equations and the irrotational condition in the Lagrangian system. As can be seen from the second-order solution, the Lagrangian mean elevation apparently differs from that in the Eulerian description. The generalized Lagrangian mean level in Eq.(36) which is a function of vertical position coincides with that presented by Ng[23] at the free surface, with the vertical label b set to zero. However, Ng[23] used a time-averaged method resembling the linear solution to deduce the free surface setup in Lagrangian expression, which cannot be extended to higher order nonlinear solutions. On the contrary, the present method can easily present the higher order solution.

In the third-order asymptotic solution, the Lagrangian wave frequency of a particle motion is equal to the Eulerian wave frequency obtained by Tadjbakhsh and Keller.[3] It implies that the Lagrangian wave frequency in standing water waves is irrelevant to the Lagrangian labels (a, b) of a particle, and this result is quite different from that in progressive water waves.[24] The third-order trajectory of particles exhibits that in each complete orbit the particle moves along inclined curves oscillating with the fluid between a node and anti-node. The particle orbit moves almost horizontally around the nodes and moves purely vertically under the anti-nodes.

The wave profile is compared with those obtained from the third-, fifth-order wave solutions in the Eulerian form and also with experimental results. The third-order Lagrangian solution accord better with experimental results than the asymptotic solutions of the same orders in the Eulerian system. The third-order Eulerian solution shows that the Eulerian wave profile presents a pronounced secondary oscillation in the trough for large waves which is not a real physical phenomenon for waves in a constant form while the present solution of the same order gives a flatter shape. Therefore, theoretical Lagrangian solution provides a much better prediction of the free surface and wave pressure than the Stokes' expansion of the same order for nonlinear standing waves.

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